VĚTA (LEVI)

$$\left.\begin{array}{l} \bullet\ f_n \geq 0, \\ \bullet\ f_n \text{ měřitelné}, \\ \bullet\ f_n \nearrow f \text{ v } \mathbb{R}^N, \end{array}\right\} \Rightarrow \int_{\mathbb{R}^N} f_n \to \int_{\mathbb{R}^N} f.$$

Důkaz. $\left(\int_{\mathbb{R}^N} f_n\right)$ je neklesající posloupnost nezáporných čísel $\Rightarrow \left[\exists \alpha \in \mathbb{R}^*: \int_{\mathbb{R}^N} f_n \nearrow \alpha\right]$

$$f_n \leq f \Rightarrow \int f_n \leq \int f \Rightarrow \boxed{\alpha \leq \int f}$$

f je nezáporná měřitelná $\Downarrow$ $\int f$ ex.

... chceme rovnost

Buď • s jednoduchá měřitelná, $\boxed{0 \leq s \leq f}$,

• $\boxed{c \in (0,1)}$.

Def. $\boxed{E_n = \{x \in \mathbb{R}^N : f_n(x) \geq c\, s(x)\}}$

Pak $E_n \subset E_{n+1}$, $E_n \in \mathcal{B}_0$, $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^N$

$(s(x) \neq 0 \Rightarrow c \cdot s(x) < f(x))$,
$f_n(x) \to f(x)$

a proto

$$\alpha \geq \int_{\mathbb{R}^N} f_n \geq \int_{E_n} f_n \geq c \int_{E_n} s \longrightarrow c \cdot \int_{\mathbb{R}^N} s$$

$$\int_{E_n} s = \sum \alpha_j \underbrace{\lambda(E_n \cap A_j)}_{\downarrow \ \lambda(\mathbb{R}^N \cap A_j)}$$

Tedy $\forall c \in (0,1)$: $\alpha \geq c \int_{\mathbb{R}^N} s \underset{(c \to 1-)}{\Longrightarrow} \alpha \geq \int_{\mathbb{R}^N} s \Rightarrow$

$$\Rightarrow \boxed{\alpha \geq \sup\left\{\int_{\mathbb{R}^N} s \ \dots\right\} = \int_{\mathbb{R}^N} f}$$

Cbd.

FATOUOVO LEMMA

- $f_n \geq 0$,
- f_n měřitelné na $\mathbb{R}^N$,

$$\left.\begin{array}{l} f_n \geq 0, \\ f_n \text{ měřitelné na } \mathbb{R}^N, \end{array}\right\} \Longrightarrow \int_{\mathbb{R}^N} \liminf f_n \leq \liminf \int_{\mathbb{R}^N} f_n .$$

Důkaz

$$\boxed{g_k \overset{\text{def.}}{=} \inf_{j \geq k} f_j} \quad \ldots \; g_k \text{ měřitelné}, \; g_k \geq 0$$

$$\Downarrow$$

$$g_k \leq f_k \Rightarrow \int g_k \leq \int f_k \Rightarrow \liminf \int g_k \leq \liminf \int f_k$$

$$g_k \nearrow \liminf f_n \text{ (z definice liminf...)} \xRightarrow{\{\text{LEVI}\}} \lim \int g_k = \int \liminf f_n$$

$$\lim \int g_k = \liminf \int g_k$$

cbd.

VĚTA (LEBESGUE)

$$\left.\begin{array}{l} \bullet\ f_n \text{ měřitelné}, \\ \bullet\ f_n \to f \text{ v } \mathbb{R}^N, \\ \bullet\ \exists g \in \mathcal{L}^1(\mathbb{R}^N)\ \forall n\ \forall x : |f_n(x)| \le g(x), \end{array}\right\} \Rightarrow \begin{array}{l} \bullet\ f \in \mathcal{L}^1(\mathbb{R}^N), \\ \bullet\ \int_{\mathbb{R}^N} |f_n - f| \to 0, \\ \bullet\ \int_{\mathbb{R}^N} f_n \to \int_{\mathbb{R}^N} f . \end{array}$$

Důkaz.

$$\left.\begin{array}{l} f \text{ je měřitelná} \\ |f| \le g \end{array}\right\} \Rightarrow \int_{\mathbb{R}^N} |f| < \infty \Rightarrow f \in \mathcal{L}^1(\mathbb{R}^N)$$

Definujme $\boxed{g_n \overset{\text{def.}}{=} 2g - |f - f_n|}$

Pak

- g_n jsou měřitelné,
- $g_n \ge 0$ (protože $|f - f_n| \le |f| + |f_n| \le g + g = 2g$)

$$\liminf_n g_n(x) = 2g(x) \quad \text{v } \mathbb{R}^N$$

Fatouovo lemma: $\int 2g = \int \liminf g_n \le \liminf \int g_n$

Odtud:

$$\int 2g \le \liminf \int (2g - |f - f_n|) = \liminf \left(\int 2g + \int -|f - f_n| \right) =$$

$$= \int 2g + \liminf \int -|f - f_n| = \underbrace{\int 2g}_{\substack{\in \mathbb{R}!! \\ (g \in \mathcal{L}^1(\mathbb{R}^N))}} - \limsup \int |f - f_n|$$

$$\Downarrow$$

$$0 \le \liminf \int |f - f_n| \le \limsup \int |f - f_n| \le 0 \Rightarrow \lim \int_{\mathbb{R}^N} |f - f_n| = 0$$

A konečně:

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \le \int |f_n - f| \to 0 \Rightarrow \int_{\mathbb{R}^N} f_n \to \int_{\mathbb{R}^N} f .$$

cbd.