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MATHEMATICAL ANALYSIS I

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INSTEAD OF PREFACE

It was all very well to say ‘Drink me,’ but the wise little Alice was not going to do *that* in a hurry. ‘No, I’ll look first,’ she said, ‘and see whether it’s marked “poison” or not’; for she had read several nice little histories about children who had got burnt, and eaten up by wild beasts and other unpleasant things, all because they *would* not remember the simple rules their friends had taught them: such as, that a red-hot poker will burn you if you hold it too long; and that if you cut your finger *very* deeply with a knife, it usually bleeds; and she had never forgotten that, if you drink much from a bottle marked ‘poison,’ it is almost certain to disagree with you, sooner or later.

However, this bottle was *not* marked ‘poison,’ so Alice ventured to taste it, and finding it very nice, (it had, in fact, a sort of mixed flavour of cherry-tart, custard, pine-apple, roast turkey, toffee, and hot buttered toast,) she very soon finished it off.

(Lewis Carroll, ‘Alice’s Adventures In Wonderland’)

Contents

A THE REAL NUMBER SYSTEM; THE SUPREMUM THEOREM

1 NUMBER SYSTEMS, THEIR NOTATION AND SOME PROPOSITIONS

1.1 $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\} \dots$ **the set of all natural numbers**.

The following theorem is useful not only when thinking about a structure of the natural numbers, but it is also a good instrument for proving many mathematical propositions.

1.2 Theorem (Principle of Mathematical Induction). *Let M be a subset of $\mathbb{N}$ such that*

i) $1 \in M$,

ii) $\forall n \in M : \; n + 1 \in M$.

Then $M = \mathbb{N}$.

1.3 Example. Prove that

$$\forall n \in \mathbb{N} : \; 1 + 2 + \ldots + n = \frac{1}{2}\, n\,(n+1).$$

PROOF. Let us denote

$$M := \left\{ k \in \mathbb{N} : \; 1 + 2 + \ldots + k = \frac{1}{2}\, k\,(k+1) \right\}.$$

The task is to prove that $M = \mathbb{N}$. According to Theorem 1.2 it is sufficient to show that the premises *i)* and *ii)* hold for M. The premise *i)* holds clearly since $1 = \frac{1}{2} \cdot 1 \cdot (1+1)$. In order to prove *ii)* let us assume that $n \in M$, i.e.,

$$1 + 2 + \ldots + n = \frac{1}{2}\, n\,(n+1).$$

We shall show that then $n + 1 \in M$, i.e.,

$$1 + 2 + \ldots + n + (n+1) = \frac{1}{2}\,(n+1)\,(n+1+1).$$

This is easy since from our assumption it follows that

$$(1 + 2 + \ldots + n) + (n+1) = \frac{1}{2}\, n\,(n+1) + (n+1) = \frac{1}{2}\,(n+1)\,(n+1+1).$$

Thus the premise *ii)* holds too.

$\square$

There is also another way how to prove this proposition. Given $n \in \mathbb{N}$,

$$\begin{aligned} 2\,(1 + 2 + 3 + \ldots + n) = \; & 1 + 2 + 3 + \ldots + n + \\ & + n + (n-1) + (n-2) + \ldots + 1 = n\,(n+1). \end{aligned}$$

$\square$

1.4 $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \bigcup_{n\in\mathbb{N}} \{n, -n, 0\} \ldots$ **the set of all integer numbers**.

1.5 $\mathbb{Q} = \{\frac{p}{q} : \ p, q \in \mathbb{Z} \ \wedge \ q \neq 0\} \ldots$ **the set of all rational numbers**.

There is a lot of ideas showing certain incompleteness of the rational number system, despite the fact that between each two - arbitrarily near - distinct rational numbers there still lies an infinite number of them. For example,

i) $(\forall \varepsilon > 0)(\exists\, p_1, p_2 \in \mathbb{Q}) : \ 2 - \varepsilon < p_1^2 < 2 < p_2^2 < 2 + \varepsilon$,

ii) there is no rational number p such that $p^2 = 2$.

Let us prove at least the second proposition.

PROOF. Conversely, we assume that there is a rational number p such that

$$p^2 = 2.$$

Since $p \in \mathbb{Q}$, there are integer coprime nonzero numbers m, n such that

$$p = \frac{m}{n}.$$

Thus we get

$$\frac{m^2}{n^2} = 2$$

and so $m^2 = 2n^2$. This implies that m has to be even (note that the square of an even number is even and the square of an odd number is odd). Therefore there is a $k \in \mathbb{Z}$ such that $m = 2k$. By inserting this relation into $m^2 = 2n^2$, we obtain $4k^2 = 2n^2$. Hence it follows that $n^2 = 2k^2$. This implies that also n has to be even which contradicts our assumption that m and n are coprime.

□

1.6 $\mathbb{R} \ldots$ **the set of all real numbers**.

Let us recall that we have number of operations defined in $\mathbb{R}$ (and also in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$): $+, \ -, \ \cdot, \ :, \ |\,|$.

An order of the real numbers is their other essential characteristic:

- for every two real numbers x, y exactly one of the following possibilities holds

  i) $x < y$,

  ii) $x = y$,

  iii) $x > y$;

- for every three real numbers x, y, z it holds that $(x < y \ \wedge \ y < z) \Rightarrow (x < z)$.

Let us also introduce the notation

$\mathbb{R}^+ = \{x \in \mathbb{R} : \ x > 0\} \ldots$ the set of all positive real numbers,
$\mathbb{R}^- = \{x \in \mathbb{R} : \ x < 0\} \ldots$ the set of all negative real numbers,
$\mathbb{R} \setminus \mathbb{Q}$ $\ldots$ the set of all irrational numbers.

1.7 $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\} \ldots$ **the extended real number system**.
Let us extend the order from $\mathbb{R}$ to $\mathbb{R}^*$:

- $\forall x \in \mathbb{R} : \ -\infty < x \ \wedge \ x < +\infty$,
- $-\infty < +\infty$.

Let us also define the following operations in $\mathbb{R}^*$:

- $\forall x > -\infty : \ x + (+\infty) = +\infty + x = +\infty$,
- $\forall x < +\infty : \ x + (-\infty) = -\infty + x = -\infty$,
- $\forall x \in \mathbb{R}^+ \cup \{+\infty\} : \ x \cdot (+\infty) = +\infty \cdot x = +\infty,$
  $x \cdot (-\infty) = -\infty \cdot x = -\infty,$
- $\forall x \in \mathbb{R}^- \cup \{-\infty\} : \ x \cdot (+\infty) = +\infty \cdot x = -\infty,$
  $x \cdot (-\infty) = -\infty \cdot x = +\infty,$
- $\forall x \in \mathbb{R} : \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0$,
- $|-\infty| = |+\infty| = +\infty$.

Instead of $x + (+\infty)$ we usually write $x + \infty$. Similarly, instead of $x + (-\infty)$ we write $x - \infty$. Also, we denote $+\infty$ briefly by ∞.

1.8 Remark. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^*$

1.9 Caution. We do not define: $+\infty - \infty, \ -\infty + \infty, \ 0 \cdot (\pm\infty), \ \frac{(\pm\infty)}{(\pm\infty)}, \ \frac{x}{0} \ (x \in \mathbb{R}^*)$.

2 MAXIMUM, MINIMUM; SUPREMUM, INFIMUM

Already since secondary school we have been used to dealing with various sets of numbers. For instance, let us consider the intervals $(-1, 1)$ and $(-1, 1]$ and number 1 which in both intervals plays role of a "right bound": any number lying to the right from it belongs neither to $(-1, 1)$ nor to $(-1, 1]$. Also, every number from the set $[1, +\infty) \cup \{+\infty\}$ has this quality (in a while: to be an upper bound). However, number 1 is the best since it is the smallest one. The fact that in the first case number 1 does not belong to the given interval and in the second case it does is an essence of the difference between the concepts "supremum" and "maximum". These concepts (together with other concepts defined below) are used for characterization of even much more complicated subsets of $\mathbb{R}^*$.

2.1 Definitions. Let $M \subset \mathbb{R}^*$.

- A number $k \in \mathbb{R}^*$ is said to be an upper bound of the set M if

$$\forall x \in M : \ x \leq k.$$

- A number $l \in \mathbb{R}^*$ is said to be a lower bound of the set M if

$$\forall x \in M: \ x \geq l.$$

2.2 Definitions. Let $M \subset \mathbb{R}^*$.

- If an upper bound of M exists and belongs to M, we call it a maximum of M and denote it by $\max M$.
- If a lower bound of M exists and belongs to M, we call it a minimum of M and denote it by $\min M$.

2.3 Definitions. Let $M \subset \mathbb{R}^*$.

- A number $s \in \mathbb{R}^*$ is called a supremum of the set M if

  i) $\forall x \in M: \ x \leq s$
  (i.e., s is the upper bound of M),

  ii) $(\forall k \in \mathbb{R}^*, \ k < s)\,(\exists x \in M): \ x > k$
  (i.e., any number smaller than s is not the upper bound of M).

  We write $s = \sup M$.

- A number $i \in \mathbb{R}^*$ is called an infimum of the set M if

  i) $\forall x \in M: \ x \geq i$
  (i.e., i is the lower bound of M),

  ii) $(\forall l \in \mathbb{R}^*, \ l > i)\,(\exists x \in M): \ x < l$
  (i.e., any number larger than i is not the lower bound of M).

  We write $i = \inf M$.

2.4 Observation. $\sup M$ is the least upper bound of M and $\inf M$ is the greatest lower bound of M.

2.5 Examples.

1) $M = (-1, 1] \ \ldots \ \min M$ does not exist, $\inf M = -1, \ \sup M = \max M = 1$.
2) $M = \mathbb{R}^+$ ... neither $\min M$ nor $\max M$ exists, $\inf M = 0, \ \sup M = +\infty$.
3) $M = \emptyset$ ... neither $\min M$ nor $\max M$ exists, $\inf M = +\infty, \ \sup M = -\infty$.
4) $M = \left\{-\frac{1}{n}: \ n \in \mathbb{N}\right\} \ \ldots \ \min M = \inf M = -1$, $\max M$ does not exist, $\sup M = 0$.

2.6 Definitions. Let $M \subset \mathbb{R}^*$.

- If $\sup M < +\infty$, we call M bounded above.
- If $\inf M > -\infty$, we call M bounded below.
- If M is bounded above and bounded below, we call it bounded.

- If M is not bounded, we call it unbounded.

2.7 Theorem (Supremum Theorem). *Every subset of* $\mathbb{R}^*$ *has exactly one supremum.*

2.8 Corollary. *Every subset of* $\mathbb{R}^*$ *has exactly one infimum.*

2.9 Exercises.

1) Think over the relation between $\max M$ and $\sup M$ ($\min M$ and $\inf M$).

2) Find out what is the relation between $\sup M$ and $\inf(-M)$, where

$$-M := \{-x : \; x \in M\} ,$$

and prove that the existence of infimum is really a consequence of Theorem 2.7.

3) Determine $\sup M$ and $\inf M$ (and also $\max M$ and $\min M$, in case they exist), if

   a) $M = \{q \in \mathbb{Q} : \; q^2 < 3\}$,
   b) $M = \left\{x \in \mathbb{R} : \; \sin \frac{1}{x} = \frac{1}{2}\right\}$,
   c) $M = \{x \in \mathbb{R} : \; x^2 + 3x - 6 \geq 0\}$.

4) Prove the proposition:

$$\text{A set } M \subset \mathbb{R}^* \text{ is bounded } \Leftrightarrow \exists k \in \mathbb{R}^+ : \; M \subset [-k, k] .$$

5) Prove that $\inf \left\{1 + \frac{2}{n} : \; n \in \mathbb{N}\right\} = 1$ and $\min \left\{1 + \frac{2}{n} : \; n \in \mathbb{N}\right\}$ fails to exist.

B REAL FUNCTIONS OF A SINGLE REAL VARIABLE

3 DEFINITION OF A FUNCTION

3.1 Definitions. We call every mapping of $\mathbb{R}$ into $\mathbb{R}$ a function (more precisely: a real function of a single real variable). In other words, a function f is a prescription that associates each element $x \in D(f) \subset \mathbb{R}$ with exactly one value $f(x) \in H(f) \subset \mathbb{R}$ ($D(f)\dots$ the domain of f; $H(f)\dots$ the range of f). If f is a real function of a single real variable, we write

$$f:\ \mathbb{R} \mapsto \mathbb{R}.$$

From now on we shall deal only with functions whose domains are not empty.

3.2 Examples.

1) $f(x) := x^2;\ D(f) = [-1,\ 1]$ ... see Fig. 1.

2) $g(x) := x^2;\ D(g) = [-1,\ 1)$ ... see Fig. 2.

   Caution: $f \neq g$
   $(f = g \Leftrightarrow [D(f) = D(g)\ \wedge\ \forall x \in D(f):\ f(x) = g(x)])$.

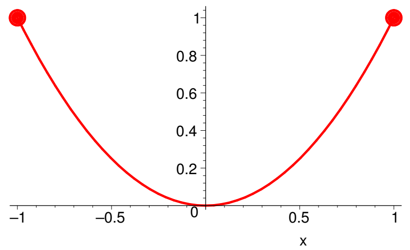

Fig. 1

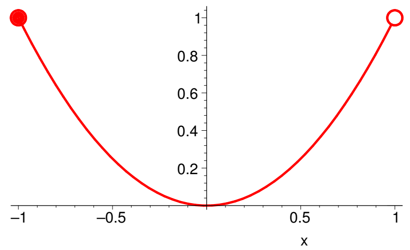

Fig. 2

3) $\eta(x) := \begin{cases} 0, & x < 0, \\ 1, & x \geq 0; \end{cases}$
   $D(\eta) = \mathbb{R},\ \eta\dots$ the so-called Heaviside function ... see Fig. 3.

4) $\mathrm{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0; \end{cases}$
   $D(\mathrm{sgn}) = \mathbb{R},\ \mathrm{sgn}\dots$ the so-called sign function ... see Fig. 4.

5) $h(x) := \lfloor x \rfloor;\ D(h) = \mathbb{R}\quad (\lfloor x \rfloor \in \mathbb{Z}:\ \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1)$,
   $\lfloor x \rfloor\dots$ the so-called lower integer part of a number x ... see Fig. 5.

6) $\mathrm{Id}(x) := x;\ D(\mathrm{Id}) = \mathbb{R},\ \mathrm{Id}\dots$ the so-called identity ... see Fig. 6.

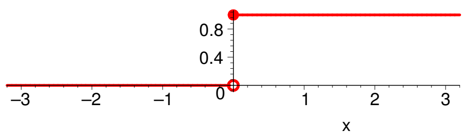

Fig. 3

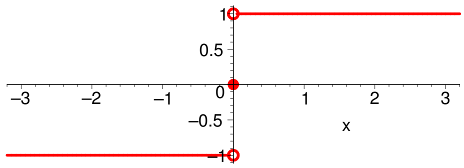

Fig. 4

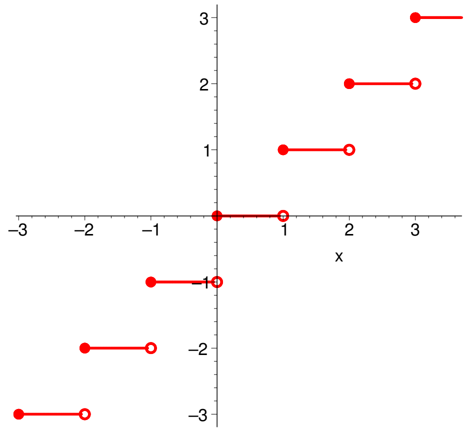

Fig. 5

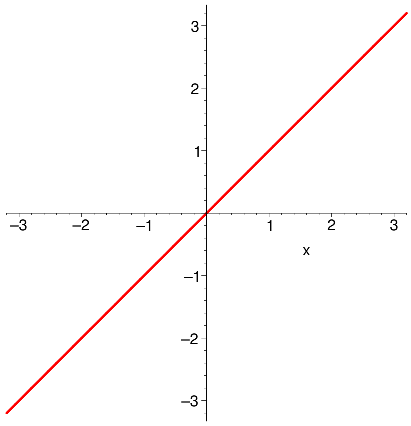

Fig. 6

7) $l(x) := |x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0; \end{cases}$

$D(l) = \mathbb{R},\ |x| \ldots$ the so-called absolute value of a number x $\quad\ldots$ see Fig. 7.

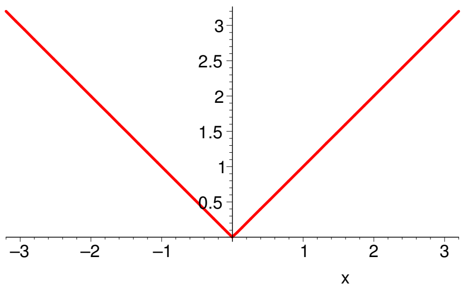

Fig. 7

8) $\chi(x) := \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q}; \end{cases}$

$D(\chi) = \mathbb{R},\ \chi \ldots$ the so-called Dirichlet function.

3.3 Definition. A graph of a function f is defined by

$$\text{Graph } f := \left\{(x, y) \in \mathbb{R} \times \mathbb{R} =: \mathbb{R}^2 :\ x \in D(f)\ \wedge\ y = f(x)\right\}.$$

3.4 Remark and convention. Now we know that a function is determined by its domain and its prescription which associates each element of the domain with exactly one value. We often determine a function only by its prescription; in this case the domain is a set of all real numbers for which the prescription is meaningful.

3.5 Example. Let us determine the domain and the graph of the function

$$k(x) := \sqrt{1-x}.$$

SOLUTION. $D(k) = \left\{x \in \mathbb{R} : \sqrt{1-x} \text{ is defined}\right\} = \{x \in \mathbb{R} : 1 - x \geq 0\} = (-\infty, 1]$.
Graph $k = \left\{(x, y) \in \mathbb{R}^2 : x \in (-\infty, 1] \wedge y = \sqrt{1-x}\right\} =$
$= \{(x, y) \in \mathbb{R}^2 : x \in (-\infty, 1] \wedge y \geq 0 \wedge y^2 = 1 - x\}$ ... see Fig. 8.

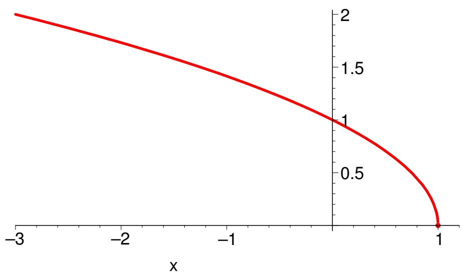

Fig. 8

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4 SOME SPECIAL PROPERTIES OF FUNCTIONS

4.1 Monotonic Functions

4.1.1 Definitions. Let $M \subset \mathbb{R}$. A function f is said to be

- increasing on the set M if

$$\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) < f(x_2),$$

- non-decreasing on the set M if

$$\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2),$$

- decreasing on the set M if

$$\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) > f(x_2),$$

- non-increasing on the set M if

$$\forall x_1, x_2 \in M : x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2).$$

A function is said to be increasing (non-decreasing, decreasing, non-increasing) if it is increasing (non-decreasing, decreasing, non-increasing) on its domain. Increasing and decreasing functions are called strictly monotonic, non-increasing and non-decreasing functions are called monotonic.

4.1.2 Examples. Let us consider the above-mentioned functions. Then

1) Id is increasing,
2) η, sgn, h, Id are non-decreasing,
3) k is decreasing,
4) k is non-increasing.

4.1.3 Remark. It is obvious that every strictly monotonic function is monotonic.

4.2 Even and Odd Functions

4.2.1 Definitions. A function f is called

- even if
$$\forall x \in D(f): \; f(-x) = f(x),$$
- odd if
$$\forall x \in D(f): \; f(-x) = -f(x).$$

4.2.2 Remark. Note that if f is even or odd, then $\forall x \in D(f): \; -x \in D(f)$.

4.2.3 Examples. Let us consider the above-mentioned functions again. Then

1) f, l, χ are even (g is not even!),
2) sgn, Id are odd.

4.2.4 Observation. Graph of an even function is symmetric to the line $x = 0$. Graph of an odd function is symmetric to the origin.

4.3 Periodic Functions

4.3.1 Definitions. A function f is said to be periodic if there exists a $T \in \mathbb{R}^+$ such that

$$\forall x \in D(f): \; f(x) = f(x+T).$$

We call such T a period of f.

4.3.2 Observation. For any periodic function f it holds: $\forall x \in D(f): \; x + T \in D(f)$.

4.3.3 Exercise. Prove the proposition:

$$T \in \mathbb{R}^+ \cap \mathbb{Q} \; \Rightarrow \; \chi \text{ is periodic with the period } T.$$

4.4 Injective Functions

4.4.1 Definition. A function is said to be injective if

$$\forall x_1, x_2 \in D(f): \; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

4.4.2 Example. Functions Id and k are injective.

4.5 Bounded Functions

4.5.1 Definitions. Let $M \subset D(f)$. A function f is said to be bounded above on the set M if a set

$$f(M) := \{f(x) : \; x \in M\}$$

is bounded above. A function f is said to be bounded above if it is bounded above on $D(f)$. Below-bounded functions and bounded functions are defined analogously.

4.5.2 Examples.

1) f, g, η, sgn are bounded,

2) l, k are bounded below.

5 OPERATIONS WITH FUNCTIONS

5.1 Sum, Difference, Product, Quotient and Composition of Functions

5.1.1 Definitions. Let f and g be functions. Then the functions $f+g,\ f-g,\ f \cdot g,\ \frac{f}{g}$ and $g \circ f$ are defined by the following prescriptions:

- $(f+g)(x) := f(x) + g(x)$,
- $(f-g)(x) := f(x) - g(x)$,
- $(f \cdot g)(x) := f(x) \cdot g(x)$,
- $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$,
- $(g \circ f)(x) := g(f(x))$.

5.1.2 If we take a close look at the previous definitions, we note certain incorrectness there. For example, in the relation $(f+g)(x) := f(x) + g(x)$ we use symbol "$+$" in two different meanings. On the left side of this equality it means the operation between two functions: the pair f and g is associated with the function $f+g$; on the right side of the equality symbol "$+$" means the sum of two real numbers $f(x)$ and $g(x)$. Similar incorrectness also appears in the definitions of the other operations.

This inaccuracy is usual in mathematical literature, but with a little attention we cannot make a mistake.

5.1.3 Examples.

1) $\mathrm{Id} = \mathrm{Id} \circ \mathrm{Id}$,

2) $k = f_2 \circ f_1$, where $f_1(x) = 1 - x$ and $f_2(x) = \sqrt{x}$,

3) $|x| = x \cdot \mathrm{sgn}(x) = \sqrt{x^2}$.

5.2 Inverse Function

5.2.1 Definition. Let f be a function. A function f_{-1} is called an <u>inverse</u> of f if

i) $D(f_{-1}) = H(f)$,

ii) $\forall x, y \in \mathbb{R}:\ f_{-1}(x) = y \Leftrightarrow x = f(y)$.

5.2.2 Theorem (Existence of an Inverse Function). *Let f be a function. Then f_{-1} exists if and only if f is injective.*

PROOF.

i) f_{-1} exists $\overset{?}{\Rightarrow}$ f is injective
Let us consider arbitrary $x_1, x_2 \in D(f)$ such that $f(x_1) = f(x_2)$ and denote the value $f(x_1) = f(x_2)$ by x. This gives $x \in H(f) = D(f_{-1})$. Hence it follows that $f_{-1}(x) = x_1$ and $f_{-1}(x) = x_2$. We thus get $x_1 = x_2$.

ii) f is injective $\overset{?}{\Rightarrow}$ f_{-1} exists
Let $x \in H(f)$. As f is injective there exists a unique $y_x \in D(f)$ such that $f(y_x) = x$. Now we define a function g on $H(f)$ by the prescription $g(x) := y_x$. It is clear that $g = f_{-1}$.

5.2.3 Example. Find the inverse, in case it exists, of the function

$$v(x) := \sqrt{1 - x^2},\ D(v) = [-1,\ 0] \quad \ldots \text{ see Fig. 9.}$$

SOLUTION.

i) $\forall x_1, x_2 \in D(v) = [-1,\ 0]$:
$v(x_1) = v(x_2) \Rightarrow \sqrt{1 - x_1^2} = \sqrt{1 - x_2^2} \Rightarrow x_1^2 = x_2^2 \Rightarrow \sqrt{x_1^2} = \sqrt{x_2^2} \Rightarrow |x_1| = |x_2| \Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$. Hence v is injective and thus v_{-1} exists!

ii) $\forall x \in D(v_{-1}) = H(v) = [0,\ 1]$:
$v_{-1}(x) = y \Leftrightarrow x = v(y) = \sqrt{1 - y^2} \Rightarrow x^2 + y^2 = 1 \Rightarrow \sqrt{y^2} = |y| = \sqrt{1 - x^2}$.
Therefore $y = v_{-1}(x) = -\sqrt{1 - x^2}$ and we get

$$v_{-1}(x) := -\sqrt{1 - x^2},\ D(v_{-1}) = [0,\ 1] \quad \ldots \text{ see Fig. 9.}$$

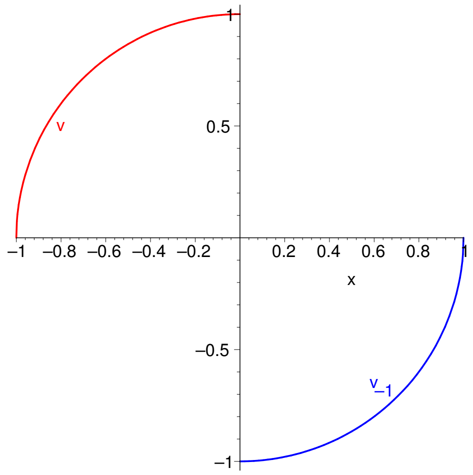

Fig. 9

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5.2.4 Observations. Let f be an injective function. Then

- $\forall x \in D(f_{-1}) : \ (f \circ f_{-1})(x) = x$,
- $\forall x \in D(f) : \ (f_{-1} \circ f)(x) = x$,
- $(f_{-1})_{-1} = f$,
- $(x, y) \in \text{Graph } f \Leftrightarrow (y, x) \in \text{Graph } f_{-1}$
  (the graphs of f and f_{-1} are symmetric to the line $y = x$).

5.3 Restriction of a Function

5.3.1 Definition. We say that a function h is a restriction of a function f to a set M (we write $h = f|_M$) if the following conditions hold:

i) $M = D(h) \subset D(f)$,

ii) $\forall x \in M : \ f(x) = h(x)$.

5.3.2 Examples.

1) $g = f|_{[-1,1)}$,

2) $\text{sgn}|_{\mathbb{R}^+} = \eta|_{\mathbb{R}^+}$.

C ELEMENTARY FUNCTIONS

6 BASIC ELEMENTARY FUNCTIONS

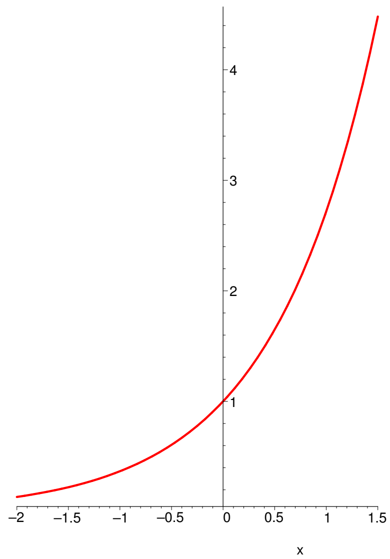

Fig. 10

6.1 An exponential function e^x (we shall denote it also by $\exp(x)$) ... see Fig. 10.
This is undoubtedly the most important function in mathematics.

This is exactly how the wonderful Walter Rudin's book "Real and Complex Analysis" starts. Then it continues with an exact definition of the exponential function and with a proof of its basic properties. However, this way is too difficult for us, in this moment we know too little. For illustration, let us note that the exponential function can be defined using a sum of a series:

$$\exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

Since we do not know now what the given sum of the series means, we have to make do with what we know about the exponential function from secondary school.

In what follows, we shall define other basic elementary functions (except the goniometric functions) exactly.

6.2 A logarithmic function is defined as the inverse of the exponential function, i.e.,

$$\log := \exp_{-1} \quad \dots \text{ see Fig. 11.}$$

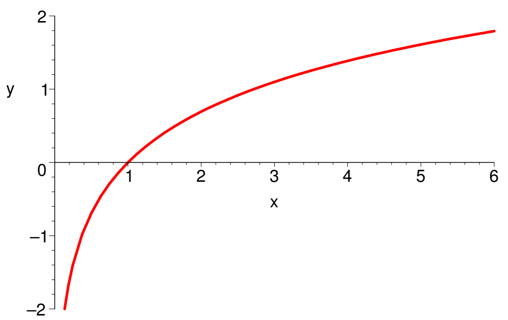

Fig. 11

6.3 A <u>constant function</u> is defined by

$$f(x) := c \quad (c \in \mathbb{R}).$$

In the case $c = 0$ we speak about a <u>zero function</u>.
For instance: $f(x) := 1$ ... see Fig. 12.

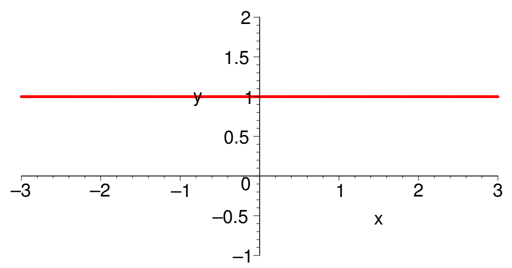

Fig. 12

6.4 The <u>power functions</u>:

- a <u>power function with a natural exponent $n \in \mathbb{N}$</u> is given by

$$f(x) = x^n := \underbrace{x \cdot x \cdot x \cdots x}_{n \text{ times}}.$$

For example: $f(x) := x$ ... see Fig. 6,
$f(x) := x^2$ ... see Fig. 13,
$f(x) := x^3$ ... see Fig. 14.

- a <u>power function with a negative integer exponent $-n$ $(n \in \mathbb{N})$</u> is given by

$$f(x) = x^{-n} := \frac{1}{x^n} = \frac{1}{x \cdot x \cdot x \cdots x}.$$

For example: $f(x) := x^{-1} = \frac{1}{x}$ ... see Fig. 15,
$f(x) := x^{-2} = \frac{1}{x^2}$ ... see Fig. 16.

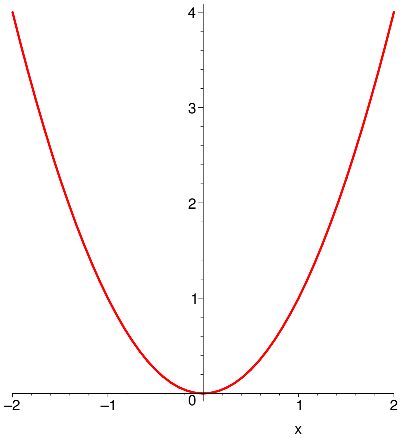

Fig. 13

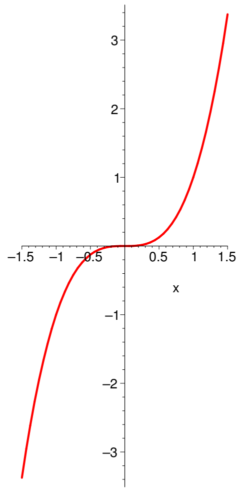

Fig. 14

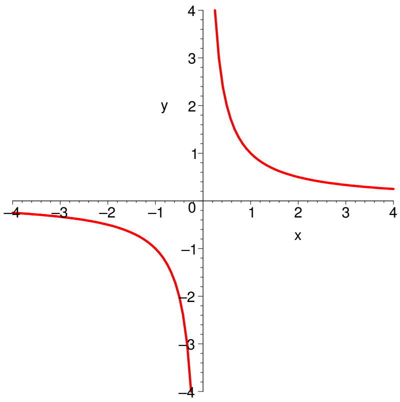

Fig. 15

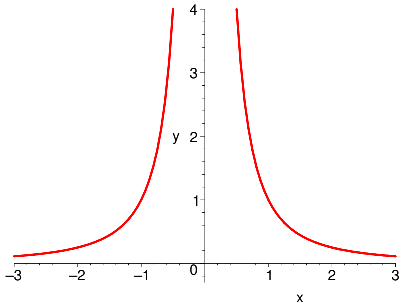

Fig. 16

- a function nth root $(n \in \mathbb{N},\ n > 1)$ is defined by

  i) $f(x) := \left(x^n|_{[0,+\infty)}\right)_{-1}$ for every even n,

  ii) $f(x) := (x^n|_{\mathbb{R}})_{-1}$ for every odd n.

  We write $f(x) = \sqrt[n]{x}$.
  For example: $f(x) := \sqrt[2]{x} =: \sqrt{x}$ ... see Fig. 17,
  $f(x) := \sqrt[3]{x}$ ... see Fig. 18.

- a power function with a real exponent $r \in \mathbb{R} \setminus \mathbb{Z}$ is defined by

$$f(x) = x^r := \mathrm{e}^{r \log x} .$$

  For example: $f(x) := x^{\sqrt{5}}$ ... see Fig. 19,
  $f(x) := x^{\frac{1}{\sqrt{5}}}$ ... see Fig. 20.

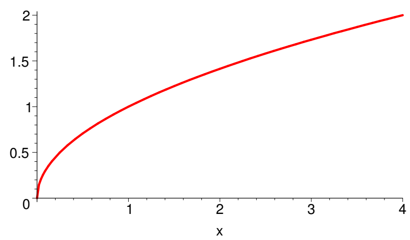

Fig. 17

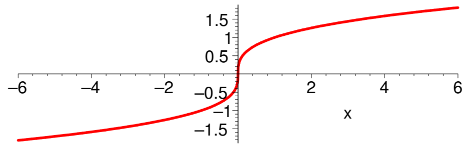

Fig. 18

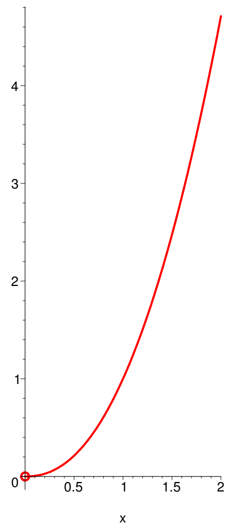

Fig. 19

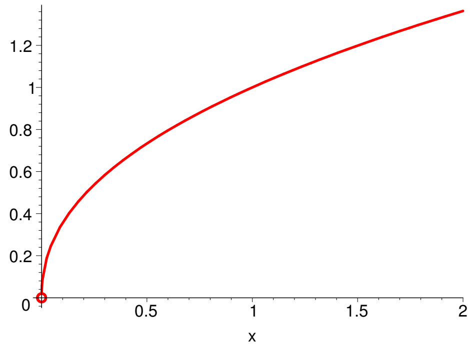

Fig. 20

- moreover, we define

$$\forall x \in \mathbb{R} : \ x^0 := 1.$$

6.5 Remark. It can be proved that

$$(\forall p, q \in \mathbb{Z}, \ q \geq 2) \left(\forall x \in \mathbb{R}^+\right) : \ x^{\frac{p}{q}} = \mathrm{e}^{\frac{p}{q} \log x} = \sqrt[q]{x^p}.$$

We can further ask why we do not define – for these p and q when, moreover, p is even – the function $x^{\frac{p}{q}}$ by the prescription $x^{\frac{p}{q}} := \sqrt[q]{x^p}$ also for a negative x. The answer is obvious. Such definition would not be correct since we could get

$$-1 = (-1)^1 = (-1)^{\frac{2}{2}} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

6.6 The goniometric functions:

- $\sin$ (sine) . . . see Fig. 21,
- $\cos$ (cosine) . . . see Fig. 22,
- $\tan := \frac{\sin}{\cos}$ (tangent) . . . see Fig. 23,
- $\cot := \frac{\cos}{\sin}$ (cotangent) . . . see Fig. 24.

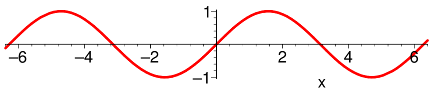

Fig. 21

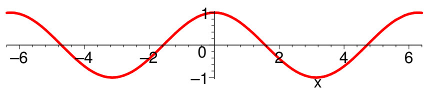

Fig. 22

We again find ourselves in a situation when we work with the functions whose definitions are beyond our comprehension in this moment. Similarly as in the case of the exponential function, let us mention that the functions sine and cosine are defined by the following sums of the infinite series:

$$\sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{+\infty} (-1)^n \, \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{+\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}.$$

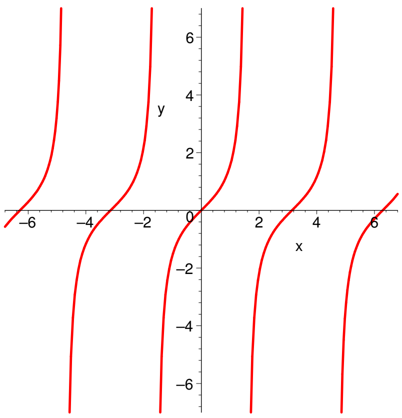

Fig. 23

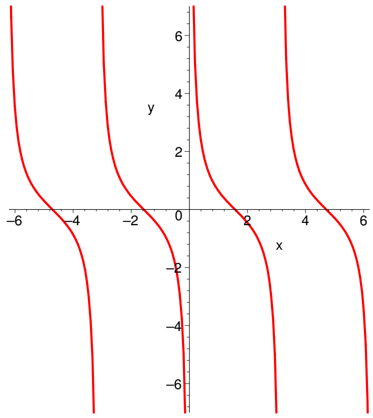

Fig. 24

6.7 Caution. Note that the domain of every goniometric function is a subset of $\mathbb{R}$; we do not use the degrees at all. In this context it is good to recall that equalities of the type $90^\circ = \frac{\pi}{2}$, $30^\circ = \frac{\pi}{6}$, etc., are meaningless.

6.8 The <u>cyclometric functions</u>:

- $\arcsin := \left(\sin|_{[-\frac{\pi}{2},\,\frac{\pi}{2}]}\right)_{-1}$ (arcsine) ... see Fig. 25,
- $\arccos := \left(\cos|_{[0,\,\pi]}\right)_{-1}$ (arccosine) ... see Fig. 26,
- $\arctan := \left(\tan|_{(-\frac{\pi}{2},\,\frac{\pi}{2})}\right)_{-1}$ (arctangent) ... see Fig. 27,
- $\mathrm{arccot} := \left(\cot|_{(0,\,\pi)}\right)_{-1}$ (arccotangent) ... see Fig. 28.

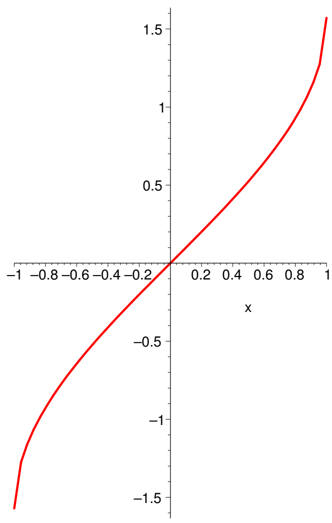

Fig. 25

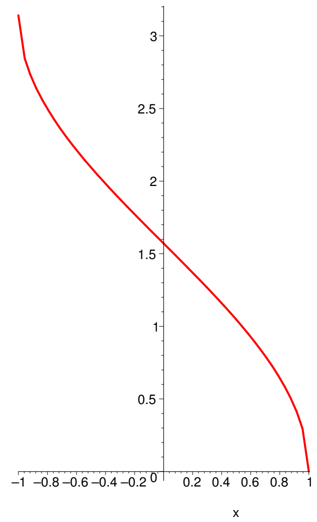

Fig. 26

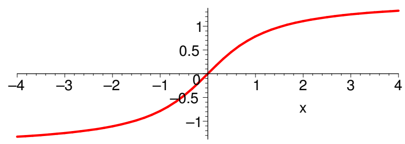

Fig. 27

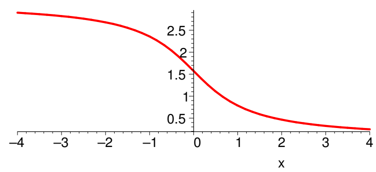

Fig. 28

6.9 The hyperbolic functions:

- $\sinh x := \frac{e^x - e^{-x}}{2}$ (hyperbolic sine) . . . see Fig. 29,
- $\cosh x := \frac{e^x + e^{-x}}{2}$ (hyperbolic cosine) . . . see Fig. 30,
- $\tanh x := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ (hyperbolic tangent) . . . see Fig. 31,
- $\coth x := \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ (hyperbolic cotangent) . . . see Fig. 32.

6.10 The hyperbolometric functions:

- $\arg\sinh := (\sinh)_{-1}$ (argument of hyperbolic sine) . . . see Fig. 33,
- $\arg\cosh := \left(\cosh|_{[0,+\infty)}\right)_{-1}$ (argument of hyperbolic cosine) . . . see Fig. 34,
- $\arg\tanh := (\tanh)_{-1}$ (argument of hyperbolic tangent) . . . see Fig. 35,
- $\arg\coth := (\coth)_{-1}$ (argument of hyperbolic cotangent) . . . see Fig. 36.

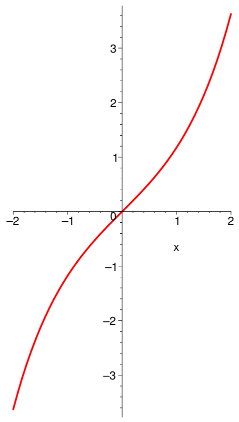

Fig. 29

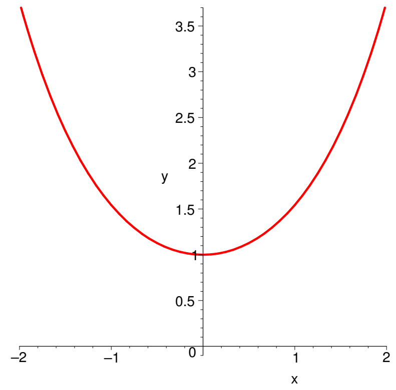

Fig. 30

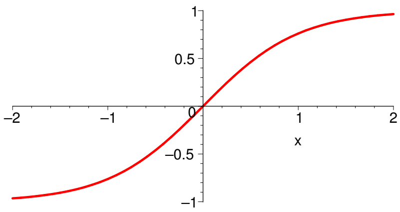

Fig. 31

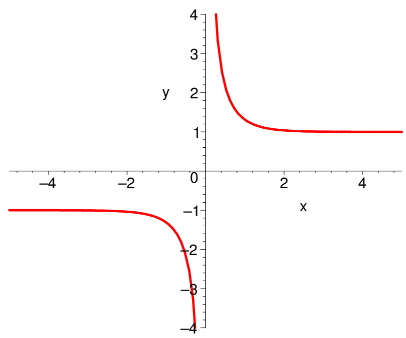

Fig. 32

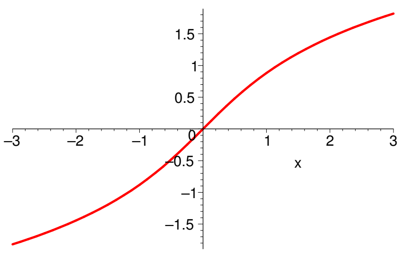

Fig. 33

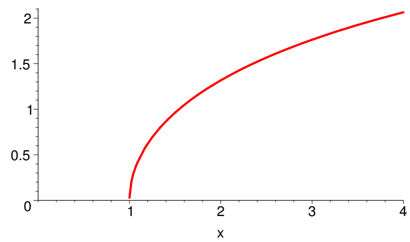

Fig. 34

7 ELEMENTARY FUNCTIONS

7.1 Definition. A function is said to be <u>elementary</u> if it is formed by the basic elementary functions using a finite number of algebraic operations ($+$, $-$, $\cdot$, $:$) and composition of functions.

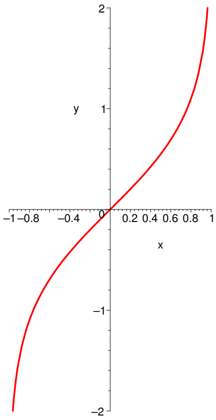

Fig. 35

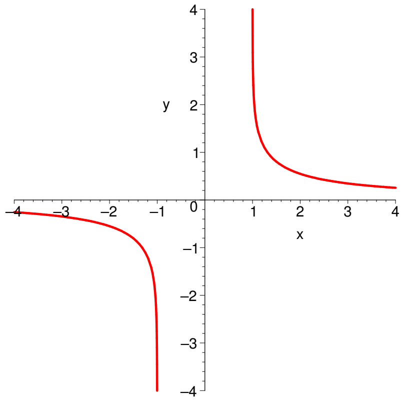

Fig. 36

7.2 Examples.

1) The function $f(x) = a^x := \mathrm{e}^{x \log a}$ $(a \in \mathbb{R}^+)$ is elementary.

2) The inverse of a^x $(a \in \mathbb{R}^+ \setminus \{1\})$ is elementary $\left(\log_a x = \frac{\log x}{\log a}\right)$.

3) sgn is not elementary.

4) $|x| = \sqrt{x^2}$ is elementary.

5) Every real polynomial p, i.e., a function given by

$$p(x) := a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (a_i \in \mathbb{R},\ i = 0, 1, \ldots, n),$$

is elementary.

7.3 Exercises. Prove the following propositions:

1) $\forall x \in [-1, 1]: \ \arcsin x + \arccos x = \frac{\pi}{2}$,

2) $\forall x \in \mathbb{R}: \ \arctan x + \operatorname{arccot} x = \frac{\pi}{2}$,

3) $\forall x \in \mathbb{R}: \ \cosh^2 x - \sinh^2 x = 1$,

4) $\forall x \in \mathbb{R}: \ \operatorname{arg\,sinh} x = \log\left(x + \sqrt{x^2 + 1}\right)$,

5) $\forall x \in [1, +\infty): \ \operatorname{arg\,cosh} x = \log\left(x + \sqrt{x^2 - 1}\right)$,

6) $\forall x \in (-1, 1): \ \operatorname{arg\,tanh} x = \frac{1}{2} \log \frac{1+x}{1-x}$,

7) $\forall x \in \mathbb{R} \setminus [-1, 1]: \ \operatorname{arg\,coth} x = \frac{1}{2} \log \frac{x+1}{x-1}$,

8) $\forall u, v \in \mathbb{R}: \ \cosh(u + v) = \cosh(u) \cosh(v) + \sinh(u) \sinh(v)$,

9) $\forall u, v \in \mathbb{R}: \ \sinh(u + v) = \sinh(u) \cosh(v) + \cosh(u) \sinh(v)$.

D SEQUENCES OF REAL NUMBERS

8 LIMIT OF A SEQUENCE

8.1 Definitions. By a sequence (more precisely: a sequence of the real numbers), we mean a function f whose domain equals to $\mathbb{N}$.

A sequence which associates every $n \in \mathbb{N}$ with a number $a_n \in \mathbb{R}$ (a_n ... the so-called nth term of the sequence (a_n)) shall be denoted by one of the following ways:

- $a_1, a_2, a_3, \ldots$;
- (a_n);
- $\{a_n\}_{n=1}^{\infty}$.

8.2 Caution. $\{a_n\}_{n=1}^{\infty} \neq \{a_n : \; n \in \mathbb{N}\}$ ... the range of a sequence.

8.3 Examples.

1) $\sqrt{13}, \sqrt{13}, \sqrt{13}, \ldots$; $a_n := \sqrt{13}$
   ... a constant sequence, $\forall n \in \mathbb{N} : \; a_{n+1} = a_n$.

2) $1, 2, 3, 4, 5, \ldots$; $a_n := n$
   ... an arithmetic sequence, $(\exists \delta \in \mathbb{R})(\forall n \in \mathbb{N}) : \; a_{n+1} = a_n + \delta$.

3) $1, 2, 4, 8, 16, \ldots$; $a_n := 2^{n-1}$
   ... a geometric sequence, $(\exists q \in \mathbb{R})(\forall n \in \mathbb{N}) : \; a_{n+1} = q\, a_n$.

4) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$; $a_n := \frac{1}{n}$
   ... a harmonic sequence.

5) $1, 1, 2, 3, 5, 8, 13, \ldots$; $a_1 = a_2 = 1, \; \forall n \in \mathbb{N} : \; a_{n+2} := a_{n+1} + a_n$
   ... a Fibonacci sequence (defined recurrently).

6) $0, 1, -1, 2, -2, 3, -3, 0, 0, -27, 27, \ldots = f(0), f(1), f(-1), \ldots, f(n), f(-n), \ldots$,
   where $f(x) := \frac{x}{1260}\,(1296 - 49x^2 + 14x^4 - x^6)$ ... see Fig. 37.

8.4 Definitions. We say that a sequence (a_n) has a limit $a \in \mathbb{R}$ (we write $\lim a_n = a$ or $a_n \to a$) if

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, \; n > n_0) : \; |a_n - a| < \varepsilon.$$

If a sequence has a (finite) limit, we call it convergent. In the opposite case we call the sequence divergent.

8.5 Examples.

1) $a_n := \sqrt{13} \to \sqrt{13}$.

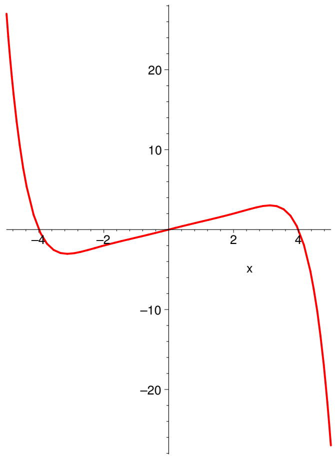

Fig. 37

PROOF. The proposition is obvious since

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ |a_n - \sqrt{13}| = 0 < \varepsilon.$$

□

2) $a_n := \frac{1}{n} \to 0$.

PROOF. We have to prove that

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ |a_n - 0| < \varepsilon,$$

i.e.,

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ \frac{1}{n} < \varepsilon.$$

First of all, let us note that for $n, \varepsilon > 0$ we have $\frac{1}{n} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n$. Now we fix $\varepsilon \in \mathbb{R}^+$ and choose an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\varepsilon} < n_0.$$

This is certainly possible. For instance, we can consider $n_0 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$. Then

$$(\forall n \in \mathbb{N},\ n > n_0):\ n > n_0 > \frac{1}{\varepsilon}.$$

□

8.6 Exercises. Prove that

1) the sequence $\{a_n\}_{n=1}^{\infty} := \{(-1)^n\}_{n=1}^{\infty}$ has no limit;

2) the sequence $\{a_n\}_{n=1}^{\infty} := \{-n\}_{n=1}^{\infty}$ is not convergent.

8.7 Theorem. *Every convergent sequence is bounded.*

PROOF. The task is to show that

$$\lim a_n = a \in \mathbb{R} \;\Rightarrow\; \exists k \in \mathbb{R}^+ :\; \{a_n : n \in \mathbb{N}\} \;\subset\; [-k,\, k]\,.$$

$$a_n \to a \;\Rightarrow\; (\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\; n > n_0) :\; |a_n - a| < \varepsilon \;\Rightarrow$$
$$\Rightarrow\; (\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\; n > n_0) :\; |a_n - a| < 1.$$

Let us take such an n_0 and put

$$k = \max\{|a_1|, |a_2|, \ldots, |a_{n_0}|, |a| + 1\}.$$

Clearly $k \in \mathbb{R}^+$.
It remains to prove that $\forall n \in \mathbb{N} :\; a_n \in [-k,\, k]$:

$$\forall n \in \{1, 2, \ldots, n_0\} :\; a_n \in \{-|a_n|, |a_n|\} \;\subset\; [-k,\, k]\,,$$

$$(\forall n \in \mathbb{N},\; n > n_0) :\; a_n \in (a - 1,\, a + 1) \;\subset\; [-k,\, k]\,.$$

□

8.8 Caution. Theorem 8.7 cannot be reversed. More precisely, not every bounded sequence is convergent. For instance, the sequence defined by $a_n := (-1)^n$ is bounded and $\lim\,(-1)^n$ fails to exist.

8.9 Definition. Let (a_n) be a sequence. Then a sequence

$$\{a_{k_n}\}_{n=1}^{\infty} = a_{k_1}, a_{k_2}, \ldots, a_{k_n}, \ldots\;,$$

where (k_n) is an increasing sequence of the natural numbers, i.e.,

$$\forall n \in \mathbb{N} :\; k_n \;<\; k_{n+1} \;\wedge\; k_n \in \mathbb{N},$$

is called a <u>subsequence</u> of (a_n).

8.10 Example.
$(a_n) = 1, 3, \sqrt{3}, 8, 12, 1, -2, \ldots\;,$
$(a_{k_n}) = 1, \sqrt{3}, 1, -2, \ldots\;,$
$(k_n) = 1, 3, 6, 7, \ldots\;.$

8.11 Theorem. *Every bounded sequence contains a convergent subsequence.*

8.12 Definitions. A sequence (a_n) is said to be a <u>Cauchy sequence</u> if it satisfies the so-called <u>Bolzano-Cauchy criterion</u>:

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n, m \in \mathbb{N};\; n, m > n_0) :\; |a_n - a_m| < \varepsilon.$$

8.13 Theorem. *A sequence is convergent if and only if it is a Cauchy sequence.*

8.14 Definitions. Let (a_n) be a sequence. Then

- (a_n) has a limit $+\infty$ (we write $\lim a_n = +\infty$ or $a_n \to +\infty$) if

$$(\forall k \in \mathbb{R})(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ a_n > k,$$

- (a_n) has a limit $-\infty$ (we write $\lim a_n = -\infty$ or $a_n \to -\infty$) if

$$(\forall l \in \mathbb{R})(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ a_n < l.$$

8.15 Examples.

1) $a_n := n^3 \to +\infty$,

2) $a_n := -n \to -\infty$.

8.16 Theorem. *Every sequence has at most one limit.*

PROOF. Let (a_n) be a sequence and $a, b \in \mathbb{R}^*$. Conversely, we suppose that $a_n \to a$, $a_n \to b$, $a \neq b$. Let, for example, $a < b$ and let us choose $c \in (a,\, b)$. Then there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(\forall n \in \mathbb{N},\ n > n_1):\ a_n < c,$$
$$(\forall n \in \mathbb{N},\ n > n_2):\ a_n > c.$$

Hence

$$(\forall n \in \mathbb{N},\ n > \max\{n_1, n_2\}):\ c < a_n < c$$

which is impossible.

□

8.17 Theorem (Limit of a Subsequence). *Let* $\lim a_n = a \in \mathbb{R}^*$ *and* (a_{k_n}) *be a subsequence of the sequence* (a_n)*. Then* $\lim a_{k_n} = a$*.*

The above-mentioned theorem can be very useful, for example, when proving that a sequence does not have any limit.

8.18 Example. The sequence $\{a_n\}_{n=1}^{\infty} := \{(-1)^n\}_{n=1}^{\infty}$ does not have any limit.

PROOF. By Theorem 8.17, it is sufficient to find two convergent subsequences of (a_n) whose limits differ. And that is quite easy:

- for a subsequence containing only even terms of (a_n) we have

$$a_{2n} = (-1)^{2n} = 1 \to 1,$$

- for a subsequence containing only odd terms of (a_n) we have

$$a_{2n-1} = (-1)^{2n-1} = -1 \to -1.$$

□

8.19 Theorem (Limit of a Monotonic Sequence). *Let* (a_n) *be a sequence.*

- *If* (a_n) *is non-decreasing, then*

$$\lim a_n = \sup \{a_n : \ n \in \mathbb{N}\}.$$

- *If* (a_n) *is non-increasing, then*

$$\lim a_n = \inf \{a_n : \ n \in \mathbb{N}\}.$$

PROOF. We assume, for example, that (a_n) is non-decreasing (if (a_n) is non-increasing, we proceed analogously, or we can employ the fact that $(-a_n)$ is non-decreasing). We put $s = \sup \{a_n : \ n \in \mathbb{N}\}$ and split the proof into two parts.

i) Let us consider primarily a situation when $s = +\infty$ (i.e., (a_n) is not bounded above). The task is to prove that $\lim a_n = +\infty$ which means that

$$(\forall k \in \mathbb{R})(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, \ n > n_0) : \ a_n > k.$$

Let $k \in \mathbb{R}$ be given. Then $k < +\infty = \sup \{a_n : \ n \in \mathbb{N}\}$, and therefore there exists an $n_0 \in \mathbb{N}$ such that $a_{n_0} > k$. Hence and from the assumption of monotonicity of (a_n) the desired proposition follows since

$$(\forall n \in \mathbb{N}, \ n > n_0) : \ a_n \geq a_{n_0} > k.$$

ii) Now if $s \in \mathbb{R}$ (i.e., (a_n) is above-bounded), we have to prove that

$$(\forall \varepsilon \in \mathbb{R}^+)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, \ n > n_0) : \ s - \varepsilon < a_n < s + \varepsilon.$$

Let $\varepsilon \in \mathbb{R}^+$ be given. Since $s - \varepsilon < s = \sup \{a_n : \ n \in \mathbb{N}\}$, there exists an $n_0 \in \mathbb{N}$ such that $a_{n_0} > s - \varepsilon$. From the monotonicity of (a_n) and from the fact that supremum is also an upper bound we finally get that

$$(\forall n \in \mathbb{N}, \ n > n_0) : \ s - \varepsilon < a_{n_0} \leq a_n \leq s < s + \varepsilon.$$

□

8.20 Examples.

1) It can be shown that the sequence (a_n), where $a_n := \left(1 + \frac{1}{n}\right)^n$, is increasing and bounded above. Therefore it is also convergent. Furthermore, it can be proved that

$$\lim \left(1 + \frac{1}{n}\right)^n = \mathrm{e} \ (\approx 2.718281828459045 \ldots).$$

2) The sequence (a_n), where

$$a_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

is clearly increasing, and therefore its limit exists. However, for every $n \in \mathbb{N}$ we have

$$|a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq \frac{1}{2n} n = \frac{1}{2}.$$

Hence (a_n) fails to be the Cauchy sequence, and therefore its finite limit does not exist, by Theorem 8.13. So

$$\lim \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) =: \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty.$$

3) The sequence (a_n), where

$$a_n := \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2},$$

is clearly increasing. Since

$$\frac{1}{k^2} < \frac{1}{k \cdot (k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

holds for every $k \in \mathbb{N} \setminus \{1\}$, we obtain

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \ldots + \frac{1}{n \cdot (n-1)} =$$

$$= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n},$$

which holds for every $n \in \mathbb{N} \setminus \{1\}$. Hence (a_n) is bounded above, and therefore it is convergent. Moreover, it can be shown that

$$\lim \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}\right) =: \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449\ldots .$$

9 CALCULATING LIMITS

9.1 Theorem. *Let* $\lim a_n = a \in \mathbb{R}^*$ *and* $\lim b_n = b \in \mathbb{R}^*$. *Then*

i) $\lim |a_n| = |a|$,

ii) $\lim (a_n \pm b_n) = a \pm b$ *whenever the right side of the equality is meaningful,*

iii) $\lim (a_n b_n) = ab$ *whenever the right side of the equality is meaningful,*

iv) $\lim \frac{a_n}{b_n} = \frac{a}{b}$ *whenever the right side of the equality is meaningful and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$,

v) $\lim \sqrt[k]{a_n} = \sqrt[k]{a}$ *whenever* $k \in \mathbb{N} \setminus \{1\}$, $a \in \mathbb{R}$ *and* $a_n \geq 0$ *for all* $n \in \mathbb{N}$.

9.2 Remarks. Let us think over the last theorem in more detail.

- Every of the mentioned propositions gives the information (of course, on the assumption that the right side is meaningful):

  i) that the relevant limit exists,

  ii) how to calculate it using numbers a and b.

- Proposition *i)* cannot be reversed for $a \neq 0$. In other words, the statement

$$\lim |a_n| \text{ exists } \Rightarrow \lim a_n \text{ exists}$$

  fails to be true. As a contrary example we can consider the sequence $\{(-1)^n\}_{n=1}^{\infty}$. However, directly from the definition of the limit it follows that

$$\lim a_n = 0 \Leftrightarrow \lim |a_n| = 0.$$

- Caution! If the right side in equalities *ii) - iv)* is meaningless, it does not imply that the relevant limit does not exist. Let us have a look at the following examples:

  i) $\left.\begin{array}{l} a_n := 2n \to +\infty \\ b_n := n \to +\infty \end{array}\right\} \Rightarrow a_n - b_n = n \to +\infty,$

  ii) $\left.\begin{array}{l} a_n := n \to +\infty \\ b_n := 2n \to +\infty \end{array}\right\} \Rightarrow a_n - b_n = -n \to -\infty,$

  iii) $\left.\begin{array}{l} a_n := n \to +\infty \\ b_n := n - a \to +\infty \end{array}\right\} \Rightarrow a_n - b_n = a \to a$ ($a \in \mathbb{R}$ can be chosen arbitrarily),

  iv) $\left.\begin{array}{l} a_n := n \to +\infty \\ b_n := n - (-1)^n \to +\infty \end{array}\right\} \Rightarrow a_n - b_n = (-1)^n$ ... this sequence has no limit.

  The examples above also show why it is not reasonable to define $(+\infty) - (+\infty)$. We can also find similar examples for other operations.

9.3 Examples.

1)

$$\lim \frac{n^2 + 6n + 7}{3n^2 - 2} = \lim \frac{1 + \frac{6}{n} + \frac{7}{n^2}}{3 - \frac{2}{n^2}} = \frac{\lim(1 + \frac{6}{n} + \frac{7}{n^2})}{\lim(3 - \frac{2}{n^2})} = \frac{1 + \frac{6}{+\infty} + \frac{7}{+\infty\cdot(+\infty)}}{3 - \frac{2}{+\infty\cdot(+\infty)}} =$$

$$= \frac{1+0+0}{3-0} = \frac{1}{3}.$$

2)

$$\lim\left(1+\frac{1}{3n}\right)^n = \lim\sqrt[3]{\left(1+\frac{1}{3n}\right)^{3n}} = \sqrt[3]{\lim\left(1+\frac{1}{3n}\right)^{3n}} = \sqrt[3]{\mathrm{e}}$$

(here we use the fact that $\left\{(1+\frac{1}{3n})^{3n}\right\}_{n=1}^{\infty}$ is a subsequence of the convergent sequence $\left\{(1+\frac{1}{n})^n\right\}_{n=1}^{\infty}$, and therefore it has the same limit e).

9.4 Convention. To say that $S(n)$ holds for all large enough $n \in \mathbb{N}$ means that

$$(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n > n_0):\ S(n).$$

9.5 Observations.

- $a_n \to a \in \mathbb{R} \ \Rightarrow\ \forall \varepsilon \in \mathbb{R}^+ :\ |a_n - a| < \varepsilon$ for all large enough $n \in \mathbb{N}$.
- Let a limit of a sequence (a_n) exist and let (b_n) be a sequence such that $a_n = b_n$ for all large enough $n \in \mathbb{N}$. Then $\lim b_n = \lim a_n$.

9.6 Definition. From now on by a sequence we shall now also mean a function defined (only) on a set $\mathbb{N} \setminus K$, where $K \subset \mathbb{N}$ is some finite set.

The above-mentioned definitions of the limit remain (without any change!) valid although we have generalized the concept of a sequence.

9.7 Examples.

1) $\lim \frac{1}{n-3} = 0$ (although $\frac{1}{n-3}$ is not defined for $n = 3$),
2) $\lim \frac{1+2n+n^3}{(n-13)(n-2007)} = +\infty$ (despite the numbers 13 and 2007 do not belong to the domain of the sequence).

9.8 Theorem (Passing a Limit in Inequalities). *Let (a_n), (b_n) and (c_n) be sequences and let* $\lim a_n = a \in \mathbb{R}^*$ *and* $\lim b_n = b \in \mathbb{R}^*$.

i) If $a < b$, then $a_n < b_n$ for all large enough $n \in \mathbb{N}$.

ii) If $a_n \le b_n$ for all large enough $n \in \mathbb{N}$, then $a \le b$.

iii) If $a_n \le c_n \le b_n$ for all large enough $n \in \mathbb{N}$ and $a = b$, then $\lim c_n$ exists and $\lim c_n = a = b$.

iv) If $a_n \le c_n$ for all large enough $n \in \mathbb{N}$ and $a = +\infty$, then $\lim c_n = +\infty$.

v) If $c_n \le b_n$ for all large enough $n \in \mathbb{N}$ and $b = -\infty$, then $\lim c_n = -\infty$.

9.9 Caution. The following proposition

$$\left.\begin{array}{l} a_n < b_n \text{ for all large enough } n \in \mathbb{N}, \\ a_n \to a, \\ b_n \to b, \end{array}\right\} \Rightarrow\ a < b$$

fails to be true (it is enough to consider, for example, $a_n = 0 \to 0$, $b_n = \frac{1}{n} \to 0$).

9.10 Examples.

1) $a_n := \frac{\sin(2007n^3 - \log n + \mathrm{e}^{3n})}{n} \to 0.$

PROOF. We first observe that

$$-\frac{1}{n} \leq \frac{\sin(2007n^3 - \log n + \mathrm{e}^{3n})}{n} \leq \frac{1}{n}$$

holds for all $n \in \mathbb{N}$. Now, since $\pm\frac{1}{n} \to 0$, we obtain, by Theorem 9.8 *iii)*, $a_n \to 0$.

□

2) $a_n := \sqrt[n]{n} \to 1.$

PROOF. Let the sequence (h_n) be defined by

$$\sqrt[n]{n} = 1 + h_n,\ n \in \mathbb{N} \setminus \{1\}.$$

By Theorem 9.1 *ii)*, it suffices now to show that $h_n \to 0$. First of all, let us note that $h_n \geq 0$ for all $n \in \mathbb{N} \setminus \{1\}$. Since

$$n = (1 + h_n)^n = \sum_{k=0}^{n} \binom{n}{k} h_n^k \geq \binom{n}{2} h_n^2 = \frac{n(n-1)}{2}\, h_n^2$$

holds for all $n \in \mathbb{N} \setminus \{1\}$, it follows that also

$$\frac{2}{n-1} \geq h_n^2 \geq 0$$

holds for all $n \in \mathbb{N} \setminus \{1\}$. As $\frac{2}{n-1} \to 0$, we have, by Theorem 9.8 *iii)*, $h_n^2 \to 0$ and hence, by Theorem 9.1 *v)*, $h_n = |h_n| = \sqrt{h_n^2} \to 0$.

□

9.11 Exercises. Let (a_n) be a sequence defined by

$$a_n := q^n,$$

where $q \in \mathbb{R}$. Prove that

1) $\lim a_n$ does not exist if $q \leq -1$,
2) $\lim a_n = 0$ if $|q| < 1$,
3) $\lim a_n = 1$ if $q = 1$,
4) $\lim a_n = +\infty$ if $q > 1$.

9.12 Theorem. *Let* $\lim a_n = 0$.

i) If $a_n > 0$ *for all large enough* $n \in \mathbb{N}$, *then* $\lim \frac{1}{a_n} = +\infty$.

ii) If $a_n < 0$ *for all large enough* $n \in \mathbb{N}$, *then* $\lim \frac{1}{a_n} = -\infty$.

E LIMIT AND CONTINUITY OF A FUNCTION

10 LIMIT OF A FUNCTION

10.1 Convention. By writing $x_0 \neq x_n \to x_0$ we mean that $x_n \to x_0$ and $x_n \neq x_0$ for all large enough $n \in \mathbb{N}$.

We understand the relations $x_0 < x_n \to x_0$ and $x_0 > x_n \to x_0$ in the similar way.

10.2 Definitions. We say that a function f has at $x_0 \in \mathbb{R}^*$

- a <u>limit</u> $a \in \mathbb{R}^*$ (we write $\lim\limits_{x \to x_0} f(x) = a$) if

$$x_0 \neq x_n \to x_0 \Rightarrow f(x_n) \to a$$

(i.e., $f(x_n) \to a$ for all sequences (x_n) satisfying $x_0 \neq x_n \to x_0$),

- a <u>limit from the right</u> $a \in \mathbb{R}^*$ (we write $\lim\limits_{x \to x_0+} f(x) = a$) if

$$x_0 < x_n \to x_0 \Rightarrow f(x_n) \to a,$$

- a <u>limit from the left</u> $a \in \mathbb{R}^*$ (we write $\lim\limits_{x \to x_0-} f(x) = a$) if

$$x_0 > x_n \to x_0 \Rightarrow f(x_n) \to a.$$

10.3 Examples. Let $f(x) := \frac{1}{x}$ (see Fig. 15). Then

1) $\lim\limits_{x \to 1} f(x) = 1,$
2) $\lim\limits_{x \to +\infty} f(x) = 0,$
3) $\lim\limits_{x \to -\infty} f(x) = 0,$
4) $\lim\limits_{x \to 0+} f(x) = +\infty,$
5) $\lim\limits_{x \to 0-} f(x) = -\infty,$
6) $\lim\limits_{x \to 0} f(x)$ does not exist since, for example, $0 \neq x_n := \frac{(-1)^n}{n} \to 0$ and $f(x_n) = (-1)^n n$ does not have any limit.

10.4 Definitions. Let $x_0 \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$. We define the following sets:

- $U(x_0, \delta) := (x_0 - \delta,\, x_0 + \delta)$
  ... a <u>neighbourhood</u> of x_0 (with radius δ),
- $U(x_0, \delta) := [x_0,\, x_0 + \delta)$
  ... a <u>right neighbourhood</u> of x_0 (with radius δ),

- $U^-(x_0, \delta) := (x_0 - \delta,\, x_0]$
  … a left neighbourhood of x_0 (with radius δ),
- $U(+\infty, \delta) := \left\{x \in \mathbb{R}^* :\ x > \frac{1}{\delta}\right\} = \left(\frac{1}{\delta},\, +\infty\right) \cup \{+\infty\}$
  … a neighbourhood of $+\infty$ (with radius δ),
- $U(-\infty, \delta) := \left\{x \in \mathbb{R}^* :\ x < -\frac{1}{\delta}\right\} = \left(-\infty,\, -\frac{1}{\delta}\right) \cup \{-\infty\}$
  … a neighbourhood of $-\infty$ (with radius δ),
- $P(x_0, \delta) := U(x_0, \delta) \setminus \{x_0\}$
  … an annular neighbourhood of x_0 (with radius δ)
  Analogously we define $P^+(x_0, \delta)$, $P^-(x_0, \delta)$, $P(+\infty, \delta)$ and $P(-\infty, \delta)$.

If we do not care about the size δ of a neighbourhood, we write briefly $U(x_0),\ P(x_0), \ldots$.

10.5 Theorem. *Let* $a \in \mathbb{R}^*$.

- *For any* $x_0 \in \mathbb{R}^*$,

$$\lim_{x \to x_0} f(x) = a \quad \Leftrightarrow \quad (\forall U(a))(\exists P(x_0))(\forall x \in P(x_0)) :\ f(x) \in U(a).$$

- *For any* $x_0 \in R$,

$$\lim_{x \to x_0+} f(x) = a \quad \Leftrightarrow \quad (\forall U(a))(\exists P^+(x_0))(\forall x \in P^+(x_0)) :\ f(x) \in U(a)$$

and

$$\lim_{x \to x_0-} f(x) = a \quad \Leftrightarrow \quad (\forall U(a))(\exists P^-(x_0))(\forall x \in P^-(x_0)) :\ f(x) \in U(a).$$

PROOF. We shall prove only the first equivalence for $x_0,\ a \in \mathbb{R}$. To check the remaining cases, it is enough to modify slightly the following steps. It is left it to the reader.

i) $\overset{?}{\Rightarrow}$

Conversely, we suppose that

$$\left(\exists \varepsilon \in \mathbb{R}^+\right) \left(\forall \delta \in \mathbb{R}^+\right) (\exists x \in P(x_0, \delta)) :\ [x \notin D(f)\ \vee\ |f(x) - a| \geq \varepsilon]\,.$$

Hence it follows that

$$(\exists \varepsilon \in \mathbb{R}^+)\,(\forall n \in \mathbb{N})\left(\exists x_n \in P\left(x_0, \tfrac{1}{n}\right)\right) :\ [x_n \notin D(f)\ \vee\ |f(x_n) - a| \geq \varepsilon]\,.$$

In this way we obtain the sequence (x_n) satisfying clearly $x_0 \neq x_n \to x_0$, but not $f(x_n) \to a$. This contradicts our assumption that $\lim\limits_{x \to x_0} f(x) = a$.

ii) $\overset{?}{\Leftarrow}$

We consider a sequence (x_n) satisfying $x_0 \neq x_n \to x_0$. Our task is to prove that $f(x_n) \to a$, i.e.,

$$\forall \varepsilon \in \mathbb{R}^+ :\ |f(x_n) - a| < \varepsilon \quad \text{for all large enough } n \in \mathbb{N}.$$

Given $\varepsilon \in \mathbb{R}^+$, there exists a $\delta > 0$ such that

$$\forall x \in P(x_0, \delta) :\ |f(x) - a| < \varepsilon.$$

Since $x_n \in P(x_0, \delta)$ (and therefore $|f(x_n) - a| < \varepsilon$) for all large enough $n \in \mathbb{N}$, the proof is actually completed.

□

10.6 Remark. Neither existence nor value of $\lim_{x\to x_0} f(x)$ depends on the existence or value of $f(x_0)$. However, if $\lim_{x\to x_0} f(x)$ exists, then the function f has to be defined on a $P(x_0, \delta)$.

10.7 Example.

$$\lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} \frac{(x-3)(x+3)}{x+3} = \lim_{x\to -3}(x-3) = -6.$$

The following three theorems are consequences of the definition of the limit of a function and corresponding theorems concerning the limit of a sequence.

10.8 Theorem. *A function f has at most one limit at $x_0 \in \mathbb{R}^*$.*

10.9 Theorem (Limit of Sum, Difference, Product and Quotient of Functions). *Let $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}^*$. Then*

i) $\lim_{x\to x_0}(f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$ *whenever the right side of the equality is meaningful,*

ii) $\lim_{x\to x_0}(f(x)g(x)) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$ *whenever the right side of the equality is meaningful,*

iii) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ *whenever the right side of the equality is meaningful.*

10.10 Examples.

1) $\lim_{x\to +\infty}(x^3 - x^2) = \lim_{x\to +\infty}(x^2(x-1)) = +\infty$,

2) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$,

3) $\lim \sin(n\pi) = 0$, but $\lim_{n\to +\infty} \sin(n\pi)$ does not exist.

10.11 Theorem. *Let*

- $f, g, h : \mathbb{R} \to \mathbb{R}$,
- $x_0, a \in \mathbb{R}^*$,
- $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = a$,
- $(\exists P(x_0))(\forall x \in P(x_0)) : f(x) \le h(x) \le g(x)$.

Then

$$\lim_{x\to x_0} h(x) = a.$$

10.12 Example.

$$\lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0.$$

PROOF. The equality follows directly from Theorem 10.11 since

- $\forall x \in \mathbb{R}\setminus\{0\} : \; -x^2 \le x^2 \sin\frac{1}{x} \le x^2$,
- $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0.$

(Note the fact that $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.)

□

10.13 Theorem. *Let $x_0 \in \mathbb{R}$ and $a \in \mathbb{R}^*$. Then $\lim\limits_{x\to x_0} f(x) = a$ if and only if*

$$\lim_{x\to x_0+} f(x) = \lim_{x\to x_0-} f(x) = a.$$

11 CONTINUITY OF A FUNCTION

11.1 Definitions. Let $x_0 \in \mathbb{R}$. A function f is said to be

- <u>continuous</u> at x_0 if
$$\lim_{x\to x_0} f(x) = f(x_0),$$
- <u>continuous from the right</u> at x_0 if
$$\lim_{x\to x_0+} f(x) = f(x_0),$$
- <u>continuous from the left</u> at x_0 if
$$\lim_{x\to x_0-} f(x) = f(x_0).$$

Note that the continuity of f

- at x_0 implies the existence of a $U(x_0)$ belonging to $D(f)$,
- from the right at x_0 implies the existence of a $U^+(x_0)$ belonging to $D(f)$,
- from the left at x_0 implies the existence of a $U^-(x_0)$ belonging to $D(f)$.

11.2 Theorem. *Let $f: \mathbb{R}\to\mathbb{R}$ and $x_0 \in \mathbb{R}$. Then the following propositions are equivalent:*

i) *f is continuous at x_0,*

ii) $x_0 \in D(f) \;\wedge\; (\forall U(f(x_0)))(\exists U(x_0))(\forall x \in U(x_0)) : \; f(x) \in U(f(x_0))$,

iii) $(\forall \varepsilon \in \mathbb{R}^+)(\exists \delta \in \mathbb{R}^+)(\forall x \in \mathbb{R}) : \; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, *iv)* $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

11.3 Examples.

1) A constant function is continuous at every $x_0 \in \mathbb{R}$.

2) The function Id is continuous at every $x_0 \in \mathbb{R}$.

3) A function f defined by $f(x) := |x|$ is continuous at every $x_0 \in \mathbb{R}$.

4) The function sgn is continuous at every $x_0 \in \mathbb{R} \setminus \{0\}$, but not at $x_0 = 0$.

5) The Dirichlet function χ is not continuous at any point.

11.4 Theorem (Continuity of Sum, Difference, Product and Quotient of Functions). Let functions f and g be continuous at $x_0 \in \mathbb{R}$. Then also functions $f + g$, $f - g$ and $f \cdot g$ are continuous at x_0. If, moreover, $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at x_0.

PROOF. From the assumptions

$$\lim_{x \to x_0} f(x) = f(x_0), \quad \lim_{x \to x_0} g(x) = g(x_0),$$

the definition of operations with functions and Theorem 10.9 it follows that

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0) = (f+g)(x_0)$$

Hence the function $f + g$ is continuous at x_0.

We proceed similarly in the case of the functions $f - g$, $f \cdot g$ and $\frac{f}{g}$.

□

11.5 Theorem (Continuity of Composition of Functions). *Let a function f be continuous at $x_0 \in \mathbb{R}$ and let a function g be continuous at $f(x_0)$. Then the function $g \circ f$ is continuous at x_0.*

PROOF. We have that

$$x_n \to x_0 \ \Rightarrow \ f(x_n) \to f(x_0) \ \Rightarrow \ g(f(x_n)) \to g(f(x_0)),$$

and therefore, according to Theorem 11.2, the function $g \circ f$ is continuous at x_0.

□

11.6 Definition. A function f is <u>continuous on an interval</u> $I \subset \mathbb{R}$ if the following conditions hold:

- f is continuous at every interior point of the interval I;
- if the basepoint of I belongs to I, then f is continuous from the right at it;
- if the endpoint of I belongs to I, then f is continuous from the left at it.

11.7 Theorem (Continuity of Basic Elementary Functions). *Let f be a basic elementary function and let $I \subset D(f)$ be an interval. Then f is continuous on I.*

11.8 Example.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

PROOF. By comparing the areas of the triangle OAC, sector of the circle OBC and the triangle OBD (see Fig. 38), we obtain the following inequalities:

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \frac{\cos x \sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}.$$

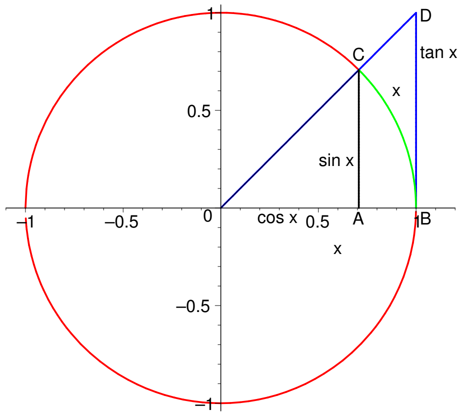

Fig. 38

Hence it follows that

$$\forall x \in \left(0, \frac{\pi}{2}\right) : \cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Furthermore, since cosine and sine are even and odd function, respectively, it holds that

$$\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\} : \cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Finally, the continuity of the functions $\cos x$ and $\frac{1}{\cos x}$ at 0, i.e.,

$$\lim_{x\to 0} \cos x = 1 = \lim_{x\to 0} \frac{1}{\cos x},$$

implies, according to Theorem 10.11, that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

□

11.9 Theorem (Limit of Composition of Functions). *Let x_0, a, $b \in \mathbb{R}^*$ and assume*

- $\lim\limits_{x\to x_0} f(x) = a$,

- $\lim\limits_{y\to a} g(y) = b$,
- $(\exists P(x_0))(\forall x \in P(x_0)):\ f(x) \neq a$ *or* g *is continuous at* a.

Then

$$\lim_{x\to x_0} g(f(x)) = b.$$

11.10 Examples. Let us show that the third assumption of the previous theorem cannot be omitted.

1) Let
$$f(x) := 0, \quad g(x) := \frac{1}{x^2}.$$
Then
$$\lim_{x\to 1} f(x) = 0, \quad \lim_{y\to 0} g(y) = +\infty,$$
but $\lim\limits_{x\to 1} g(f(x))$ does not exist since $D(g\circ f) = \emptyset$.

2) Let
$$f(x) := 0, \quad g(x) := \begin{cases} \frac{1}{x^2} & x \neq 0, \\ 1918 & x = 0. \end{cases}$$
Then
$$\lim_{x\to 1} f(x) = 0, \quad \lim_{y\to 0} g(y) = +\infty,$$
but
$$\lim_{x\to 1} g(f(x)) = \lim_{x\to 1} g(0) = \lim_{x\to 1} 1918 = 1918 \neq +\infty.$$

11.11 Exercises.

1) Prove Theorem 11.9.

2) Modify (and prove) Theorem 11.9 for the case of one-sided limits.

11.12 Examples.

1)
$$\lim_{x\to 0} \frac{\sin\left(\sqrt{5}x\right)}{\sqrt{5}x} = 1$$
since

- $\lim\limits_{x\to 0} \left(\sqrt{5}x\right) = 0$,
- $\lim\limits_{y\to 0} \frac{\sin y}{y} = 1$,
- $\forall x \in \mathbb{R}\setminus\{0\}:\ \sqrt{5}x \neq 0$.

2)
$$\lim_{x\to 0} \cos\left(x^2 \sin\frac{1}{x}\right) = 1$$

since

- $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$ (see Example 10.12),
- the function cosine is continuous at 0 (i.e., $\lim\limits_{y\to 0} \cos y = 1$).

(Note that $(\forall P(0))(\exists x \in P(0)) :\ x^2 \sin\frac{1}{x} = 0$.)

F DIFFERENTIAL AND DERIVATIVE OF A FUNCTION

12 MOTIVATION

12.1 It is often useful to substitute a function f (at least locally, i.e., on a neighbourhood) by a simpler function, preferably linear. However, this simplification (for a non-linear f) causes a certain error.

Let us try to find a linear function approximating a function f on a neighbourhood of a point c so that the approximation error is small. More precisely, we try to find a $k \in \mathbb{R}$ such that $f(c+h) \approx f(c) + kh$ for any small h. Let us define a function $\omega(h)$ (an error) by

$$\omega(h) := f(c+h) - f(c) - kh.$$

Thus we want $w(h)$ to be small for any small h. We could require

$$\lim_{h \to 0} \omega(h) = 0.$$

However, this is not very reasonable, since for a continuous function f any choice of $k \in \mathbb{R}$ complies with this accuracy rate. It is more reasonable to ask for

$$\lim_{h \to 0} \frac{\omega(h)}{h} = 0,$$

i.e.,

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - kh}{h} = \lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} - k \right) = 0.$$

It follows that

$$k = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} =: f'(c).$$

If $f'(c) \in \mathbb{R}$, then we call a function df_c defined by

$$df_c(h) := kh = f'(c)h$$

a differential of f at c. Note, by the way, that the line

$$y = f(c) + f'(c)(x - c)$$

is said to be a tangent of a graph of f at $(c,\ f(c))$. Number $f'(c)$ is the slope of the tangent.

12.2 Now let us consider a mass point moving along a line and let us denote its position in a time t by $s(t)$. An average velocity of the point on a time interval $[c,\ c+h]$ can be expressed by

$$\frac{s(c+h) - s(c)}{h}.$$

If h tends to zero, then the average velocity clearly tends to an immediate velocity of given mass point in the time c, i.e.,

$$v(c) = \lim_{h \to 0} \frac{s(c+h) - s(c)}{h} =: s'(c).$$

13 DIFFERENTIAL AND DERIVATIVE; CALCULATING DERIVATIVES

13.1 Definitions. Let $f: \mathbb{R} \mapsto \mathbb{R}$ and $x \in \mathbb{R}$.

- If
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
exists, we denote it by $f'(x)$ and we call it a derivative of the function f at the point x.

- If
$$\lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$$
exists, we denote it by $f'_+(x)$ and we call it a derivative from the right of the function f at the point x.

- If
$$\lim_{h\to 0-} \frac{f(x+h)-f(x)}{h}$$
exists, we denote it by $f'_-(x)$ and we call it a derivative from the left of the function f at the point x.

13.2 Convention. Unless otherwise stated, we shall use the concept of derivation in the meaning of the proper (i.e., finite) derivation.

13.3 Observations.

- If $f'(x)$ exists (proper or improper), then there is a $U(x)$ such that $U(x) \subset D(f)$.
- $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ whenever one side of the equality is meaningful.

13.4 Examples.

1) If f is constant, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

PROOF. Let us assume that $(\exists c \in \mathbb{R})(\forall x \in \mathbb{R}): f(x) = c$, and therefore for all $x \in \mathbb{R}$ we have
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$
□

2) $(\mathrm{Id})' = 1$ in $\mathbb{R}$.

PROOF. For all $x \in \mathbb{R}$, we have
$$(\mathrm{Id})'(x) = \lim_{h\to 0} \frac{(x+h)-(x)}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$$
□

13.5 Definitions. If there is a $k \in \mathbb{R}$ such that for a function ω defined by

$$\omega(h) := f(c+h) - f(c) - kh$$

it holds that

$$\lim_{h \to 0} \frac{\omega(h)}{h} = 0,$$

then the function f is said to be <u>differentiable</u> at the point c. A linear function df_c defined by

$$df_c(h) := kh$$

is called a <u>differential</u> of the function f at the point c.

13.6 Theorem (Existence of a Differential). *A function f is differentiable at a point $c \in \mathbb{R}$ if and only if the (finite!) derivative of the function f at the point c exists. Moreover, in such case $\forall h \in \mathbb{R}: \ df_c(h) = f'(c)\, h$.*

13.7 Theorem (Continuity of a Differentiable Function). *If a function f is differentiable at a point x_0, then it is continuous at x_0.*

PROOF. The task is to prove that

$$\lim_{x \to x_0} (f(x) - f(x_0)) = 0.$$

First, we note that for all $x, x_0 \in D(f)$ such that $x \neq x_0$ we have

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0).$$

Hence it follows that

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right] = f'(x_0) \cdot 0 = 0.$$

(From the assumption and Theorem 13.6 it follows that $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$.)

□

13.8 Examples.

1) The function sgn has an (improper) derivative at the point 0, but it is not continuous at 0.

PROOF.

$$\operatorname{sgn}'(0) = \lim_{x \to 0} \frac{\operatorname{sgn}(x) - \operatorname{sgn}(0)}{x - 0} = \lim_{x \to 0} \frac{\operatorname{sgn}(x)}{x} = \lim_{x \to 0} \frac{1}{|x|} = +\infty.$$

□

2) The function $f(x) := \sqrt[3]{x}$ has an (improper) derivative at the point 0 and it is continuous at 0.

PROOF.

$$f'(0) = \lim_{x\to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

□

3) The function $f(x) := |x|$ is continuous at the point 0, but $f'(0)$ does not exist.

PROOF. We can easily calculate that

$$f'_+(0) = 1 \quad \text{and} \quad f'_-(0) = -1,$$

and therefore, by Theorem 10.13, $f'(0)$ does not exist.

□

13.9 Theorem (Derivative of Sum, Difference, Product and Quotient of functions). *Let $x \in \mathbb{R}$. Then*

i) $(f \pm g)'(x) = f'(x) \pm g'(x)$ *whenever the right side of the equality is meaningful,*

ii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ *whenever $f'(x)$ and $g'(x)$ exist finite,*

iii) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ *whenever $f'(x)$ and $g'(x)$ exist finite and $g(x) \neq 0$.*

PROOF.

i)

$$(f \pm g)'(x) = \lim_{h\to 0} \frac{(f \pm g)(x+h) - (f \pm g)(x)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \frac{g(x+h) - g(x)}{h} = f'(x) \pm g'(x)$$

whenever $f'(x) \pm g'(x)$ is meaningful (see Theorem 10.9).

□

ii)

$$(fg)'(x) = \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \pm \frac{f(x)g(x+h)}{h} \right) =$$

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right) = f'(x)g(x) + f(x)g'(x),$$

where the last equality follows from

$$g'(x) \in \mathbb{R} \;\Rightarrow\; g \text{ is continuous at } x \;\Rightarrow\; \lim_{h\to 0} g(x+h) = g(x)$$

and the fact that $f'(x) \in \mathbb{R}$.

□

iii)

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} =$$

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)}\left(\frac{f(x+h)-f(x)}{h}g(x) - f(x)\frac{g(x+h)-g(x)}{h}\right)\right] =$$

$$= \frac{1}{g^2(x)}\left(f'(x)g(x) - f(x)g'(x)\right).$$

□

13.10 Remark. Analogous propositions hold also for the one-sided derivatives.

13.11 Theorem (Derivative of Composition of Functions). *Let $x \in \mathbb{R}$ and let $f'(x)$ and $g'(f(x))$ exist finite. Then*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

13.12 Remark. For the sake of lucidity, we shall write, not very correctly, $(f(x))'$ instead of $f'(x)$.

13.13 Examples.

1) $\forall x \in \mathbb{R}:\ \sin' x = \cos x.$

PROOF. First, we recall that

$$\forall x \in \mathbb{R}:\ \sin^2 x = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Therefore for all $x \in \mathbb{R}$ we get

$$\sin' x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} =$$

$$= \lim_{h\to 0}\left(\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right) = \lim_{h\to 0}\left(\cos x\frac{\sin h}{h} - \sin x\frac{2\sin^2\frac{h}{2}}{h}\right) =$$

$$= \lim_{h\to 0}\left(\cos x\frac{\sin h}{h} - \sin x\frac{\sin\frac{h}{2}}{\frac{h}{2}}\sin\frac{h}{2}\right) = \cos x\cdot 1 - \sin x \cdot 1\cdot 0 = \cos x.$$

□

2) $\forall x \in \mathbb{R}:\ \cos' x = -\sin x.$

PROOF. From Theorem 13.11 and the previous example it follows that for all $x \in \mathbb{R}$ we have

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)' = \sin'\left(\frac{\pi}{2}-x\right)\left(\frac{\pi}{2}-x\right)' = \cos\left(\frac{\pi}{2}-x\right)(0-1) = -\sin x.$$

□

3) $\forall x \in D(\tan): \ \tan' x = \frac{1}{\cos^2 x}$.

PROOF. From Theorem 13.9 and the previous examples, it follows that for all $x \in D(\tan)$ we have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

□

4) $\forall x \in D(\cot): \ \cot' x = -\frac{1}{\sin^2 x}$.

PROOF.

$$\forall x \in D(\cot): \ (\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

□

5) $\forall x \in \mathbb{R}: \ (\mathrm{e}^x)' = \mathrm{e}^x$.

(Let us leave this proposition without proof . . .)

13.14 Theorem (Derivative of an Inverse Function). *Let a function f be continuous and strictly monotonic on an interval $I \subset \mathbb{R}$, let x be an interior point of I, and let $f'(f^{-1}(x))$ exist. Then $(f^{-1})'(x)$ exists and is defined by*

$$(f^{-1})'(x) = \begin{cases} \frac{1}{f'(f^{-1}(x))} & \text{if } f'(f^{-1}(x)) \neq 0, \\ +\infty & \text{if } f'(f^{-1}(x)) = 0 \text{ and } f \text{ is increasing on } I, \\ -\infty & \text{if } f'(f^{-1}(x)) = 0 \text{ and } f \text{ is decreasing on } I. \end{cases}$$

13.15 Examples.

1) $\forall x \in \mathbb{R}^+: \ \log' x = \frac{1}{x}$.

PROOF.

$$\forall x \in \mathbb{R}^+: \ \log' x = \frac{1}{\exp'(\log x)} = \frac{1}{\exp(\log x)} = \frac{1}{x}.$$

□

2) Let $n \in \mathbb{N}$. Then $\forall x \in \mathbb{R}: \ (x^n)' = nx^{n-1}$.

PROOF. We shall use the mathematical induction.

i) $\forall x \in \mathbb{R}: \ (x)' = \mathrm{Id}'(x) = 1$.

ii) The task is to prove the implication

$$\big(n \in \mathbb{N} \ \wedge \ \forall x \in \mathbb{R}: \ (x^n)' = nx^{n-1}\big) \ \Rightarrow \ \forall x \in \mathbb{R}: \ (x^{n+1})' = (n+1)x^n.$$

$$(x^{n+1})' = (x^n x)' = (x^n)'x + x^n x' = nx^{n-1}x + x^n = (n+1)x^n.$$

$\square$

3) Let $-n \in \mathbb{N}$. Then $\forall x \in \mathbb{R} \setminus \{0\} : \ (x^n)' = nx^{n-1}$.

PROOF.

$$\forall x \in \mathbb{R} \setminus \{0\} : \ (x^n)' = \left(\frac{1}{x^{-n}}\right)' = \frac{(1)'x^{-n} - 1(x^{-n})'}{(x^{-n})^2} =$$

$$= \frac{-(-n)x^{-n-1}}{x^{-2n}} = nx^{-n-1-(-2n)} = nx^{n-1}.$$

$\square$

4) Let $r \in \mathbb{R}$. Then $\forall x \in \mathbb{R}^+ : \ (x^r)' = rx^{r-1}$.

PROOF.

$$\forall x \in \mathbb{R}^+ : \ (x^r)' = (e^{r \log x})' = e^{r \log x}(r \log x)' = x^r r \frac{1}{x} = rx^{r-1}.$$

$\square$

5) Let $x \in \mathbb{R}$ and let f and g be functions satisfying $f(x) > 0$ and $f'(x), \ g'(x) \in \mathbb{R}$. Then

$$(f(x)^{g(x)})' = \left(\mathrm{e}^{g(x) \log f(x)}\right)' = \mathrm{e}^{g(x) \log f(x)}(g(x) \log f(x))' =$$

$$= f(x)^{g(x)} \left(g'(x) \log f(x) + g(x) \frac{f'(x)}{f(x)}\right).$$

6) $\forall x \in (-1, \ 1) : \ \arcsin' x = \frac{1}{\sqrt{1-x^2}}$.

PROOF. First, let us recall that $\forall x \in (-1, \ 1) : \ \arcsin x \in (-\frac{\pi}{2}, \ \frac{\pi}{2})$ and $\forall x \in (-\frac{\pi}{2}, \ \frac{\pi}{2}) :$ $\cos x > 0$. Therefore

$$\forall x \in (-1, \ 1) : \ (\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} =$$

$$= \frac{1}{|\cos(\arcsin x)|} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}.$$

$\square$

7) $\forall x \in (-1, \ 1) : \ \arccos' x = -\frac{1}{\sqrt{1-x^2}}$.

PROOF. Since $\forall x \in (-1, \ 1) : \ \arccos x \in (0, \ \pi)$ and $\forall x \in (0, \ \pi) : \ \sin x > 0$, we get

$$\forall x \in (-1, \ 1) : \ (\arccos x)' = \frac{1}{\cos'(\arccos x)} = -\frac{1}{\sin(\arccos x)} =$$

$$= -\frac{1}{|\sin(\arccos x)|} = -\frac{1}{\sqrt{1 - \cos^2(\arccos x)}} = -\frac{1}{\sqrt{1 - x^2}}.$$

$\square$

7) $\forall x \in \mathbb{R}: \ \arctan' x = \frac{1}{1+x^2}.$

PROOF. First, let us observe that $\forall x \in \mathbb{R}: \ \arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}):$ $\frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x$. Therefore

$$\forall x \in \mathbb{R}: \ \arctan' x = \frac{1}{\tan'(\arctan x)} = \frac{1}{\frac{1}{\cos^2(\arctan x)}} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}.$$

□

8) $\forall x \in \mathbb{R}: \ \operatorname{arccot}' x = -\frac{1}{1+x^2}.$

PROOF. First, let us observe that $\forall x \in \mathbb{R}: \ \operatorname{arccot} x \in (0, \pi)$ and $\forall x \in (0, \pi):$ $\frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x$. Therefore

$$\forall x \in \mathbb{R}: \ \operatorname{arccot}' x = \frac{1}{\cot'(\operatorname{arccot} x)} = \frac{1}{\frac{-1}{\sin^2(\operatorname{arccot} x)}} = \frac{-1}{1+\cot^2(\operatorname{arccot} x)} = \frac{-1}{1+x^2}.$$

□

13.16 Definition. Let f be a function. A function f' defined by

$$f'(x) := f'(x)$$

is said to be a <u>derivative of the function</u> f. Analogously we define functions f'_+ and f'_-.

13.17 Definitions. Let $I \subset \mathbb{R}$ be an interval with end points $a, b \in \mathbb{R}^*$, $a < b$. A function f is said to be

- <u>differentiable</u> on the interval I if the following three conditions hold:
  - *i)* $\forall x \in (a, b): \ f'(x) \in \mathbb{R}$,
  - *ii)* if $a \in I$, then $f'_+(a) \in \mathbb{R}$,
  - *iii)* if $b \in I$, then $f'_-(b) \in \mathbb{R}$;
- <u>continuously differentiable</u> on the interval I if the following three conditions hold:
  - *i)* the function f' is continuous on (a, b),
  - *ii)* if $a \in I$, then the function f'_+ is continuous from the right at the point a,
  - *iii)* if $b \in I$, then the function f'_- is continuous from the left at the point b.

13.18 Definitions. Let $n \in \mathbb{N}$. Let us define a function called <u>$(n+1)$th derivative of a function f</u> by induction

$$f^{(n+1)} := \left(f^{(n)}\right)'.$$

Moreover, let us define a function $f^{(0)}$ by

$$f^{(0)}(x) := f(x).$$

13.19 Example.
$\sin^{(0)} x = \sin x,$
$\sin' x = \cos x,$
$\sin'' x = (\sin' x)' = (\cos x)' = -\sin x,$
$\sin''' x = (\sin'' x)' = (-\sin x)' = -\cos x,$
$\sin^{(4)} x = (\sin''' x)' = (-\cos x)' = \sin x,$
$\sin^{(5)} x = (\sin^{(4)} x)' = (\sin x)' = \cos x,$
$\ldots .$

G BASIC THEOREMS OF DIFFERENTIAL CALCULUS

14 THEOREMS ON FUNCTION INCREMENT

14.1 Theorem (Rolle). *Let a function f be continuous on an interval $[a, b]$ and differentiable on (a, b), and let $f(a) = f(b)$. Then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

PROOF. We shall carry out the proof later (see 17.2.6).

□

The meaning of Rolle's theorem is illustrated in Fig. 39.

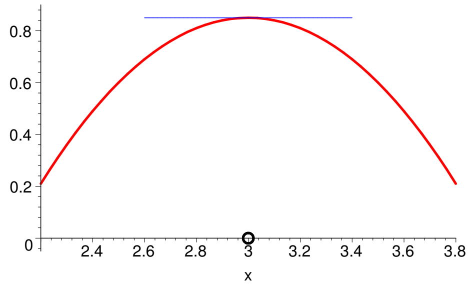

Fig. 39

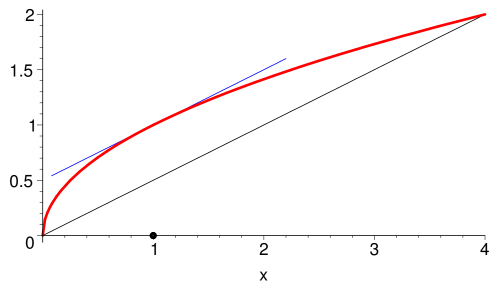

Fig. 40

14.2 Theorem (Lagrange's Mean Value Theorem). *Let a function f be continuous on an interval $[a, b]$ and differentiable on (a, b). Then there is a $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

PROOF. Let us define a function F by

$$F(x) := f(x) - \frac{f(b) - f(a)}{b - a}\,(x - a).$$

From the assumptions it follows that F is continuous on the interval $[a, b]$ and differentiable on (a, b). Moreover, since $F(a) = F(b)\,(= f(a))$, there is (see Rolle's theorem) a $\xi \in (a, b)$ such that $F'(\xi) = 0$. For all $x \in (a, b)$, the derivative of F is given by

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Hence and from $F'(\xi) = 0$ it follows that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

□

The meaning of the latter theorem is illustrated in Fig. 40.

14.3 Theorem (Cauchy's Mean Value Theorem). *Let functions f and g be continuous on an interval $[a, b]$ and differentiable on (a, b). Let g' be finite and nonzero on (a, b). Then there is a $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

PROOF. First, note that $g(b) - g(a) \neq 0$. (If it were true that $g(a) = g(b)$, there would be, by Rolle's theorem, a point $\xi \in (a, b)$ such that $g'(\xi) = 0$, which contradicts the assumption.) Let us define a function F by

$$F(x) := f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

It can be easily checked that this function satisfies all assumptions of Rolle's theorem. Therefore there is a $\xi \in (a, b)$ such that

$$F'(\xi) = f'(\xi)(g(b) - g(a)) - g'(\xi)(f(b) - f(a)) = 0.$$

Hence and from $g'(\xi) \neq 0$ and $g(b) - g(a) \neq 0$ we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

□

15 L'HOSPITAL'S RULE

15.1 Theorem (l'Hospital's Rule). *Let*

- $x_0, a \in \mathbb{R}^*$,
- $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ *or* $\lim_{x\to x_0} |g(x)| = +\infty$,
- $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = a$.

Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = a.$$

PROOF. We give the proof only for the case

$$x_0 \in \mathbb{R} \quad \text{and} \quad \lim_{x\to x_0+} f(x) = \lim_{x\to x_0+} g(x) = 0.$$

In other words, we prove only the implication

$$\left.\begin{aligned} \lim_{x\to x_0+} f(x) = \lim_{x\to x_0+} g(x) = 0, \\ \lim_{x\to x_0+} \tfrac{f'(x)}{g'(x)} = a \in \mathbb{R}^*, \end{aligned}\right\} \Rightarrow \lim_{x\to x_0+} \frac{f(x)}{g(x)} = a.$$

Let us first define (or redefine) functions f and g at the point x_0 by

$$f(x_0) = g(x_0) = 0.$$

(This step affects neither existence nor value of the studied limit since the limit depends neither on the value $f(x_0)$ nor on the value $g(x_0)$.) The assumption

$$\lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = a$$

yields the existence of a $\delta > 0$ such that for every $x \in (x_0,\, x_0 + \delta)$ there are $f'(x), g'(x) \in \mathbb{R}$ and, moreover, $g'(x) \neq 0$. Hence (see Theorem 13.7) it follows that functions f and g are continuous on $[x_0,\, x_0 + \delta)$. Therefore (see Theorem 14.3)

$$(\forall x \in (x_0,\, x_0 + \delta))\, (\exists \xi \in (x_0,\, x)) : \ \frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Hence it is easily seen that

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = \lim_{\xi \to x_0+} \frac{f'(\xi)}{g'(\xi)} = a.$$

□

If $x_0 \in \mathbb{R}$, then the analogous propositions hold also for the one-sided limits.

15.2 Examples.

1)
$$\lim_{x \to 0} \frac{\tan x}{3x} \overset{\text{l'H.}}{=} \lim_{x \to 0} \frac{\frac{1}{\cos^2 x}}{3} = \frac{1}{3}.$$

2)
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} \overset{\text{l'H.}}{=} \lim_{x \to 0} \frac{\sin x}{2x} \overset{\text{l'H.}}{=} \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

3)
$$\lim_{x \to +\infty} \frac{x^3 - 2x^2 + x - 1}{2x^2 + 3x + 1} \overset{\text{l'H.}}{=} \lim_{x \to +\infty} \frac{3x^2 - 4x + 1}{4x + 3} \overset{\text{l'H.}}{=} \lim_{x \to +\infty} \frac{6x - 4}{4} = +\infty.$$

4) Caution!
$$\lim_{x \to +\infty} \frac{x + \sin x}{x} = \lim_{x \to +\infty} \left(1 + \frac{\sin x}{x}\right) = 1,$$

but
$$\lim_{x \to +\infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to +\infty} \frac{1 + \cos x}{1}$$

does not exist.

5)
$$\lim_{x \to 0+} (x \log x) = \lim_{x \to 0+} \frac{\log x}{\frac{1}{x}} \overset{\text{l'H.}}{=} \lim_{x \to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0+} (-x) = 0.$$

6)

$$\lim_{x\to 0+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0+}\frac{x-\sin x}{x\sin x}\overset{\text{l'H.}}{=}\lim_{x\to 0+}\frac{1-\cos x}{\sin x+x\cos x}\overset{\text{l'H.}}{=}$$

$$\overset{\text{l'H.}}{=}\lim_{x\to 0+}\frac{\sin x}{\cos x+\cos x-x\sin x}=0.$$

7)

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=\mathrm{e}^{1}=\mathrm{e},$$

since (see Theorem 11.9)

a) $\forall x\in\mathbb{R}^{+}:\ \left(1+\frac{1}{x}\right)^{x}=\mathrm{e}^{x\log\left(1+\frac{1}{x}\right)}$,

b) $\lim\limits_{x\to+\infty}\left(x\log\left(1+\frac{1}{x}\right)\right)=\lim\limits_{x\to+\infty}\frac{\log\frac{x+1}{x}}{\frac{1}{x}}\overset{\text{l'H.}}{=}\lim\limits_{x\to+\infty}\frac{x}{x+1}=\lim\limits_{x\to+\infty}\frac{1}{1+\frac{1}{x}}=1,$

c) the exponential function is continuous at 1.

15.3 Observation. If a function f is continuous from the right at a point $x_0\in\mathbb{R}$ and if $\lim\limits_{x\to x_0+}f'(x)$ exists, then

$$f'_{+}(x_0)=\lim_{x\to x_0+}\frac{f(x)-f(x_0)}{x-x_0}\overset{\text{l'H.}}{=}\lim_{x\to x_0+}f'(x).$$

Analogous proposition holds also for $f'_{-}(x_0)$.

15.4 Example. Let the function f be defined by

$$f(x):=|\sin x|.$$

Decide whether $f'(0)$ exists.

SOLUTION. The function f is clearly continuous on $\mathbb{R}$, and therefore

$$f'_{+}(0)=\lim_{x\to 0+}f'(x)=\lim_{x\to 0+}(\sin x)'=\lim_{x\to 0+}\cos x=1,$$

$$f'_{-}(0)=\lim_{x\to 0-}f'(x)=\lim_{x\to 0-}(-\sin x)'=\lim_{x\to 0-}(-\cos x)=-1.$$

Since $f'_{+}(0)\neq f'_{-}(0)$, the derivative of f at the point 0 does not exist.

■

15.5 Caution. $+\infty=\operatorname{sgn}'(0)=\operatorname{sgn}'_{+}(0)\neq\lim\limits_{x\to 0+}\operatorname{sgn}'(x)=\lim\limits_{x\to 0+}0=0$ ... see Fig. 4.

H FUNCTION BEHAVIOUR

16 INTERVALS OF STRICT MONOTONICITY

16.1 Theorem. *Let a function f be continuous on an interval I with end points $a, b \in \mathbb{R}^*$, $a < b$.*

- *If $f'(x) > 0$ for all $x \in (a,\, b)$, then the function f is increasing on the interval I.*
- *If $f'(x) \geq 0$ for all $x \in (a,\, b)$, then the function f is non-decreasing on the interval I.*
- *If $f'(x) < 0$ for all $x \in (a,\, b)$, then the function f is decreasing on the interval I.*
- *If $f'(x) \leq 0$ for all $x \in (a,\, b)$, then the function f is non-increasing on the interval I.*
- *If $f'(x) = 0$ for all $x \in (a,\, b)$, then the function f is constant on the interval I.*

PROOF. We give the proof only for the first statement of the above theorem (the other ones can be proved analogously). Let x_1 and x_2 be arbitrary points of the interval I such that $x_1 < x_2$. The task is to show that $f(x_1) < f(x_2)$. From the assumption it follows that f is continuous on $[x_1,\, x_2]$ and differentiable on $(x_1,\, x_2)$. Therefore (see Theorem 14.2) there is a $\xi \in (x_1,\, x_2) \subset (a,\, b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) > 0.$$

Hence it is evident that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) > 0,$$

so that

$$f(x_1) < f(x_2).$$

□

16.2 Example. Find the intervals of strict monotonicity of the function f defined by

$$f(x) := 2x^3 - 3x^2 - 12x + 1.$$

SOLUTION. The function f is continuous on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}: \ f'(x) = 6x^2 - 6x - 12 = 6\,(x+1)(x-2).$$

Since it is obvious that

- $f'(x) > 0 \ \Leftrightarrow \ x \in (-\infty,\, -1) \cup (2,\, +\infty)$,
- $f'(x) < 0 \ \Leftrightarrow \ x \in (-1,\, 2)$,

it follows from Theorem 16.1 that

- f is increasing on $(-\infty,\, -1]$ and on $[2,\, +\infty)$,

- f is decreasing on $[-1, 2]$.

It is important to note that f is not increasing on $(-\infty, -1] \cup [2, +\infty)$!

■

16.3 Theorem (Darboux's Property of a Continuous Function and its Derivative). *Let a function f be continuous on an interval $[a, b]$.*

- *If $f(a)f(b) < 0$, then there is a $\xi \in (a, b)$ such that $f(\xi) = 0$.*
- *If $f'(x)$ exists for all $x \in [a, b]$ and if $f'(a)f'(b) < 0$, then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

16.4 Corollaries.

i) If a function f is continuous on an interval J, then $f(J)$ is either a one-point set or an interval.

ii) If a continuous function f has a nonzero derivative at all points of an interval J, then f is strictly monotonic on J.

PROOF.

i) It is sufficient to prove the implication

$$(a, b \in f(J) \text{ and } a < c < b) \;\Rightarrow\; c \in f(J).$$

So let us assume that $a, b \in f(J)$ and $a < c < b$. Hence it follows that there are points $x_1, x_2 \in J$ such that $a = f(x_1) < c < f(x_2) = b$ which implies $(f(x_1) - c)(f(x_2) - c) < 0$. Since, by the assumption, the function $g(x) := f(x) - c$ is continuous on the interval $[x_1, x_2]$, it follows from the first proposition of Theorem 16.3 that there is an $x \in (x_1, x_2) \subset J$ such that $g(x) = f(x) - c = 0$. Hence $c = f(x) \in f(J)$.

□

ii) Note that f' does not change the sign on J, in other words, f' is either positive or negative on J. (If there were numbers $a, b \in J$ satisfying $f'(a) < 0 < f'(b)$, then the second proposition of Theorem 16.3 would imply that there is a $\xi \in (a, b) \subset J$ such that $f'(\xi) = 0$. But this contradicts the assumption that f' is nonzero on J.) The proposition now clearly follows from Theorem 16.1.

□

16.5 Remark. Proposition *i)* certainly corresponds to our notion of a continuous function. Proposition *ii)* is more interesting since the function f' need not be continuous!

16.6 Remark. Let us focus on solving the equation $f(x) = 0$ numerically. Let a function f be continuous on an interval $[a, b]$ and satisfying $f(a)f(b) < 0$. We look for a point $\xi \in (a, b)$ such that $f(\xi) = 0$ (such point does exist!). We put $c := \frac{a+b}{2}$, so that exactly one of the following cases happens:

i) $f(c) = 0$,

ii) $f(a)f(c) < 0$,

iii) $f(c)f(b) < 0$.

In situation *i)* we choose $\xi := c$ (we have found the root). If the second or the third case happens, we are in the same situation as at the beginning, but on a half interval $[a,\, c]$ or $[c,\, b]$. After certain number of these steps (we speak of the bisection method) we shall either find the desired root of the equation $f(x) = 0$, or get arbitrarily near to it (i.e., we shall find an interval of an arbitrary pre-given small length where the root lies).

17 EXTREMES OF FUNCTIONS

17.1 Local Extremes

17.1.1 Definitions. Let $x_0 \in \mathbb{R}$. A function f has

- a local maximum at x_0 if there is a $P(x_0)$ such that

$$\forall x \in P(x_0) : \ f(x) \leq f(x_0);$$

- a local minimum at x_0 if there is a $P(x_0)$ such that

$$\forall x \in P(x_0) : \ f(x) \geq f(x_0);$$

- a strict local maximum at x_0 if there is a $P(x_0)$ such that

$$\forall x \in P(x_0) : \ f(x) < f(x_0);$$

- a strict local minimum at x_0 if there is a $P(x_0)$ such that

$$\forall x \in P(x_0) : \ f(x) > f(x_0).$$

17.1.2 Observation. If a function f has a local extreme (i.e., local maximum or local minimum) at a point x_0, then there is a $U(x_0)$ such that

$$U(x_0) \subset D(f).$$

17.1.3 Examples.

1) Function $f(x) := |x|$ has a strict local minimum at 0 (note that $f'(0)$ does not exist) . . . see Fig. 7.

2) Function $f(x) := x^2$ has a strict local minimum at 0 (note that $f'(0) = 0$) . . . see Fig. 13.

3) Function $f(x) := x^3$ does not have a local extreme at 0 (note that $f'(0) = 0$) . . . see Fig. 14.

17.1.4 Theorem (Necessary Condition for Existence of a Local Extreme). *If a function f has a local extreme at $x_0 \in \mathbb{R}$, then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.*

PROOF. It is sufficient to prove the validity of the implication

$$0 \neq f'(x_0) \in \mathbb{R}^* \Rightarrow \ f \text{ does not have a local extreme at } x_0.$$

Assume $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} > 0$ (in case $f'(x_0) < 0$ it is possible to proceed analogously, or get to the function $-f$). Hence it follows

$$(\exists P(x_0))(\forall x \in P(x_0)) : \ \frac{f(x) - f(x_0)}{x - x_0} > 0,$$

and thus ($P(x_0) = P^+(x_0) \cup P^-(x_0)$):

- $f(x) > f(x_0)$ for every $x \in P^+(x_0)$,
- $f(x) < f(x_0)$ for every $x \in P^-(x_0)$.

Therefore f does not have a local extreme at x_0.

□

17.1.5 Theorem (Sufficient Condition for Existence of a Local Extreme). *Assume $f'(x_0) = 0$. If*

- *$f''(x_0) > 0$, then the function f has a strict local minimum at the point x_0,*
- *$f''(x_0) < 0$, then the function f has a strict local maximum at the point x_0.*

17.1.6 Remarks. Let $n \in \mathbb{N}$ and

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \neq f^{(n)}(x_0).$$

Then

- if n is odd, the function f does not have a local extreme at the point x_0;
- if n is even and $f^{(n)}(x_0) > 0$, the function f has a strict local minimum at the point x_0;
- if n is even and $f^{(n)}(x_0) < 0$, the function f has a strict local maximum at the point x_0.

SKETCH OF THE PROOF. Similarly as in the proof of Theorem 17.1.4, it can be shown that the assumption

$$f'(x_0) = 0 < f''(x_0)$$

implies that there is a $\delta > 0$ such that

i) $f'(x) > 0$ for every $x_0 < x \in P(x_0, \delta)$,
ii) $f'(x) < 0$ for every $x_0 > x \in P(x_0, \delta)$.

Moreover, the function f is clearly continuous at the point x_0 ($f'(x_0) = 0 \in \mathbb{R}$). Therefore (see Theorem 16.1) f is increasing in $[x_0, x_0 + \delta)$ and decreasing in $(x_0 - \delta, x_0]$. Hence we get that f has a strict local minimum at x_0. Now we can employ the mathematical induction to finish the proof.
(The previous two statements can be proved analogously.)

□

17.1.7 Definition. If $f'(x_0) = 0$, then we call x_0 a stationary point of the function f.

17.1.8 Example. Find all local extremes of the function f defined by

$$f(x) := x^3(x - 7).$$

SOLUTION. First, f is differentiable on $\mathbb{R}$, and therefore (see Theorem 17.1.4) it can have local extremes only at stationary points. It is easily seen that

- $\forall x \in \mathbb{R} : \ f'(x) = x^2(4x - 21)$,
- $f'(x) = 0 \ \Leftrightarrow \ \left[x = 0 \ \vee \ x = \frac{21}{4}\right]$,
- $f''(0) = 0 \neq -42 = f'''(0)$,
- $f''(\frac{21}{4}) = \frac{441}{4} > 0$,

and therefore (see Theorem 17.1.5) the function f has a unique local extreme: f has a strict local minimum at the point $\frac{21}{4}$. The situation is illustrated in Fig. 41.

■

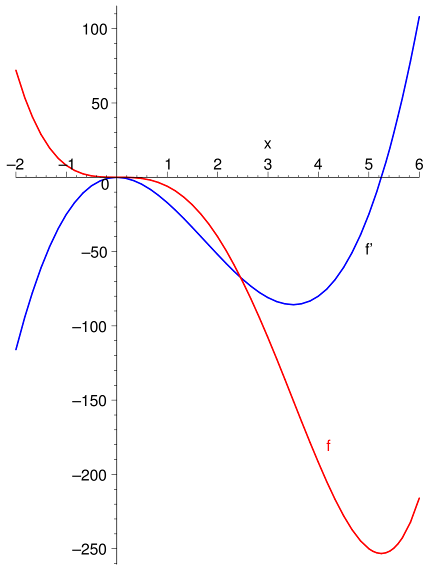

Fig. 41

17.2 Global Extremes

17.2.1 Definitions. A function f attains its maximum on a set $M \subset D(f)$ at a point x_0 if

$$x_0 \in M \ \wedge \ f(x_0) = \max\ \{f(x) :\ x \in M\} =: \max_{x \in M} f(x).$$

A function f attains its minimum on a set $M \subset D(f)$ at a point x_0 if

$$x_0 \in M \ \wedge \ f(x_0) = \min\ \{f(x) :\ x \in M\} =: \min_{x \in M} f(x).$$

17.2.2 Example. Let the function f be defined by $f(x) := \arctan x$ ... see Fig. 27. Then

- neither $\min\limits_{x \in \mathbb{R}} f(x)$ nor $\max\limits_{x \in \mathbb{R}} f(x)$ exists,
- $\min\limits_{x \in [0, +\infty)} f(x) = 0$, $\max\limits_{x \in [0, +\infty)} f(x)$ does not exist,
- $\min\limits_{x \in [-1, 1]} f(x) = -\frac{\pi}{4}$, $\max\limits_{x \in [-1, 1]} f(x) = \frac{\pi}{4}$.

17.2.3 Theorem (Weierstrass). *If a function f is continuous in an interval $[a, b]$ ($a, b \in \mathbb{R}$, $a < b$), then both $\max\limits_{x \in [a, b]} f(x)$ and $\min\limits_{x \in [a, b]} f(x)$ exist.*

PROOF. Let us show that

$$\max_{x \in [a, b]} f(x)$$

exists (the existence of the minimum can be shown analogously).

We know (see Theorem 2.7) that

$$\sup\ \{f(x) :\ x \in [a, b]\} =: s$$

exists, so that we have to prove that

$$\exists x_0 \in [a, b] :\ f(x_0) = s.$$

From the definition of the supremum it follows that there is a sequence (x_n) such that $x_n \in [a, b]$ for every $n \in \mathbb{N}$ and $f(x_n) \to s$.

Now let us choose a convergent subsequence (x_{k_n}) of the bounded sequence (x_n) (it is possible due to Theorem 8.11), i.e., there is an $x_0 \in \mathbb{R}$ such that $x_{k_n} \to x_0$. Since for every $n \in \mathbb{N}$ it holds that $x_{k_n} \in [a, b]$, we get

$$x_0 \in [a, b].$$

To finish the proof, it suffices to note that since $(f(x_{k_n}))$ is a subsequence of $(f(x_n))$, it has to have the same limit s (see Theorem 8.17). Moreover,

$$[a, b] \ni x_{k_n} \to x_0 \in [a, b],$$

and therefore (f is continuous in $[a, b]$) $f(x_{k_n}) \to f(x_0)$. Since any sequence can have no more than one limit (see Theorem 8.16), we get

$$f(x_0) = s.$$

□

17.2.4 Examples.

1) Let $f(x) := x^2$ and $M = (0,\,1)$. Then neither $\min\limits_{x\in M} f(x)$ nor $\max\limits_{x\in M} f(x)$ exists.

2) Let $f(x) := -x + \operatorname{sgn}(x)$ and $M = [-1,\,1]$. Then neither $\min\limits_{x\in M} f(x)$ nor $\max\limits_{x\in M} f(x)$ exists.

3) Let $f(x) := \operatorname{sgn}(x)$ and $M = \mathbb{R}$. Then $\min\limits_{x\in M} f(x) = -1$ and $\max\limits_{x\in M} f(x) = 1$.

17.2.5 Example. Find global extremes of the function

$$f(x) := x^3 - 2x^2$$

on the set $M = \left[-\frac{1}{2},\,3\right]$.

SOLUTION. From the Weierstrass theorem it follows that the desired extremes exist!

If f attains its extreme on M at a point x_0, then either $x_0 \in \left\{-\frac{1}{2},\,3\right\}$ or $x_0 \in (-\frac{1}{2},\,3)$. If the second possibility happens, x_0 is obviously also a local extreme, and since f is differentiable on $\mathbb{R}$, we get $f'(x_0) = 0$ (see Theorem 17.1.4). It holds that

$$f'(x) = 3x^2 - 4x = 0 \Leftrightarrow \left[x = 0 \,\vee\, x = \frac{4}{3}\right] \quad \text{and} \quad 0,\,\frac{4}{3} \in \left(-\frac{1}{2},\,3\right).$$

Thus $-\frac{1}{2}, 0, \frac{4}{3}, 3$ are the so-called "suspicious points" (points at which f can have an extreme on M). By comparison of the function values

$$f\left(-\frac{1}{2}\right) = -\frac{5}{8} \approx -0.6;\;\; f(0) = 0;\;\; f\left(\frac{4}{3}\right) = -\frac{32}{27} \approx -1.2;\;\; f(3) = 9,$$

we conclude that f attains its maximum on M at the point 3 and minimum on M at the point $\frac{4}{3}$.

■

17.2.6 Proof of the Rolle Theorem (see Theorem 14.1). At least one of the following possibilities is bound to happen:

i) there is an $x_1 \in [a,\,b]$ such that $f(x_1) > f(a)$,

ii) there is an $x_2 \in [a,\,b]$ such that $f(x_2) < f(a)$,

iii) for every $x \in [a,\,b]$ it holds that $f(x) = f(a)$.

If the first case happens, let us consider $\xi \in [a,\,b]$ such that f attains its maximum on $[a,\,b]$ at it. (It follows from the Weierstrass theorem that such ξ exists!) Then obviously $f(\xi) \geq f(x_1) > f(a) = f(b)$, and therefore $\xi \in (a,\,b)$. Hence it follows that f has a local maximum at ξ, and thus (according to the assumption that $f'(\xi)$ exists) $f'(\xi) = 0$ (see Theorem 17.1.4).

If the second case happens, we choose ξ so that f attains its minimum on $[a,\,b]$ at ξ. Similarly as in the first case, it can be shown that $\xi \in (a,\,b)$ and $f'(\xi) = 0$.

If the third case happens, $f'(\xi) = 0$ even for every $\xi \in (a,\,b)$ since f is constant in $[a,\,b]$.

□

18 CONVEX AND CONCAVE FUNCTIONS

18.1 Definitions. Assume $I \subset \mathbb{R}$ is an interval. A function f is

- strictly convex in I if

$$\forall x_1, x_2, x_3 \in I;\ x_1 < x_2 < x_3:\ f(x_2) < f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1).$$

- convex in I if

$$\forall x_1, x_2, x_3 \in I;\ x_1 < x_2 < x_3:\ f(x_2) \leq f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1).$$

- strictly concave in I if

$$\forall x_1, x_2, x_3 \in I;\ x_1 < x_2 < x_3:\ f(x_2) > f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1).$$

- concave if

$$\forall x_1, x_2, x_3 \in I;\ x_1 < x_2 < x_3:\ f(x_2) \geq f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1).$$

18.2 Observation. Assume $I \subset \mathbb{R}$ is an interval, x_1, x_2, $x_3 \in I$, $x_1 < x_2 < x_3$, and f is a function defined on I. Let p is a line passing through the points $(x_1, f(x_1))$ and $(x_3, f(x_3))$, i.e.,

$$p:\ y = f(x_1) + \frac{f(x_3) - f(x_1)}{x_3 - x_1}\,(x - x_1).$$

If the function f is strictly convex in I, then the point $(x_2, f(x_2))$ "lies under the line" p. Similarly, if the function f is strictly concave in I, then the point $(x_2, f(x_2))$ "lies above the line" p.

18.3 Example. Function $f(x) := x^3$ is strictly concave in the interval $(-\infty,\ 0]$ and strictly convex in the interval $[0,\ +\infty)$ ... see Fig. 14.

18.4 Theorem. *Assume f is a continuous function in an interval I and $f''(x)$ exists at every interior point x of I. Then f is*

- *convex in I if and only if $f''(x) \geq 0$ at every interior point x of I;*
- *concave in I if and only if $f''(x) \leq 0$ at every interior point x of I.*

Moreover,

- *if $f''(x) > 0$ at every interior point x of I, then f is strictly convex in I;*
- *if $f''(x) < 0$ at every interior point x of I, then f is strictly concave in I.*

18.5 Remark. It is useful to note that

- if $f''(x) > 0$ for every $x \in I$, then the function f' is increasing in I;
- if $f''(x) < 0$ for every $x \in I$, then the function f' is decreasing in I.

18.6 Definition. We say that a function f has an inflection point at a point x_0 if the derivative $f'(x_0)$ exists finite and there is a neighbourhood $P(x_0) = P^+(x_0) \cup P^-(x_0)$ such that either

- f is strictly convex in $P^-(x_0)$ and strictly concave in $P^+(x_0)$, or
- f is strictly concave in $P^-(x_0)$ and strictly convex in $P^+(x_0)$.

In other words, f has an inflection point at a point at which the derivative exists proper and at which the function f changes from being strictly convex to being strictly concave, or vice versa.

18.7 Example. A function $f(x) := \arctan x$ has an inflection point at the point 0
$\dots$ see Fig. 27.

18.8 Exercise. Prove the following proposition: if a function f has an inflection point at a point x_0 and $f''(x_0)$ exists, then $f''(x_0) = 0$.

19 ASYMPTOTES OF (A GRAPH OF) A FUNCTION

19.1 Definition. A line $x = x_0$ ($x_0 \in \mathbb{R}$) is called a vertical asymptote of a function f if at least one of the one-sided limits of the function f at the point x_0 is improper (i.e., equal to $+\infty$ or $-\infty$).

19.2 Example. The line $x = 1$ is a vertical asymptote of the function f defined by

$$f(x) := \begin{cases} 1, & x \le 1, \\ \frac{1}{x-1}, & x > 1, \end{cases}$$

since

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \frac{1}{x-1} = +\infty.$$

The situation is depicted in Fig. 42.

19.3 Definition. A line $y = ax + b$ ($a, b \in \mathbb{R}$) is called

- an asymptote of a function f at $+\infty$ if $\lim\limits_{x\to+\infty} (f(x) - (ax + b)) = 0$,
- an asymptote of a function f at $-\infty$ if $\lim\limits_{x\to-\infty} (f(x) - (ax + b)) = 0$.

19.4 Example. Let the function f be defined by

$$f(x) := \begin{cases} 20\,\mathrm{e}^x, & x \le 0, \\ 2x + \frac{10\sin(2x)}{x}, & x > 0. \end{cases}$$

Then

- $\lim_{x \to -\infty} (f(x) - 0) = \lim_{x \to -\infty} (20\,\mathrm{e}^x) = 0$,
- $\lim_{x \to +\infty} (f(x) - 2x) = \lim_{x \to +\infty} \frac{10\sin(2x)}{x} = 0$,

and therefore

- the line $y = 0$ is the asymptote of f at $-\infty$,
- the line $y = 2x$ is the asymptote of f at $+\infty$.

The situation is depicted in Fig. 43.

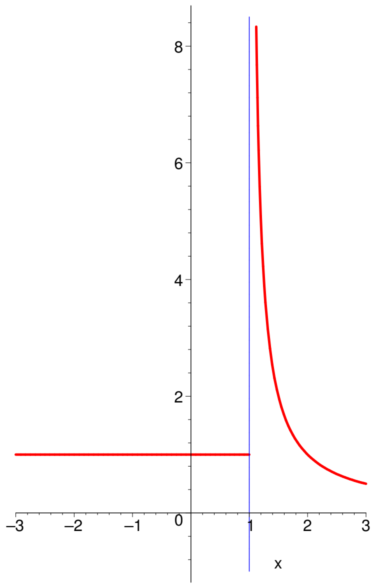

Fig. 42

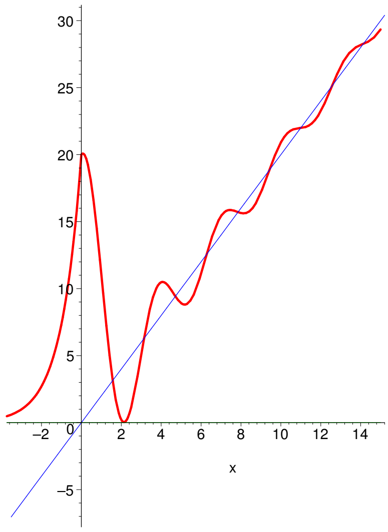

Fig. 43

19.5 Observation. Assume a line $y = ax + b$ is an asymptote of f at $+\infty$, i.e.,

$$\lim_{x \to +\infty} (f(x) - (ax + b)) = 0.$$

Hence it follows

$$\lim_{x \to +\infty} (f(x) - ax) = b \ (\in \mathbb{R}),$$

and therefore

$$0 = \lim_{x \to +\infty} \frac{f(x) - ax}{x} = \lim_{x \to +\infty} \left(\frac{f(x)}{x} - a \right),$$

so that

$$\lim_{x \to +\infty} \frac{f(x)}{x} = a \ (\in \mathbb{R}).$$

(Similar relations can be derived also for an asymptote of f at $-\infty$.)

19.6 Theorem (Existence of Asymptotes at $+\infty$ and $-\infty$). *A line $y = ax + b$ is an asymptote of a function f at $+\infty$ if and only if*

- $\lim_{x\to+\infty} \frac{f(x)}{x} = a \in \mathbb{R}$,
- $\lim_{x\to+\infty} (f(x) - ax) = b \in \mathbb{R}$.

A line $y = ax + b$ is an asymptote of a function f at $-\infty$ if and only if

- $\lim_{x\to-\infty} \frac{f(x)}{x} = a \in \mathbb{R}$,
- $\lim_{x\to-\infty} (f(x) - ax) = b \in \mathbb{R}$.

Note that the statements of the theorem include formulas saying how to find the asymptotes of a given function f at $+\infty$ and at $-\infty$.

19.7 Example. Find all asymptotes of the function f defined by

$$f(x) := \frac{x}{\arctan x}.$$

SOLUTION. The function f is clearly continuous at every point of its domain $D(f) = \mathbb{R} \setminus \{0\}$, and since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\arctan x} \overset{\text{l'H.}}{=} \lim_{x\to 0} \frac{1}{\frac{1}{1+x^2}} = 1,$$

there is no vertical asymptote of the function f. Moreover, since

- $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{1}{\arctan x} = \frac{1}{\pm\frac{\pi}{2}} = \pm\frac{2}{\pi}$,
- $\lim_{x\to\pm\infty} \left(f(x) - \left(\pm\frac{2}{\pi}x\right)\right) = \lim_{x\to\pm\infty} \frac{x(\pi\mp 2\arctan x)}{\pi \arctan x} =$

$$= \left(\lim_{x\to\pm\infty} \frac{1}{\pi \arctan x}\right)\left(\lim_{x\to\pm\infty} \frac{\pi\mp 2\arctan x}{\frac{1}{x}}\right) \overset{\text{l'H.}}{=} \left(\pm\frac{2}{\pi^2}\right)(\pm 2) = +\frac{4}{\pi^2},$$

it follows from Theorem 19.6 that

- the line $y = \frac{2}{\pi}x + \frac{4}{\pi^2}$ is the asymptote of f at $+\infty$,
- the line $y = -\frac{2}{\pi}x + \frac{4}{\pi^2}$ is the asymptote of f at $-\infty$.

The situation is depicted in Fig. 44.

■

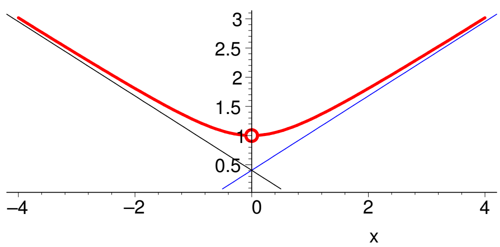

Fig. 44

20 EXAMINATION OF A FUNCTION BEHAVIOUR

20.1 To examine the function behaviour means to determine (calculate)

- domain of the function;
- points (and intervals!) in which the function is continuous;
- whether the function is even or odd; whether the function is periodic;
- one-sided limits at the end points of the domain of the function and at the points of discontinuity;
- derivative of the function;
- intervals of the strict monotonicity of the function;
- intervals of the strict convexity and strict concavity of the function; inflexion points of the function;
- asymptotes of the function;
- eventually other properties of the function (for example: function values at significant points, one-sided derivatives at significant points, intersection points with axis, ...);

and to draw the graph of the function with all essential qualitative characteristics.

20.2 Example. Examine the behaviour of the function f defined by

$$f(x) := \arcsin \frac{2x}{1+x^2}.$$

SOLUTION. It can be easily checked that

- $D(f) = \mathbb{R}$,
- f is continuous in $\mathbb{R}$,
- f is odd,
- $\lim\limits_{x \to +\infty} f(x) = 0$,
- $\forall x \in [0,\, 1): \ f'(x) = \frac{2}{1+x^2}$,

  $\forall x \in (1,\, +\infty): \ f'(x) = -\frac{2}{1+x^2}$,
- $\forall x \in [0,\, 1): \ f''(x) = \frac{-4x}{(1+x^2)^2}$,

  $\forall x \in (1,\, +\infty): \ f''(x) = \frac{4x}{(1+x^2)^2}$,
- $f(0) = 0,\ f(1) = \frac{\pi}{2},\ f'(0) = 2,\ f'_+(1) = -1,\ f'_-(1) = 1.$

Now let us think over the validity of the following statements:

- f is continuous in $\mathbb{R}$, and therefore f has no vertical asymptote;
- f is odd and $\lim_{x \to +\infty} f(x) = 0$, and therefore the line $y = 0$ is the asymptote of f at $+\infty$ and at $-\infty$;
- f is continuous in $\mathbb{R}$, f is odd, $\forall x \in [0, 1) : f'(x) > 0$, $\forall x \in (1, +\infty) : f'(x) < 0$, and therefore f is increasing in the interval $[-1, 1]$ and decreasing in the intervals $(-\infty, -1]$ and $[1, +\infty)$;
- f is continuous in $\mathbb{R}$, f is odd, $\forall x \in (0, 1) : f''(x) < 0$, $\forall x \in (1, +\infty) : f''(x) > 0$, and therefore f is strictly convex in the intervals $[-1, 0]$ and $[1, +\infty)$ and strictly concave in the intervals $(-\infty, -1]$ and $[0, 1]$; moreover, since $f'(0) \in \mathbb{R}$, f has the inflection point at 0.

Finally, let us calculate that $f(0) = 0$, $f(1) = \frac{\pi}{2}$, $f'(0) = 2$, $f'_+(1) = -1$, $f'_-(1) = 1$. To finish the task, it remains to draw the graph of the function f:

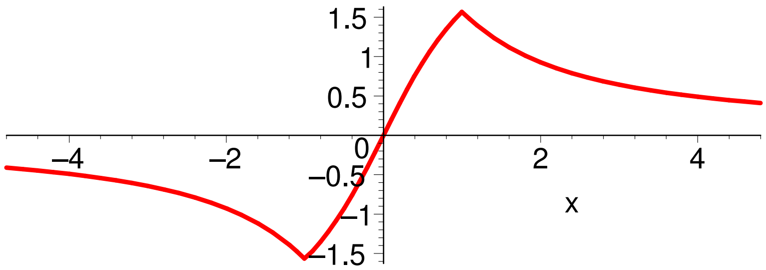

Fig. 45

■

I APPROXIMATION OF A FUNCTION BY A POLYNOMIAL

21 TAYLOR'S POLYNOMIAL (APPROXIMATION OF A FUNCTION IN A NEIGHBOURHOOD OF A POINT)

21.1 Taylor's Polynomial

21.1.1 Let us try to approximate a function f on a neighbourhood of a point x_0 by a polynomial p. We already know (see Chapter F) that the best polynomial of the first degree approximating a differentiable function f on a neighbourhood of a point x_0 is

$$p(x) := f(x_0) + f'(x_0)(x - x_0)$$

(graph of p is the tangent of graph of the function f at x_0). We also know that for the function

$$R(x) := f(x) - p(x)$$

(the so-called approximation error) it holds that

$$\lim_{x \to x_0} \frac{R(x)}{x - x_0} = 0.$$

It is certainly natural to expect that in the case of approximation of a function by a polynomial of the degree greater than 1, it can sometimes happen that the approximation error is reduced. Let us try to find a polynomial p (of the degree at most n) so that the approximation error R is in the neighbourhood of x_0 as small as possible; more precisely, for a $k \in \mathbb{N}$ as large as possible we want

$$\lim_{x \to x_0} \frac{R(x)}{(x - x_0)^k} = 0. \tag{1}$$

(If (1) holds, we say that the function R is infinitely small of order greater than k at the point x_0.) Note that (1) holds if

$$R(x_0) = R'(x_0) = R''(x_0) = \ \dots \ = R^{(k-1)}(x_0) = R^{(k)}(x_0) = 0.$$

PROOF.

$$\lim_{x \to x_0} \frac{R(x)}{(x - x_0)^k} \overset{\text{l'H.}}{=} \lim_{x \to x_0} \frac{R'(x)}{k\,(x - x_0)^{k-1}} \overset{\text{l'H.}}{=} \dots \overset{\text{l'H.}}{=} \lim_{x \to x_0} \frac{R^{(k-1)}(x)}{k\,(k-1)\cdots 2(x - x_0)} =$$

$$= \lim_{x \to x_0} \left[\frac{1}{k!} \frac{R^{(k-1)}(x) - R^{(k-1)}(x_0)}{x - x_0} \right] = \frac{1}{k!} R^{(k)}(x_0) = 0.$$

□

Thus let us look for a polynomial p of the degree at most n so that for a k as large as possible it holds:

- $R(x_0) = f(x_0) - p(x_0) = 0,$
- $R'(x_0) = f'(x_0) - p'(x_0) = 0,$
- $R''(x_0) = f''(x_0) - p''(x_0) = 0,$

$\vdots$

- $R^{(k)}(x_0) = f^{(k)}(x_0) - p^{(k)}(x_0) = 0.$

(Let us assume that the function f has "needed number of derivatives" at the point x_0.)

1) Let $n = 1$ and let us look for a polynomial p in the form

$$p(x) = ax + b,$$

where $a, b \in \mathbb{R}$. It is easily seen that the first two requirements

$$f(x_0) = p(x_0) = ax_0 + b, \quad f'(x_0) = p'(x_0) = a$$

already uniquely determine the numbers a and b, so that

$$p(x) = f'(x_0)x + f(x_0) - f'(x_0)x_0 = f(x_0) + f'(x_0)(x - x_0).$$

(We are certainly not surprised by this result ...)

2) Let $n = 2$ and let us look for a polynomial p in the form

$$p(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$. In this case, the polynomial p is determined by these three conditions:

- $f(x_0) = p(x_0) = ax_0^2 + bx_0 + c,$
- $f'(x_0) = p'(x_0) = 2ax_0 + b,$
- $f''(x_0) = p''(x_0) = 2a.$

Hence we get

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

3) Let $n \in \mathbb{N}$ and let us look for a polynomial p in the form

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n,$$

where $a_0, \ldots, a_n \in \mathbb{R}$. Let us remark that every polynomial of the degree at most n can be written in such way. It can be checked again that the conditions

- $f(x_0) = p(x_0) = a_0,$
- $f'(x_0) = p'(x_0) = a_1,$
- $f''(x_0) = p''(x_0) = 2a_2,$

$\vdots$

- $f^{(j)}(x_0) = p^{(j)}(x_0) = j!\, a_j$,

$\vdots$

- $f^{(n)}(x_0) = p^{(n)}(x_0) = n!\, a_n$

yield

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n =$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k =: T_n(x).$$

T_n is the so-called Taylor polynomial of the order n of the function f at the point x_0. We already know that the function

$$R_{n+1}(x) := f(x) - T_n(x)$$

(the so-called remainder after nth term) is infinitely small of order greater than n at the point x_0, i.e.,

$$\lim_{x \to x_0} \frac{R_{n+1}(x)}{(x - x_0)^n} = 0.$$

The following theorem shall give us more accurate information about the size of $R_{n+1}(x)$.

21.1.2 Theorem (Taylor). *Assume a function f has (finite) $(n+1)$th derivative in a neighbourhood $U(x_0, \delta)$ and $x \in P(x_0, \delta)$. Then there exists a ξ lying between points x and x_0 such that*

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x - x_0)^{n+1}.$$

By "ξ lies between points x and x_0" we mean that either

- $\xi \in (x_0,\, x)$ if $x > x_0$, or
- $\xi \in (x,\, x_0)$ if $x < x_0$.

21.1.3 Remarks.

- The term $\frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x - x_0)^{n+1}$ is called the Lagrange form of the remainder.
- If we put $h = x - x_0$, we can write

$$T_n(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1}h + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n.$$

- If $x_0 = 0$, we obtain the so-called Maclaurin polynomial.

21.1.4 Examples. Find the Maclaurin polynomial of the function

1) e^x.

SOLUTION. For every $x \in \mathbb{R}$, we have $\mathrm{e}^x = (\mathrm{e}^x)' = (\mathrm{e}^x)'' = \dots$, and therefore

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

■

2) $\sin x$.

SOLUTION.

- $\sin 0 = 0$,
- $\sin' 0 = \cos 0 = 1$,
- $\sin'' 0 = -\sin 0 = 0$,
- $\sin''' 0 = -\cos 0 = -1$,
- $\sin^{(4)} 0 = \sin 0 = 0$,

$\vdots$

and therefore

$$T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

■

3) $\cos x$.

SOLUTION.

- $\cos 0 = 1$,
- $\cos' 0 = -\sin 0 = 0$,
- $\cos'' 0 = -\cos 0 = -1$,
- $\cos''' 0 = \sin 0 = 0$,
- $\cos^{(4)} 0 = \cos 0 = 1$,

$\vdots$

and therefore

$$T_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

■

21.1.5 Approximation of a function by the Taylor polynomial is useful in many applications. Let us show – for illustration – at least one of them.

21.2 Numerical Derivative

21.2.1 In applications we often work with a differentiable function given only by values at certain points (for example, obtained by measurement).

For simplicity, let us assume that values of a function f are given at the points:

$$a = x_0,\ x_1,\ x_2,\ \ldots,\ x_n = b \in [a,\ b],$$

where $x_k = x_0 + kh$ and $h = \frac{b-a}{n}$, i.e., we know $f(x_k)$ ($n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$).

We shall be concerned with the following problem: *How to approximate the derivative of the function f at the points x_k, where $k \in \{1, \ldots, n-1\}$, so that the approximation error is for $h \to 0$ (i.e., for $n \to +\infty$) as small as possible?*

The first idea could be a use of the definition of the derivative:

$$f'(x_k) = \lim_{h\to 0} \frac{f(x_k+h) - f(x_k)}{h} \approx \frac{f(x_k+h) - f(x_k)}{h} = \frac{f(x_{k+1}) - f(x_k)}{h}.$$

Note that if the Taylor polynomial of the first order of the function f at the point x_k is used, i.e., if we employ the equality

$$f(x_k+h) = f(x_k) + f'(x_k)h + R_2(x_k+h),$$

we get the same result since

$$f'(x_k) \approx \frac{f(x_k+h) - f(x_k)}{h} = \frac{f(x_{k+1}) - f(x_k)}{h}. \tag{2}$$

(The aim of this subsection is only to show one way how we can think about the given problem. Therefore we shall not pay attention to a precise formulation of the assumptions under which our ideas are possible and correct. On the other hand, the reader is capable of correction of this inaccuracy.) How large is the error for $h \to 0$? The answer is obvious (see Subsection 21.1):

$$\mathcal{R}^1(h) := \frac{f(x_k+h) - f(x_k)}{h} - f'(x_k) = \frac{R_2(x_k+h)}{h} \to 0 \quad \text{for } h \to 0.$$

If we use the Taylor polynomial of the second order:

$$f(x_k+h) = f(x_k) + f'(x_k)h + \frac{1}{2}f''(x_k)h^2 + R_3(x_k+h),$$

$$f(x_k-h) = f(x_k) - f'(x_k)h + \frac{1}{2}f''(x_k)h^2 + R_3(x_k-h),$$

we obtain by subtracting these equations and neglecting the remainders after the second term the following approximation:

$$f'(x_k) \approx \frac{f(x_k+h) - f(x_k-h)}{2h} = \frac{f(x_{k+1}) - f(x_{k-1})}{2h}. \tag{3}$$

Let us again estimate the extent of the error:

$$\mathcal{R}^2(h) := \frac{f(x_k+h) - f(x_k-h)}{2h} - f'(x_k) = \frac{R_3(x_k+h)}{2h} - \frac{R_3(x_k-h)}{2h},$$

and thus even

$$\frac{\mathcal{R}^2(h)}{h} = \frac{R_3(x_k+h)}{2h^2} - \frac{R_3(x_k-h)}{2h^2} \to 0 \quad \text{for } h \to 0.$$

21.2.2 Remark. We showed that the approximation (3) causes (for $h \to 0$) an error of lower order than the approximation (2).

Let us also mention another advantage of (3). If the function f is linear (i.e., $f(x) = ax + b$), the approximation (2) is accurate, furthermore, the approximation (3) is accurate even for functions at most quadratic ($f(x) = ax^2 + bx + c$).

22 INTERPOLATION POLYNOMIALS (APPROXIMATION OF A FUNCTION IN AN INTERVAL)

22.1 Let us start with the similar situation as in Subsection 21.2. Let a function f be given by values at mutually distinct points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Now we shall deal with the task: *How to approximate values of f also at another points of $[a, b]$?*

It can be shown that there is a unique (the so-called Lagrange) polynomial φ of the degree at most n such that $\varphi(x_k) = f(x_k)$ for every $k \in \{0, 1, \ldots, n\}$.

22.2 Remark. The approximation of a function by the Lagrange polynomial carries many disadvantages. One of them is the fact that when $n \in \mathbb{N}$ is large, it happens that polynomial φ attains in the intervals (x_k, x_{k+1}) also values "differing considerably" from the values $f(x_k)$ and $f(x_{k+1})$ (see Fig. 46). This drawback can be removed if we approximate the function f in $[a, b]$ by functions that are piecewise polynomial.

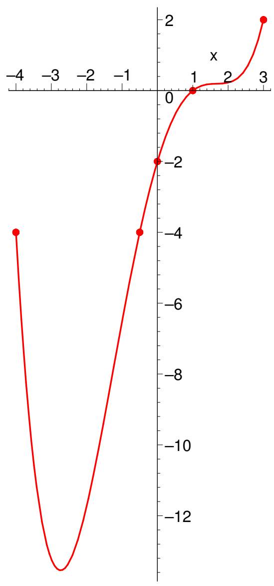

Fig. 46

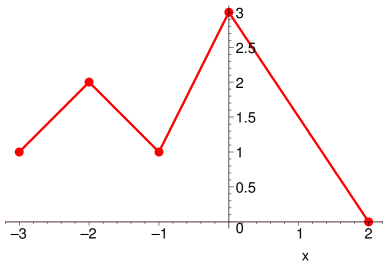

Fig. 47

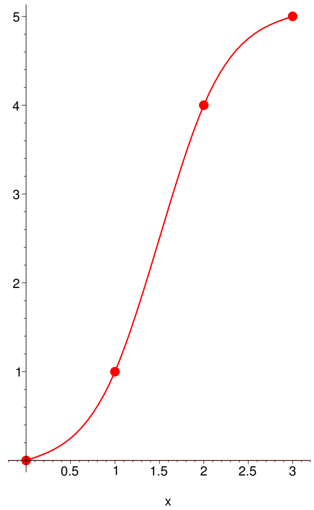

Fig. 48

22.3 We speak of a <u>linear interpolation</u> when we approximate the function f in every interval $[x_k,\, x_{k+1}]$ by a linear function given by points $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$ (the so-called <u>Lagrange polynomial of the first order</u>) ... see Fig. 47.

22.4 Remark. We removed the above-described disadvantage of the Lagrange polynomials though, but at the cost of obtaining a function which is (generally) not differentiable. Moreover, the differentiability fails exactly at the points x_k.

22.5 Now let us describe an <u>interpolation by cubic spline-functions</u> ... see Fig. 48.
It can be shown that there is a function φ such that

i) it is equal to a polynomial of at most third degree in every $[x_k,\, x_{k+1}]$,

ii) φ, φ', φ'' are continuous in $[a,\, b]$,

iii) $\varphi(x_k) = f(x_k)$ for every $k \in \{0, 1, \ldots, n\}$.

Conditions *i)*, *ii)*, *iii)* do not determine the function φ uniquely, however, if we add (for example) the condition

iv) $\varphi''(a) = \varphi''(b) = 0$,

then there exists a unique function φ satisfying *i)*, *ii)*, *iii)*, *iv)*.

22.6 Remark. In all mentioned cases we required equality of the function values of f with its approximation φ at the points $x_0, x_1, \ldots, x_n$; i.e., $f(x_k) = \varphi(x_k)$ for every $k \in \{0, 1, \ldots, n\}$.

However, if the values $f(x_k)$ are obtained, for instance, by measurement, i.e., they are loaded in certain error, this requirement is too restrictive. It often suffices to require that the values of f and φ do not differ "too much". The similar ideas lead to many other approximations, however, we shall not be focused on them.

J ANTIDERIVATIVE (INDEFINITE INTEGRAL)

23 ANTIDERIVATIVE

23.1 Definition. A function $F : \mathbb{R} \mapsto \mathbb{R}$ is said to be an antiderivative of a function $f : \mathbb{R} \mapsto \mathbb{R}$ in an open interval $I \subset \mathbb{R}$ if

$$\forall x \in D(F) = I : \; F'(x) = f(x).$$

23.2 Example. The function $F(x) := \sin x$ is the antiderivative of the function $f(x) := \cos x$ in $\mathbb{R}$.

23.3 Theorem (Existence of an Antiderivative). *If a function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous on an open interval $I \subset \mathbb{R}$, then f has an antiderivative in I.*

Continuity of a function is the sufficient, but not necessary condition for existence of an antiderivative.

23.4 Theorem. *Suppose F is an antiderivative of a function f in an open interval $I \subset \mathbb{R}$. Then the functions $G_c : \mathbb{R} \mapsto \mathbb{R}$ defined by*

$$G_c(x) := F(x) + c \quad (c \in \mathbb{R})$$

are exactly all antiderivatives of f in I.

PROOF. Note that we need to prove these two statements:

i) G_c is an antiderivative of f in I for any $c \in \mathbb{R}$;

ii) if G is an antiderivative of f in I, then there is a $c \in \mathbb{R}$ such that $G(x) = F(x) + c$ for every $x \in I$.

The proof of the first statement is straightforward since for all $x \in I$ we have $(G_c(x))' = (F(x) + c)' = F'(x) + c' = f(x) + 0 = f(x)$.

To prove the second statement, let us consider the function $G - F$. Our assumptions imply that $(G - F)'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$ for all $x \in I$, and therefore $G - F$ is constant in I (see Theorem 16.1). Thus there is a $c \in \mathbb{R}$ such that $(G - F)(x) = G(x) - F(x) = c$ for every $x \in I$.

□

23.5 Convention. If a function F is an antiderivative of a function f, then we shall write

$$F(x) = \int f(x) \, \mathrm{d}x$$

and speak of an indefinite integral of the function f.

23.6 Remarks.

- This notation, used by historical reasons, is not fully correct. We have already observed that the antiderivative F is uniquely determined, except for an additive constant, by the function f. The question is how to understand the equalities $\int \cos x \, \mathrm{d}x = \sin x$, $\int \cos x \, \mathrm{d}x = \sin x + 5$, etc.
- Let us make a convention that $\int f(x) \, \mathrm{d}x$ stands for some antiderivative of f (we would get any other by adding an arbitrary constant). Thus the equality $\int f(x) \, \mathrm{d}x = G(x)$ shall be understood so that there is an antiderivative F of f satisfying $F(x) = G(x)$ for every $x \in D(F)$.
- In speaking of an antiderivative (or indefinite integral), we have always in mind some open interval too. (See Definition 23.1.)

23.7 Theorem. *On every open interval that is a subset of the domain of the corresponding integrated function it holds that*

- $\int k \, \mathrm{d}x = kx \quad (k \in \mathbb{R})$,
- $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} \quad (n \in \mathbb{R} \setminus \{-1\})$,
- $\int \frac{1}{x} \, \mathrm{d}x = \log |x|$,
- $\int \sin x \, \mathrm{d}x = -\cos x$,
- $\int \cos x \, \mathrm{d}x = \sin x$,
- $\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \tan x$,
- $\int \frac{1}{\sin^2 x} \, \mathrm{d}x = -\cot x$,
- $\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x$,
- $\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \operatorname{arccot} x$,
- $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x$.

24 TECHNIQUES OF INTEGRATION (METHODS OF CALCULATING ANTIDERIVATIVES)

24.1 Theorem (Linearity of Indefinite Integral). *Assume $\alpha, \beta \in \mathbb{R}$ and functions f and g are continuous on an open interval $I \subset \mathbb{R}$. Then*

$$\int (\alpha f(x) + \beta g(x)) \, \mathrm{d}x = \alpha \int f(x) \, \mathrm{d}x + \beta \int g(x) \, \mathrm{d}x \quad \textit{in } I.$$

PROOF. The existence of all mentioned integrals is ensured by the continuity of the functions f and g and Theorem 23.3. Moreover, by Theorem 13.9,

$$\left(\alpha\int f(x)\,\mathrm{d}x+\beta\int g(x)\,\mathrm{d}x\right)'=\alpha\left(\int f(x)\,\mathrm{d}x\right)'+\beta\left(\int g(x)\,\mathrm{d}x\right)'=\alpha f(x)+\beta g(x)\quad\text{in } I.$$

□

24.2 Examples.

1)

$$\int\frac{\sqrt{x}-\sqrt[3]{x^5}+x^2}{\sqrt[6]{x}}\,\mathrm{d}x=\int\left(x^{\frac12-\frac16}-x^{\frac53-\frac16}+x^{2-\frac16}\right)\,\mathrm{d}x=$$

$$=\int x^{\frac13}\,\mathrm{d}x-\int x^{\frac32}\,\mathrm{d}x+\int x^{\frac{11}{6}}\,\mathrm{d}x=\frac{x^{\frac43}}{\frac43}-\frac{x^{\frac52}}{\frac52}+\frac{x^{\frac{17}{6}}}{\frac{17}{6}}=\frac34\sqrt[3]{x^4}-\frac25\sqrt{x^5}+\frac{6}{17}\sqrt[6]{x^{17}}\quad(\text{in } \mathbb{R}^+).$$

2)

$$\int\tan^2 x\,\mathrm{d}x=\int\frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x=\int\frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x=\int\left(\frac{1}{\cos^2 x}-1\right)\,\mathrm{d}x=$$

$$=\int\frac{1}{\cos^2 x}\,\mathrm{d}x-\int 1\,\mathrm{d}x=\tan x-x$$

$\Big($in every $\left(-\frac{\pi}{2}+k\pi,\ \frac{\pi}{2}+k\pi\right)$, where $k\in\mathbb{Z}\Big)$.

24.3 Theorem (Integration by Parts). *Suppose functions u and v have continuous first derivatives in an open interval $I\subset\mathbb{R}$. Then*

$$\int u(x)v'(x)\,\mathrm{d}x=u(x)v(x)-\int u'(x)v(x)\,\mathrm{d}x\quad\textit{in } I.$$

PROOF. The existence of both integrals is ensured by Theorem 23.3. Moreover, by Theorem 13.9, for all $x\in I$

$$(u(x)v(x))'=u'(x)v(x)+u(x)v'(x),$$

and therefore, by Theorem 24.1,

$$u(x)v(x)=\int(u'(x)v(x)+u(x)v'(x))\,\mathrm{d}x=\int u'(x)v(x)\,\mathrm{d}x+\int u(x)v'(x)\,\mathrm{d}x.$$

□

24.4 Examples.

1)

$$\underset{\substack{u=x,\quad v'=\sin x\\ u'=1,\quad v=-\cos x}}{\int x\sin x\,\mathrm{d}x}=-x\cos x+\int\cos x\,\mathrm{d}x=-x\cos x+\sin x\quad(\text{in } \mathbb{R}).$$

2)

$$\int \log x \,\mathrm{d}x = \underset{\substack{u=\log x, \quad v'=1 \\ u'=\frac{1}{x}, \quad v=x}}{\int 1 \cdot \log x \,\mathrm{d}x} = x \log x - \int x \,\frac{1}{x}\,\mathrm{d}x = x\log x - x \quad \text{(in } \mathbb{R}^+\text{)}.$$

3)

$$\underset{\substack{u=\sin x, \quad v'=\mathrm{e}^x \\ u'=\cos x, \quad v=\mathrm{e}^x}}{\int \mathrm{e}^x \sin x \,\mathrm{d}x} = \mathrm{e}^x \sin x - \underset{\substack{u=\cos x, \quad v'=\mathrm{e}^x \\ u'=-\sin x, \quad v=\mathrm{e}^x}}{\int \mathrm{e}^x \cos x \,\mathrm{d}x} = \mathrm{e}^x \sin x - \left(\mathrm{e}^x \cos x - \int \mathrm{e}^x(-\sin x)\,\mathrm{d}x\right) =$$

$$= \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x \,\mathrm{d}x,$$

and thus

$$\int \mathrm{e}^x \sin x \,\mathrm{d}x = \frac{1}{2}\,\mathrm{e}^x(\sin x - \cos x) \quad \text{(in } \mathbb{R}\text{)}.$$

4) Let us derive a recurrent formula for calculation of the integral

$$I_n(x) = \int \frac{\mathrm{d}x}{(1+x^2)^n},$$

where $n \in \mathbb{N}$.

- $I_1(x) = \int \frac{\mathrm{d}x}{1+x^2} = \arctan x$,
- $I_n(x) = \int 1 \cdot \frac{\mathrm{d}x}{(1+x^2)^n} = \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2+1-1}{(1+x^2)^{n+1}} = \frac{x}{(1+x^2)^n} + 2nI_n(x) - 2nI_{n+1}(x)$.

  Hence it is easily seen that

$$I_{n+1}(x) = \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n} I_n(x) \quad \text{(in } \mathbb{R}\text{)}.$$

24.5 Theorem (First Substitution Rule). *Let*

- *a function φ have a finite derivative in an interval $(a,\, b)$ and $\varphi(x) \in (\alpha,\, \beta)$ for all $x \in (a,\, b)$,*
- *a function f be continuous on an interval $(\alpha,\, \beta)$.*

Assume F is an arbitrary antiderivative of f in $(\alpha,\, \beta)$. Then

$$\int f(\varphi(x))\,\varphi'(x)\,\mathrm{d}x = F(\varphi(x)) \quad \textit{in } (a,\, b).$$

PROOF. The statement follows directly from Theorem 13.11 since

$$(F(\varphi(x)))' = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x) \quad \text{(in } (a,\, b)\text{)}.$$

□

24.6 Remark. When we use the first substitution rule, we write it formally in this way:

$$\int\limits_{\substack{\varphi(x)=t\\ \varphi'(x)\,\mathrm{d}x=\,\mathrm{d}t}} f(\varphi(x))\,\varphi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t = F(t) = F(\varphi(x)).$$

24.7 Examples.

1)

$$\int \cot x\,\mathrm{d}x = \int\limits_{\substack{\sin x=t\\ \cos x\,\mathrm{d}x=\,\mathrm{d}t}} \frac{\cos x}{\sin x}\,\mathrm{d}x = \int \frac{1}{t}\,\mathrm{d}t = \log|t| = \log|\sin x|$$

(in every $(k\pi,\, \pi+k\pi)$, where $k\in\mathbb{Z}$).

2)

$$\int \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x = \int\limits_{\substack{x^2-1=t\\ 2x\,\mathrm{d}x=\,\mathrm{d}t}} \frac{2x}{2\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{\mathrm{d}t}{2\sqrt{t}} = \sqrt{t} = \sqrt{x^2-1}$$

(in $(-\infty,\,-1)$ and in $(1,\,+\infty)$).

3)

$$\int\limits_{\substack{3x+2007=t\\ 3\,\mathrm{d}x=\,\mathrm{d}t\\ \mathrm{d}x=\frac{1}{3}\,\mathrm{d}t}} \frac{\mathrm{d}x}{3x+2007} = \int \frac{1}{3}\frac{\mathrm{d}t}{t} = \frac{1}{3}\log|t| = \frac{1}{3}\log|3x+2007|$$

$\Big($in $\big(-\infty,\, -\frac{2007}{3}\big)$ and in $\big(-\frac{2007}{3},\, +\infty\big)\Big)$.

24.8 Theorem (Second Substitution Rule). *Let*

- *a function φ map an interval $(\alpha,\,\beta)$ onto an interval $(a,\,b)$ and let φ have a continuous and nonzero derivative in $(\alpha,\,\beta)$,*
- *a function f be continuous on $(a,\,b)$.*

Assume F is an arbitrary antiderivative of $(f\circ\varphi)\,\varphi'$ in $(\alpha,\,\beta)$. Then

$$\int f(x)\,\mathrm{d}x = F(\varphi^{-1}(x)) \quad \textit{in } (a,\,b).$$

PROOF. First, let us note that $F(\varphi^{-1}(x))$ is well defined (our assumptions imply that the function $\varphi:\ (\alpha,\,\beta) \overset{\text{onto}}{\mapsto} (a,\,b)$ is strictly monotonous, and thus injective; hence we see that the inverse of φ exists and $\varphi^{-1}:\ (a,\,b) \overset{\text{onto}}{\mapsto} (\alpha,\,\beta)$). By Theorems 13.11 and 13.14, we get

$$(F(\varphi^{-1}(x)))' = F'(\varphi^{-1}(x))(\varphi^{-1}(x))' = f(\varphi(\varphi^{-1}(x)))\varphi'(\varphi^{-1}(x))\frac{1}{\varphi'(\varphi^{-1}(x))} = f(x)$$

for every $x\in(a,\,b)$.

□

24.9 Remark. When we use the second substitution rule, we write it formally in this way:

$$\int\limits_{\substack{x=\varphi(t)\\ \mathrm{d}x=\varphi'(t)\,\mathrm{d}t}} f(x)\,\mathrm{d}x = \int f(\varphi(t))\,\varphi'(t)\,\mathrm{d}t = F(t) = F(\varphi^{-1}(x)).$$

24.10 Examples. Let $a \in \mathbb{R}^+$.

1)

$$\int\limits_{\substack{x=at\\ \mathrm{d}x=a\,\mathrm{d}t}} \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{a\,\mathrm{d}t}{\sqrt{a^2-a^2t^2}} = \int \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arcsin t = \arcsin\frac{x}{a} \quad (\text{in } (-a,\,a)).$$

2)

$$\int\limits_{\substack{x=a\sin t\\ \mathrm{d}x=a\cos t\,\mathrm{d}t\\ x\in(-a,\,a),\ t\in(-\pi/2,\,\pi/2)}} \sqrt{a^2-x^2}\,\mathrm{d}x = \int\limits_{t=\arcsin\frac{x}{a}} \sqrt{a^2-a^2\sin^2 t}\cdot a\cos t\,\mathrm{d}t = a^2\int |\cos t|\cos t\,\mathrm{d}t =$$

$$= a^2\int\cos^2 t\,\mathrm{d}t = a^2\int\frac{1+\cos(2t)}{2}\,\mathrm{d}t = a^2\int\frac{1}{2}\,\mathrm{d}t + a^2\int\frac{\cos(2t)}{2}\,\mathrm{d}t =$$

$$= \frac{a^2}{2}t + a^2\frac{\sin(2t)}{4} = \frac{a^2}{2}t + \frac{a^2}{2}\sin t\sqrt{1-\sin^2 t} =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\sin(\arcsin\frac{x}{a})\sqrt{1-\sin^2(\arcsin\frac{x}{a})} =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} \quad (\text{in } (-a,\,a)).$$

24.11 Exercise. Try to calculate the second example using the integration by parts.

25 INTEGRATION OF RATIONAL FUNCTIONS

25.1 Partial Fractions Decomposition of Rational Functions

25.1.1 Let us recall that every function q given by

$$q(x) := a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_0,\, a_1,\, \ldots,\, a_n \in \mathbb{R}$, $a_n \neq 0$ and $n \in \mathbb{N}$, is called a polynomial of the degree n with real coefficients. Such a polynomial can be also written in the form

$$q(x) = a_n(x-\alpha_1)^{n_1}\cdots(x-\alpha_k)^{n_k}\,(x^2+\beta_1 x+\gamma_1)^{m_1}\cdots(x^2+\beta_l x+\gamma_l)^{m_l},$$

where

- α_i are mutually distinct real numbers,
- $\beta_j,\, \gamma_j \in \mathbb{R}$,
- polynomials $(x^2 + \beta_j x + \gamma_j)$ have mutually distinct non-real roots,
- $n_i,\, m_j \in \mathbb{N} \cup \{0\}$.

25.1.2 Theorem (Partial Fractions Decomposition of a Rational Function). *Let p and q be polynomials with real coefficients such that the degree of p is less than the degree of q. Let us write q in the form*

$$q(x) = a_n(x-\alpha_1)^{n_1} \cdots (x-\alpha_k)^{n_k}\,(x^2+\beta_1 x+\gamma_1)^{m_1} \cdots (x^2+\beta_l x+\gamma_l)^{m_l}.$$

Then there are real numbers $a_{ij},\, b_{rs},\, c_{rs}$ such that

$$\frac{p(x)}{q(x)} = \frac{a_{11}}{x-\alpha_1} + \frac{a_{12}}{(x-\alpha_1)^2} + \ldots + \frac{a_{1n_1}}{(x-\alpha_1)^{n_1}} + \ldots + \frac{a_{k1}}{x-\alpha_k} + \frac{a_{k2}}{(x-\alpha_k)^2} + \ldots + \frac{a_{kn_k}}{(x-\alpha_k)^{n_k}} +$$

$$+ \frac{b_{11}x + c_{11}}{x^2+\beta_1 x+\gamma_1} + \frac{b_{12}x + c_{12}}{(x^2+\beta_1 x+\gamma_1)^2} + \ldots + \frac{b_{1m_1}x + c_{1m_1}}{(x^2+\beta_1 x+\gamma_1)^{m_1}} + \ldots +$$

$$+ \frac{b_{l1}x + c_{l1}}{x^2+\beta_l x+\gamma_l} + \frac{b_{l2}x + c_{l2}}{(x^2+\beta_l x+\gamma_l)^2} + \ldots + \frac{b_{lm_l}x + c_{lm_l}}{(x^2+\beta_l x+\gamma_l)^{m_l}}\,.$$

25.1.3 Examples.

- $\frac{x+2}{3x^2+3x-18} = \frac{x+2}{3(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3}$,
- $\frac{3x^2-2x+12}{(x+4)^2(x-1)^3} = \frac{a}{x+4} + \frac{b}{(x+4)^2} + \frac{c}{x-1} + \frac{d}{(x-1)^2} + \frac{e}{(x-1)^3}$,
- $\frac{x^3+2x-13}{x^2(x^2+2x+3)} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx+d}{x^2+2x+3}$,
- $\frac{x^3+2x-13}{x^2(x^2+2x+1)} = \frac{x^3+2x-13}{x^2(x+1)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$,
- $\frac{x}{(x-1)(2x^2+x+3)^2} = \frac{a}{x-1} + \frac{bx+c}{2x^2+x+3} + \frac{dx+e}{(2x^2+x+3)^2}$.

25.1.4 Definitions. Functions of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials, are said to be rational functions. Functions of the form $\frac{a}{(x-\alpha)^n}$ and $\frac{bx+c}{(x^2+\beta x+\gamma)^m}$ are called partial fractions ($a,\, b,\, c,\, \alpha,\, \beta,\, \gamma \in \mathbb{R}$; $n,\, m \in \mathbb{N}$; $x^2 + \beta x + \gamma$ has no real root).

25.1.5 Example. Decompose the rational function $\frac{x^2+x+1}{x^4-1}$ into the partial fractions.

SOLUTION.

$$\frac{x^2+x+1}{x^4-1} = \frac{x^2+x+1}{(x+1)(x-1)(x^2+1)} = \frac{a}{x+1} + \frac{b}{x-1} + \frac{cx+d}{x^2+1}$$

for some $a,\ b,\ c,\ d \in \mathbb{R}$. After multiplying this equation by the expression $x^4 - 1$, we obtain

$$x^2 + x + 1 = a(x-1)(x^2+1) + b(x+1)(x^2+1) + (cx+d)(x^2-1).$$

By comparison of the coefficients of the powers of x we get the following system of linear equations

$$\begin{aligned} x^3&: \quad 0 = a + b + c,\\ x^2&: \quad 1 = -a + b + d,\\ x^1&: \quad 1 = a + b - c,\\ x^0&: \quad 1 = -a + b - d. \end{aligned}$$

By solving this system we (uniquely) obtain the desired numbers $a,\ b,\ c,\ d$. The result is

$$\frac{x^2+x+1}{x^4-1} = -\frac{1}{4(x+1)} + \frac{3}{4(x-1)} - \frac{x}{2(x^2+1)}.$$

■

25.1.6 Observation. Solving the above-mentioned system of equations can be accelerated if we insert into

$$x^2 + x + 1 = a(x-1)(x^2+1) + b(x+1)(x^2+1) + (cx+d)(x^2-1)$$

the real roots of the denominator $x^4 - 1$:

$$x = 1: \quad 3 = 4b \ \Rightarrow\ b = \frac{3}{4},$$

$$x = -1: \quad 1 = -4a \ \Rightarrow\ a = -\frac{1}{4}.$$

25.1.7 Remark. If the degree of a polynomial p is not less than the degree of a non-constant polynomial q in $\frac{p(x)}{q(x)}$, we perform the polynomial division $p(x) : q(x)$ with remainder, so that we obtain

$$\frac{p(x)}{q(x)} = u(x) + \frac{v(x)}{q(x)},$$

where u and v are polynomials, and the degree of v is less than the degree of q.

25.1.8 Example. Decompose the rational function $\frac{x^4}{x^4-x^3-x+1}$.

SOLUTION. $x^4 : (x^4 - x^3 - x + 1) = 1 + \frac{x^3+x-1}{x^4-x^3-x+1}$, and therefore there are $a, b, c, d \in \mathbb{R}$

$$\begin{array}{c} -(x^4 - x^3 - x + 1) \\ \hline x^3 + x - 1 \end{array}$$

such that

$$1 + \frac{x^3+x-1}{x^4-x^3-x+1} = 1 + \frac{x^3+x-1}{(x-1)^2(x^2+x+1)} = 1 + \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+x+1}.$$

Hence it follows

$$x^3 + x - 1 = a(x-1)(x^2+x+1) + b(x^2+x+1) + (cx+d)(x-1)^2.$$

By comparison of the coefficients of the powers of x we get the system

$$\begin{aligned} x^3\colon &\quad 1 = a + c, \\ x^2\colon &\quad 0 = b - 2c + d, \\ x^1\colon &\quad 1 = b + c - 2d, \\ x^0\colon &\quad -1 = -a + b + d, \end{aligned}$$

whose solution is $a = 1,\ b = \frac{1}{3},\ c = 0,\ d = -\frac{1}{3}$, and therefore

$$\frac{x^4}{x^4 - x^3 - x + 1} = 1 + \frac{1}{x-1} + \frac{1}{3(x-1)^2} - \frac{1}{3(x^2+x+1)}.$$

■

25.2 Integration of Partial Fractions

25.2.1 Integrals of the type

$$\int \frac{a}{(x-\alpha)^n}\,\mathrm{d}x$$

can be calculated using the substitution $x - \alpha = t$.

25.2.2 Examples.

1)

$$\int \frac{\mathrm{d}x}{x-6} \underset{\substack{x-6=t \\ \mathrm{d}x=\mathrm{d}t}}{=} \int \frac{\mathrm{d}t}{t} = \log|t| = \log|x-6| \quad \text{(in } (-\infty,\, 6) \text{ and in } (6,\, \infty)\text{)}.$$

2)

$$\int \frac{\mathrm{d}x}{(x-6)^3} = \int \frac{\mathrm{d}t}{t^3} = -\frac{1}{2t^2} = -\frac{1}{2(x-6)^2} \quad \text{(in } (-\infty,\, 6) \text{ and in } (6,\, \infty)\text{)}.$$

25.2.3 In order to calculate integrals of the type

$$\int \frac{bx + c}{(x^2 + \beta x + \gamma)^m}\,\mathrm{d}x,$$

we first use the decomposition into addition:

$$\int \frac{bx+c}{(x^2+\beta x+\gamma)^m}\,\mathrm{d}x = \frac{b}{2}\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^m}\,\mathrm{d}x + \left(c - \frac{b\beta}{2}\right)\int \frac{\mathrm{d}x}{(x^2+\beta x+\gamma)^m}.$$

Then

- the first integral can be calculated using the substitution $x^2 + \beta x + \gamma = t$ (since then $(2x+\beta)\,\mathrm{d}x = \mathrm{d}t$),
- by modifying the denominator in order to "complete the square" and using the appropriate (linear) substitution, we transform the second integral into $\int \frac{\mathrm{d}t}{(1+t^2)^m}$, for which we derived (by the integration by parts) the recurrent formula.

25.2.4 Example.

$$\int \frac{6x-1}{x^2+2x+3}\,\mathrm{d}x = 3\int \frac{2x+2}{x^2+2x+3}\,\mathrm{d}x - 7\int \frac{\mathrm{d}x}{x^2+2x+3} =$$

$$\begin{array}{l} x^2+2x+3=t \\ (2x+2)\,\mathrm{d}x = \mathrm{d}t \end{array}$$

$$= 3\int \frac{\mathrm{d}t}{t} - 7\int \frac{\mathrm{d}x}{(x+1)^2+2} = 3\log|t| - \frac{7}{2}\int \frac{\mathrm{d}x}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2} =$$

$$\begin{array}{l} \frac{x+1}{\sqrt{2}} = u \\ \mathrm{d}x = \sqrt{2}\,\mathrm{d}u \end{array}$$

$$= 3\log(x^2+2x+3) - \frac{7}{2}\sqrt{2}\int \frac{\mathrm{d}u}{1+u^2} = 3\log(x^2+2x+3) - \frac{7}{\sqrt{2}}\arctan u =$$

$$= 3\log(x^2+2x+3) - \frac{7}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) \quad (\text{in } \mathbb{R}).$$

25.2.5 Exercise. Calculate

$$\int \frac{6x-1}{(x^2+2x+3)^2}\,\mathrm{d}x.$$

Let us disclose the solution: $-\frac{3}{x^2+2x+3} - \frac{7}{4}\frac{x+1}{x^2+2x+3} - \frac{7}{4\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right)$.

25.2.6 Example. Compare the length of the following two calculations:

- $$\int \frac{x^3}{x^4+1}\,\mathrm{d}x = \int \frac{x^3}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}\,\mathrm{d}x =$$

$$= \int \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}\,\mathrm{d}x = \int \frac{2x+\sqrt{2}}{4(x^2+\sqrt{2}x+1)} + \frac{2x-\sqrt{2}}{4(x^2-\sqrt{2}x+1)}\,\mathrm{d}x =$$

$$= \frac{1}{4}\log(x^2+\sqrt{2}x+1) + \frac{1}{4}\log(x^2-\sqrt{2}x+1) = \frac{1}{4}\log(x^4+1),$$

- $$\int \frac{x^3}{x^4+1}\,\mathrm{d}x = \frac{1}{4}\int \frac{\mathrm{d}t}{t} = \frac{1}{4}\log|t| = \frac{1}{4}\log(x^4+1).$$

$$\begin{array}{l} x^4+1=t \\ 4x^3\,\mathrm{d}x = \mathrm{d}t \end{array}$$

25.2.7 Exercise. Calculate

$$\int \frac{2x}{3x^4+4}\,\mathrm{d}x.$$

26 INTEGRATION OF SOME OTHER SPECIAL FUNCTIONS

26.1 Integrals of the Type $\int \sin^n x\, \cos^m x\, \mathrm{d}x$

26.1.1 Let $n,\, m \in \mathbb{N} \cup \{0\}$. We distinguish two cases. First, suppose n or m is odd. Then we use the first substitution rule; see the following example.

26.1.2 Example.

$$\int \sin^3 x\, \cos^2 x\, \mathrm{d}x = \int \sin x\,(1 - \cos^2 x)\cos^2 x\, \mathrm{d}x = \underset{\substack{\cos x = t \\ -\sin x\, \mathrm{d}x = \mathrm{d}t}}{}$$

$$= -\int (1-t^2)\, t^2\, \mathrm{d}t = -\frac{t^3}{3} + \frac{t^5}{5} = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} \quad \text{(in } \mathbb{R}\text{)}.$$

26.1.3 If both n and m are even, we can employ the equalities

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \cos^2 x = \frac{1 + \cos(2x)}{2};$$

see the following example.

26.1.4 Example.

$$\int \sin^2 x\, \cos^2 x\, \mathrm{d}x = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}\, \mathrm{d}x = \frac{1}{4}\int 1 - \cos^2(2x)\, \mathrm{d}x =$$

$$= \frac{1}{4}\int 1\, \mathrm{d}x - \frac{1}{4}\int \frac{1+\cos(4x)}{2}\, \mathrm{d}x = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\frac{\sin(4x)}{4} = \frac{x}{8} - \frac{\sin(4x)}{32} \quad \text{(in } \mathbb{R}\text{)}.$$

26.1 By $R(u,v)$ we mean a fraction in whose numerator and denominator there are only finite sums of expressions of the type $k\, u^n v^m$, where $k, u, v \in \mathbb{R};\ n, m \in \mathbb{N} \cup \{0\}$. A mapping $(u,v) \mapsto R(u,v)$ is called a <u>rational function of two variables</u>. Examples of such mappings:

$$R(u,v) = \frac{3u^2v^0 + 2uv^0 + 1u^0v^0}{1u^0v^0} = 3u^2 + 2u + 1,$$

$$R(u,v) = \frac{1u^3v^2 + 3uv^3 + 2u^0v^0}{1\, u^0v^0} = u^3v^2 + 3uv^3 + 2,$$

$$R(u,v) = \frac{2u^0v^2 + 1u^0v^0}{1u^2v^0 + 1u^0v^3} = \frac{2v^2+1}{u^2+v^3}.$$

26.2 Integrals of the Type $\int R(\sin x, \cos x)\,\mathrm{d}x$

26.2.1 Using the substitution $\tan\frac{x}{2} = t$ we can transform these integrals into those of rational functions. To show this, let us express $\sin x$ and $\cos x$ by t. Since

$$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \;\Rightarrow\; \tan^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}} - 1 \;\Rightarrow\; \cos^2\frac{x}{2} = \frac{1}{1+\tan^2\frac{x}{2}} = \frac{1}{1+t^2},$$

we get

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = \frac{2t}{1+t^2}, \\ \cos x &= 2\cos^2\frac{x}{2} - 1 = 2\frac{1}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}, \\ \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}}\,\mathrm{d}x &= \mathrm{d}t, \end{aligned}$$

so that

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,\mathrm{d}t.$$

Note that the integrals of the type $\int \sin^n x \cos^m x\,\mathrm{d}x$ are also of the type $\int R(\sin x, \cos x)\,\mathrm{d}x$ – it is sufficient to set $R(u,v) := u^n v^m$. However, the method of calculation of these integrals examined in Subsection 26.1 takes often less labour than application of the substitution $\tan\frac{x}{2} = t$.

26.2.2 Example.

$$\int \frac{1-\sin x}{1+\cos x}\,\mathrm{d}x = \int \frac{1-\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{t^2-2t+1}{t^2+1}\,\mathrm{d}t = \int 1 - \frac{2t}{t^2+1}\,\mathrm{d}t =$$

$$= t - \log(t^2+1) = \tan\frac{x}{2} - \log\left(\tan^2\frac{x}{2} + 1\right) \quad \text{(in, for example, } (-\pi,\,\pi)\text{)}.$$

26.2.3 Example. Sometimes, though, a less laborious way of solving can be invented. Compare:

- $$\int \frac{\sin^3 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{\left(\frac{2t}{1+t^2}\right)^3}{\left(\frac{1-t^2}{1+t^2}\right)^2}\frac{2}{1+t^2}\,\mathrm{d}t = \int \frac{16t^3}{(1+t^2)^2(t-1)^2(t+1)^2}\,\mathrm{d}t = \dots,$$

- $$\int \frac{\sin^3 x}{\cos^2 x}\,\mathrm{d}x = \underset{\substack{\cos x = t \\ -\sin x\,\mathrm{d}x = \mathrm{d}t}}{\int} \frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,\mathrm{d}x = -\int \frac{1-t^2}{t^2}\,\mathrm{d}t = \frac{1}{t} + t = \frac{1}{\cos x} + \cos x.$$

26.3 Integrals of the Type $\int R\left(x, \sqrt[s]{\frac{ax+b}{cx+d}}\right)\,\mathrm{d}x$

26.3.1 These integrals, where $a, b, c, d \in \mathbb{R},\ s \in \mathbb{N} \setminus \{1\},\ ad \neq bc$, can be calculated using the substitution $\sqrt[s]{\frac{ax+b}{cx+d}} = t$.

26.3.2 Exercise. Why do we assume $ad \neq bc$?

26.3.3 Example.

$$\int \frac{\sqrt{2x+3}+x}{\sqrt{2x+3}-x}\,\mathrm{d}x = \int \frac{t+\frac{t^2-3}{2}}{t-\frac{t^2-3}{2}}\,t\,\mathrm{d}t = \int \frac{t^2+2t-3}{-t^2+2t+3}\,t\,\mathrm{d}t =$$

$$\sqrt{2x+3} = t,\quad x = \frac{t^2-3}{2},\quad \mathrm{d}x = t\,\mathrm{d}t$$

$$= \int -t-4+\frac{8t+12}{-(t+1)(t-3)}\,\mathrm{d}t = -\frac{t^2}{2}-4t+\int -\frac{9}{t-3}+\frac{1}{t+1}\,\mathrm{d}t =$$

$$= -\frac{t^2}{2}-4t-9\log|t-3|+\log|t+1| =$$

$$= -\frac{2x+3}{2}-4\sqrt{2x+3}-9\log|\sqrt{2x+3}-3|+\log|\sqrt{2x+3}+1|.$$

26.4 Integrals of the Type $\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x$

26.4.1 Let us focus only on the situation when $a, b, c \in \mathbb{R},\ a \neq 0$, and ax^2+bx+c has two distinct (generally complex) roots α_1, α_2. If $a > 0$, we choose the so-called Euler substitution

$$\sqrt{ax^2+bx+c} = \sqrt{a}\,x + t$$

which is suitable on every open interval that is a part of the domain of the integrated function.

26.4.2 Example.

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2+2x-1}} = \int \frac{1}{\frac{t^2+1}{2(1-t)}\left(\frac{t^2+1}{2(1-t)}+t\right)}\,\frac{4t-2t^2+2}{4(1-t)^2}\,\mathrm{d}t =$$

$$\sqrt{x^2+2x-1} = \sqrt{1}\,x+t,\quad x = \frac{t^2+1}{2(1-t)},\quad \mathrm{d}x = \frac{4t-2t^2+2}{4(1-t)^2}\,\mathrm{d}t$$

$$= \int \frac{2}{t^2+1}\,\mathrm{d}t = 2\arctan t = 2\arctan(\sqrt{x^2+2x-1}-x)$$

$$\left(\text{in } \left(-\infty, -1-\sqrt{2}\right) \text{ and in } \left(-1+\sqrt{2}, \infty\right)\right).$$

26.4.3 If $a < 0$, it is reasonable to consider only the case when $\alpha_1, \alpha_2 \in \mathbb{R}$ (otherwise $ax^2 + bx + c < 0$ in $\mathbb{R}$). If $\alpha_1 < \alpha_2$, then for every $x \in (\alpha_1, \alpha_2)$ (and another one we cannot be concerned with)

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha_1)(x-\alpha_2)} = \sqrt{(-a)(x-\alpha_1)(\alpha_2-x)} =$$

$$\sqrt{(-a)}\,(x-\alpha_1)\sqrt{\frac{\alpha_2-x}{x-\alpha_1}},$$

and therefore we can rewrite the calculated integral as

$$\int R\left(x, \sqrt{-a}\,(x-\alpha_1)\sqrt{\frac{\alpha_2-x}{x-\alpha_1}}\right)\,\mathrm{d}x.$$

We are already familiar with integrals of this type – we choose the substitution

$$\sqrt{\frac{\alpha_2-x}{x-\alpha_1}} = t.$$

26.4.4 Exercise. Using the above-mentioned method, show that

$$\int \frac{\mathrm{d}x}{2+\sqrt{3-2x-x^2}} = 2\arctan\sqrt{\frac{x+3}{1-x}} + \frac{2\sqrt{1-x}}{\sqrt{x+3}+\sqrt{1-x}} \quad \text{(in } (-3,\,1)).$$

K RIEMANN'S (DEFINITE) INTEGRAL

27 MOTIVATION AND INSPIRATION

27.1 Let a function f be non-negative and continuous in an interval $[a, b]$. Let us try to find a solution of the problem:

How to calculate an area of the region $\{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge 0 \le y \le f(x)\}$*?*

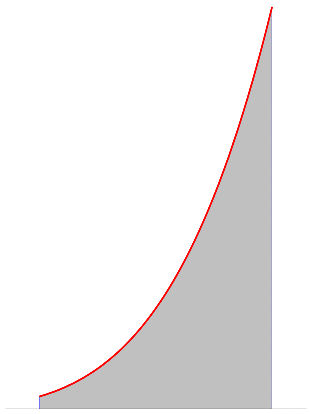

Fig. 49

We can use the following idea: let us divide the interval $[a, b]$ into n subintervals of the same length. In each of them we approximate the function f by a constant function whose value equals to the value of f at, for example, left end point. Then we approximate the desired area by the sum of areas of the relevant rectangles (see the figures below; the left one corresponds to $n = 4$, and the right one to $n = 16$).

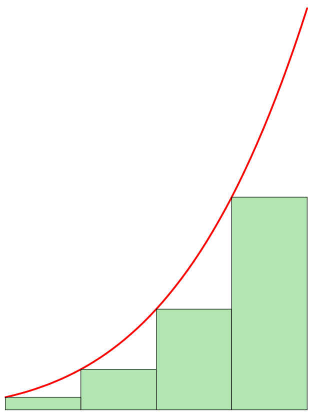

Fig. 50

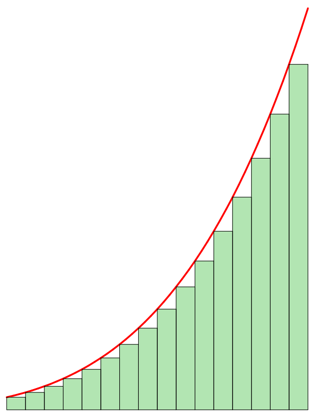

Fig. 51

It can be guessed that if the number of subintervals n increases to infinity, then the sum of areas of the rectangles tends to the area of the region. We shall denote this area by $\int_a^b f(x)\,\mathrm{d}x$.

28 DEFINITION OF A DEFINITE INTEGRAL

28.1 Now let us generalize the ideas from the preceding section also for some non-continuous (and not necessarily non-negative) functions. Let a function f be bounded in an interval $[a,\, b]$ ($a, b \in \mathbb{R}$; $a < b$), i.e., there are numbers $m, M \in \mathbb{R}$ such that

$$\forall x \in [a,\, b] :\; m \le f(x) \le M.$$

For every partition $D :\; a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a,\, b]$, we define a lower and upper sum corresponding to the partition D as

$$s(D) := \sum_{k=1}^{n} \left(\inf_{x \in [x_{k-1},\, x_k]} f(x) \right) (x_k - x_{k-1})$$

and

$$S(D) := \sum_{k=1}^{n} \left(\sup_{x \in [x_{k-1},\, x_k]} f(x) \right) (x_k - x_{k-1}),$$

respectively.

28.2 Definitions. By a lower and upper Riemann's integral of a function f from a to b we mean the numbers

$$\underline{\int_a^b} f(x)\,\mathrm{d}x := \sup\{s(D) :\; D \text{ is a partition of } [a,\, b]\}$$

and

$$\overline{\int_a^b} f(x)\,\mathrm{d}x := \inf\{S(D) :\; D \text{ is a partition of } [a,\, b]\},$$

respectively. If

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x,$$

we call this number the Riemann integral of the function f from a to b and we denote it by

$$\int_a^b f(x)\,\mathrm{d}x.$$

28.3 Observation. Note that numbers $\underline{\int_a^b} f(x)\,\mathrm{d}x$ and $\overline{\int_a^b} f(x)\,\mathrm{d}x$ are defined for every function f that is bounded in $[a,\, b]$. Moreover, the inequalities

$$m(b-a) = \sum_{k=1}^{n} m(x_k - x_{k-1}) \le s(D) \le S(D) \le \sum_{k=1}^{n} M(x_k - x_{k-1}) = M(b-a)$$

imply that

$$\underline{\int_a^b} f(x)\,\mathrm{d}x,\; \overline{\int_a^b} f(x)\,\mathrm{d}x \in \mathbb{R}.$$

28.4 Examples.

1) If a function f is defined by $f(x) := 1$ in $[a,\, b]$, where $a,\, b \in \mathbb{R},\ a < b$, we get

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup_D \left\{ \sum_k 1 \cdot (x_k - x_{k-1}) \right\} = \sup_D \{b-a\} = b-a.$$

Similarly,

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf_D \left\{ \sum_k 1 \cdot (x_k - x_{k-1}) \right\} = \inf_D \{b-a\} = b-a,$$

and therefore

$$\int_a^b f(x)\,\mathrm{d}x = b-a.$$

2) If χ is the Dirichlet function, i.e.,

$$\chi(x) := \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

then for all $a,\, b \in \mathbb{R};\ a < b$

$$\underline{\int_a^b} \chi(x)\,\mathrm{d}x = 0, \qquad \overline{\int_a^b} \chi(x)\,\mathrm{d}x = b-a,$$

and thus

$$\int_a^b \chi(x)\,\mathrm{d}x$$

fails to exist.

28.5 Remark. Let us consider a "sequence" of partitions of an interval $[a,\, b]$

$$D_1,\ D_2,\ \ldots,\ D_m,\ \ldots\ ,$$

where

$$D_m:\ a = x_{m,0} < x_{m,1} < \ldots < x_{m,n_m-1} < x_{m,n_m} = b,$$

such that

$$||D_m|| := \max_{k \in \{1,\ldots,n_m\}} (x_{m,k} - x_{m,k-1}) \ \to\ 0.$$

Then the following statements hold:

- if the function f is bounded in the interval $[a,\, b]$, then

$$\lim s(D_m) = \underline{\int_a^b} f(x)\,\mathrm{d}x, \quad \lim S(D_m) = \overline{\int_a^b} f(x)\,\mathrm{d}x;$$

- if $\int_a^b f(x)\,\mathrm{d}x$ exists, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim s(D_m) = \lim S(D_m) = \lim I(D_m),$$

  where

$$I(D_m) := \sum_{k=1}^{n_m} f(\xi_{m,k})(x_{m,k} - x_{m,k-1})$$

  for arbitrarily chosen points $\xi_{m,k} \in [x_{m,k-1},\, x_{m,k}]$.

28.6 Theorem (Existence of the Riemann Integral). *Let a function f be continuous or monotonous in an interval $[a,\, b]$. Then $\int_a^b f(x)\,\mathrm{d}x$ exists.*

28.7 Theorem (Properties of the Riemann Integral). *Assume $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Then*

i)

$$\forall \alpha, \beta \in \mathbb{R} : \int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x,$$

ii)

$$\forall c \in (a,\, b) : \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x,$$

iii)

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x,$$

iv)

$$\textit{if } f(x) \leq g(x) \textit{ for all } x \in [a,\, b], \textit{ then } \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

28.8 Remark. Statements *i) - iv)* of Theorem 28.7 also contain the fact that all integrals appearing there exist!

28.9 Corollary. *If a function f is continuous in an interval $[a,\, b]$, then*

$$m(b-a) = \int_a^b m\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b M\,\mathrm{d}x = M(b-a),$$

where

$$m := \min_{x \in [a,\, b]} f(x) \quad \textit{and} \quad M := \max_{x \in [a,\, b]} f(x).$$

Since a continuous function f attains on $[a,\, b]$ all values lying between numbers m and M (see Corollaries 16.4 i)), there exists a $\xi \in (a,\, b)$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

(This proposition is sometimes called the <u>mean value theorem of the integral calculus</u>.)

28.10 Definitions. In what follows, the following extensions of the definition of the Riemann integral shall be helpful:

- $\int_a^a f(x)\,\mathrm{d}x := 0$ if $f(a)$ is defined,
- $\int_b^a f(x)\,\mathrm{d}x := -\int_a^b f(x)\,\mathrm{d}x$ if $a < b$ and $\int_a^b f(x)\,\mathrm{d}x$ exists.

29 INTEGRAL WITH VARIABLE UPPER BOUND

29.1 Theorem. *Assume f is a continuous function in an interval $[a,\, b]$ and*

$$F(x) := \int_a^x f(t)\,\mathrm{d}t.$$

Then

i) $F'(x) = f(x)$ *for every* $x \in (a,\, b)$,

ii) $F'_+(a) = f(a)$,

iii) $F'_-(b) = f(b)$.

PROOF. The task is to show
a) $x_0 \in [a,\, b) \Rightarrow F'_+(x_0) = f(x_0)$,
b) $x_0 \in (a,\, b] \Rightarrow F'_-(x_0) = f(x_0)$.
We prove only the statement a) since the proof of b) is absolutely analogous. It is left to the reader. First, let us note that for $x_0 \in [a,\, b)$ and $x_0 < x < b$

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{\int_a^x f(t)\,\mathrm{d}t - \int_a^{x_0} f(t)\,\mathrm{d}t}{x-x_0} - f(x_0)\right| =$$

$$= \frac{1}{x-x_0}\left|\int_{x_0}^x f(t)\,\mathrm{d}t - f(x_0)(x-x_0)\right| = \frac{1}{x-x_0}\left|\int_{x_0}^x f(t)\,\mathrm{d}t - \int_{x_0}^x f(x_0)\,\mathrm{d}t\right| =$$

$$= \frac{1}{x-x_0}\left|\int_{x_0}^x (f(t)-f(x_0))\,\mathrm{d}t\right| \le \frac{1}{x-x_0}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t \le$$

$$\le \frac{1}{x-x_0}\int_{x_0}^x \max_{s\in[x_0,\,x]} |f(s)-f(x_0)|\,\mathrm{d}t = \frac{1}{x-x_0}\max_{s\in[x_0,\,x]} |f(s)-f(x_0)|(x-x_0) =$$

$$= \max_{s\in[x_0,\,x]} |f(s)-f(x_0)|.$$

Under the assumptions, the function f is continuous from the right at the point x_0, so that

$$\lim_{x\to x_0+}\max_{s\in[x_0,\,x]} |f(s)-f(x_0)| = 0.$$

Hence

$$\lim_{x\to x_0+}\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = 0,$$

i.e.,

$$F'_+(x_0) = f(x_0).$$

□

29.2 Theorem. *Assume f is a continuous function in an open interval $I \subset \mathbb{R}$ and $c \in I$ is an arbitrary point. Then the function F defined on I by*

$$F(x) := \int_c^x f(t)\,\mathrm{d}t$$

is an antiderivative of f in I (i.e., $F'(x) = f(x)$ for every $x \in I = DF$).

30 METHODS OF CALCULATING DEFINITE INTEGRALS

30.1 Theorem. *Let functions f and F be continuous in an interval $[a, b]$, where $-\infty < a < b < +\infty$, and $F'(x) = f(x)$ for every $x \in (a, b)$. Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) =: [F(x)]_a^b.$$

PROOF. Redefine (or change) the function f on $\mathbb{R} \setminus [a, b]$ so that it is continuous in $\mathbb{R}$ (this is certainly possible and the calculated integral remains unchanged). By Theorems 23.4 and 29.2, it follows that for every $x \in [a, b]$ we have $F(x) = \int_a^x f(t)\,\mathrm{d}t + c$. The rest of the proof is now easy:

$$F(b) - F(a) = \left(\int_a^b f(t)\,\mathrm{d}t + c\right) - \left(\int_a^a f(t)\,\mathrm{d}t + c\right) = \int_a^b f(t)\,\mathrm{d}t.$$

□

30.2 Examples.

1)
$$\int_0^1 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

2)
$$\int_0^\pi \sin x\,\mathrm{d}x = [-\cos x]_0^\pi = -[\cos x]_0^\pi = -(-1-1) = 2.$$

3)
$$\int_0^{\frac{\pi}{2}} \sin(2+x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} (\sin 2\cos x + \cos 2\sin x)\,\mathrm{d}x =$$
$$= \sin 2 \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x + \cos 2 \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x =$$
$$= \sin 2\ [\sin x]_0^{\frac{\pi}{2}} + \cos 2\ [-\cos x]_0^{\frac{\pi}{2}} = \sin 2 + \cos 2.$$

4)
$$\int_{-2}^3 |x^2-1|\,\mathrm{d}x = \int_{-2}^{-1} |x^2-1|\,\mathrm{d}x + \int_{-1}^1 |x^2-1|\,\mathrm{d}x + \int_1^3 |x^2-1|\,\mathrm{d}x =$$
$$= \int_{-2}^{-1} (x^2-1)\,\mathrm{d}x + \int_{-1}^1 (-x^2+1)\,\mathrm{d}x + \int_1^3 (x^2-1)\,\mathrm{d}x =$$
$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[-\frac{x^3}{3} + x\right]_{-1}^1 + \left[\frac{x^3}{3} - x\right]_1^3 = \frac{28}{3}.$$

30.3 Theorem (Integration by Parts). *Suppose u and v have their first derivatives continuous in an interval $[a,\, b]$. Then*

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

30.4 Examples.

1)

$$\int_0^\pi \underset{\substack{u = x, \quad v' = \sin x \\ u' = 1, \quad v = -\cos x}}{x \sin x}\,\mathrm{d}x = [-x\cos x]_0^\pi + \int_0^\pi \cos x\,\mathrm{d}x = \pi + [\sin x]_0^\pi = \pi.$$

2)

$$\int_1^{\mathrm{e}} \log x\,\mathrm{d}x = \int_1^{\mathrm{e}} \underset{\substack{u = \log x, \quad v' = 1 \\ u' = \frac{1}{x}, \quad v = x}}{1 \cdot \log x}\,\mathrm{d}x = [x \log x]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x\,\frac{1}{x}\,\mathrm{d}x = \mathrm{e} - [x]_1^{\mathrm{e}} = 1.$$

30.5 Theorem (Substitution Rule). *Let*

- *a function φ have its first derivative continuous in an interval $[a,\, b]$,*
- *a function φ map an interval $[a,\, b]$ into an interval $J \subset \mathbb{R}$,*
- *a function f be continuous in an interval J.*

Then

$$\int_a^b f(\varphi(x))\,\varphi'(x)\,\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t.$$

30.6 Remark. We use the proposition of this theorem for calculation of the integral on the left side (analogy to the first substitution rule; see Theorem 24.5) as well as for calculation of the integral on the right side (see Theorem 24.8 concerning the second substitution rule). Calculation mechanism is the same as when dealing with indefinite integrals, we only have to be aware of changing the bounds. Moreover, it is necessary to verify that all assumptions of the substitution theorem hold since – in contrast to indefinite integrals – we have no chance to verify the correctness of the result by differentiating. Finally, let us note that a substitution is chosen (according to the type of the integrated function) the same as when calculating indefinite integrals.

30.7 Examples.

1)

$$\int_0^{\frac{\pi}{2}} \underset{\substack{\sin x = t \\ \cos x\,\mathrm{d}x = \mathrm{d}t}}{\sin^4 x \cos x\,\mathrm{d}x} = \int_0^1 t^4\,\mathrm{d}t = \left[\frac{t^5}{5}\right]_0^1 = \frac{1}{5}$$

$$\left(\varphi(x) = \sin x,\ [a,\, b] = \left[0,\, \frac{\pi}{2}\right],\ J = \mathbb{R},\ f(t) = t^4\right).$$

2)

$$\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{4\,(1-\sin^2 t)}\,2\cos t\,\mathrm{d}t =$$

$x = 2\sin t$
$\mathrm{d}x = 2\cos t\,\mathrm{d}t$

$$= \int_0^{\frac{\pi}{2}} 4\cos^2 t\,\mathrm{d}t = 4\int_0^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2}\,\mathrm{d}t = \pi$$

$$\left(\varphi(t) = 2\sin t,\ [a,\,b] = \left[0,\ \frac{\pi}{2}\right],\ J = [-2,\,2],\ f(x) = \sqrt{4-x^2}\right).$$

Remark:
We could choose, for example, $[a,\,b] = \left[-\pi,\ \frac{\pi}{2}+2\pi\right]$; the choice $[a,\,b] = \left[0,\ \frac{\pi}{2}\right]$ is more suitable, however, since $\sqrt{\cos^2 t} = |\cos t| = \cos t$ for every $t \in \left[0,\ \frac{\pi}{2}\right]$.

31 NUMERICAL CALCULATION OF THE RIEMANN INTEGRAL

31.1 Let us introduce two methods – rectangle and trapezoidal – that are used for approximation of $\int_a^b f(x)\,\mathrm{d}x$.

Let us divide the interval $[a,\,b]$ into n subintervals of the same length $h = \frac{b-a}{n}$, i.e., let us choose a partition

$$D:\ a = x_0 < x_1 = x_0 + h < \ldots < x_n = x_{n-1} + h = b$$

of the interval $[a,\,b]$, and denote

$$y_0 = f(x_0),\ y_1 = f(x_1),\ \ldots,\ y_n = f(x_n).$$

We speak of a <u>rectangle</u> and <u>trapezoidal</u> method when we use the approximations

$$\int_a^b f(x)\,\mathrm{d}x \approx hy_0 + hy_1 + \ldots + hy_{n-1} = \frac{b-a}{n}\,(y_0 + \ldots + y_{n-1}) \quad \ldots \text{ see Fig. 52}$$

and

$$\int_a^b f(x)\,\mathrm{d}x \approx h\,\frac{y_0+y_1}{2} + h\,\frac{y_1+y_2}{2} + \ldots + h\,\frac{y_{n-1}+y_n}{2} =$$

$$= \frac{b-a}{2n}\,(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n) \quad \ldots \text{ see Fig. 53,}$$

respectively.

Now we shall try to calculate an error estimate for the rectangle method on an assumption that f has its first derivative bounded on $[a,\,b]$. We shall employ the mean value theorem of the integral calculus (see Theorem 28.9):

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\frac{b-a}{n}$$

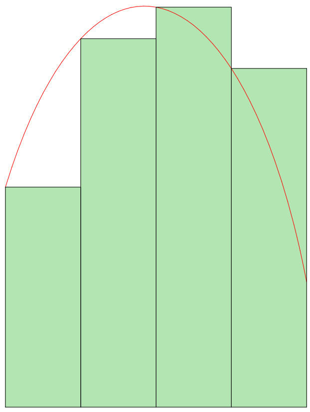

Fig. 52

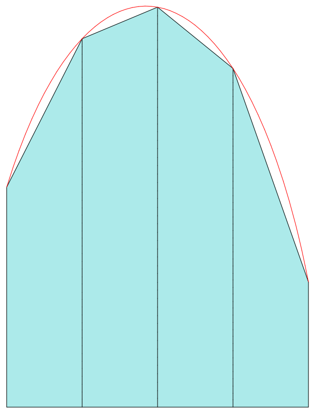

Fig. 53

for suitable $\xi_k \in (x_{k-1}, x_k)$. Hence, by the Lagrange mean value theorem (see Theorem 14.2), we get (for suitable $d_k \in (x_{k-1}, \xi_k)$)

$$\left| \int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{n} \sum_{k=1}^n f(x_{k-1}) \right| = \left| \frac{b-a}{n} \sum_{k=1}^n (f(\xi_k) - f(x_{k-1})) \right| =$$

$$= \frac{b-a}{n} \left| \sum_{k=1}^n f'(d_k)(\xi_k - x_{k-1}) \right| \leq \frac{b-a}{n} \sum_{k=1}^n \left(\sup_{x \in [a,\, b]} |f'(x)| \frac{b-a}{n} \right) =$$

$$= \left(\frac{b-a}{n} \right)^2 \sup_{x \in [a,\, b]} |f'(x)| n = \frac{(b-a)^2}{n} \sup_{x \in [a,\, b]} |f'(x)|.$$

It can be shown that if the second derivative of the function f is bounded on $[a, b]$, then the error is of lower order. For instance, in case of the trapezoidal method the error is estimated by $\frac{(b-a)^3}{12n^2} \sup\limits_{x \in [a,\, b]} |f''(x)|$.

31.2 Example. Approximate the integral

$$I := \int_2^3 \frac{\mathrm{d}x}{x-1}$$

using the trapezoidal method so that the error is at most equal to the number $\frac{1}{1000}$.

SOLUTION. Due to the relations

$$\left| \left(\frac{1}{x-1} \right)'' \right| = \left| \left(-\frac{1}{(x-1)^2} \right)' \right| = \left| 2 \frac{1}{(x-1)^3} \right| \leq \frac{2}{1^3} \quad \text{holding in } [2,\, 3]$$

and the above estimate, we know that the number of subintervals n can be chosen so that

$$\frac{1}{12n^2} 2 \leq \frac{1}{1000}.$$

Hence

$$n \geq \sqrt{\frac{500}{3}} \approx 12.9.$$

If we put $n = 13$, we obtain $I \approx 0.69352$. (The exact value is $I = [\log|x-1|]_2^3 = \log 2 = 0.69314\ldots$.)

■

32 APPLICATIONS OF A DEFINITE INTEGRAL

32.1 Area of a Plane Region

32.1.1 Let us return to the ideas from the beginning of Section 27 and form the problem:

Is it possible to assign a non-negative number $P(f; a, b)$ to every non-negative and continuous function f on an interval $[a,\, b]$ so that

i) $\forall c \in \mathbb{R}^+ :\ [f(x) = c$ *in* $[a,\, b] \ \Rightarrow\ P(f; a, b) = c\,(b - a)]$,

ii) $\forall \xi \in (a,\, b) :\ P(f; a, b) = P(f; a, \xi) + P(f; \xi, b)$,

iii) if g *is a continuous function and* $f \leq g$ *in* $[a,\, b]$*, then* $P(f; a, b) \leq P(g; a, b)$ *?*

32.1.2 Exercise. Prove (it is not too difficult) that

- the answer to the above question is *yes*,
- the number $P(f; a, b)$ is uniquely determined by *i)*, *ii)* and *iii)*, and

$$P(f; a, b) = \int_a^b f(x)\,\mathrm{d}x.$$

32.1.3 Definition. Assume f is a continuous and non-negative in $[a,\, b]$, where $-\infty < a < b < +\infty$. By an area of the surface

$$\left\{(x, y) \in \mathbb{R}^2 :\ x \in [a,\, b]\ \wedge\ 0 \leq y \leq f(x)\right\}$$

we mean the number

$$P(f; a, b) := \int_a^b f(x)\,\mathrm{d}x.$$

32.1.4 Examples.

1) Calculate the area $\mathcal{O}$ of the half-circle

$$\left\{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 \leq 4 \wedge y \geq 0\right\}.$$

SOLUTION. $\mathcal{O} = P(f; -2, 2)$, where $f(x) := \sqrt{4 - x^2}$, and therefore

$$\mathcal{O} = \int_{-2}^{2} \sqrt{4 - x^2}\,\mathrm{d}x = 2\int_0^2 \sqrt{4 - x^2}\,\mathrm{d}x = 2\pi.$$

■

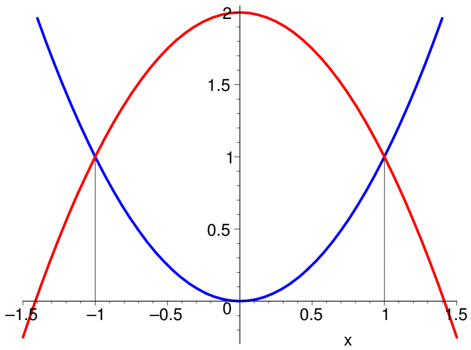

Fig. 54

2) Calculate the area $\mathcal{O}$ of the surface

$$\tau := \left\{(x, y) \in \mathbb{R}^2 : \; x^2 \leq y \leq 2 - x^2\right\}.$$

SOLUTION. From Fig. 54 it is easily seen that

$$\mathcal{O} = P(2 - x^2; -1, 1) - P(x^2; -1, 1) = \int_{-1}^{1} (2 - x^2)\,\mathrm{d}x - \int_{-1}^{1} x^2\,\mathrm{d}x =$$

$$= \left[2x - \frac{x^3}{3}\right]_{-1}^{1} - \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{3}.$$

■

32.1.5 Observations.

- If a function f is continuous and even on $[-a,\, a]$, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$$

- If a function f is continuous and odd on $[-a,\, a]$, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$$

- If a function f is continuous in $\mathbb{R}$ and periodic with a period $T \in \mathbb{R}^+$, then

$$\forall \alpha \in \mathbb{R} : \int_{0}^{T} f(x)\,\mathrm{d}x = \int_{\alpha}^{\alpha+T} f(x)\,\mathrm{d}x.$$

32.1.6 Example.

$$\int_{2}^{4} \sin(\pi x)\,\mathrm{d}x = \int_{-1}^{1} \sin(\pi x)\,\mathrm{d}x = 0$$

since the function $f(x) := \sin(\pi x)$ is continuous in $\mathbb{R}$, periodic with the period 2, and odd.

32.2 Length of a Plane Curve

32.2.1 In this subsection by a curve we mean a graph of a continuous function in a closed bounded interval.

Assume f is a continuous function in an interval $[a, b]$. We shall be concerned with the question: *How to define and calculate the length of the curve*

$$k := \{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge y = f(x)\} ?$$

The problem shall be solved gradually:

- Let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ be points of the plane $\mathbb{R}^2$. By the length of the line segment with the end points α, β (let us denote such segment by $[\alpha; \beta]$) we mean the non-negative number
$$\lambda([\alpha; \beta]) := \sqrt{(\beta_1 - \alpha_1)^2 + (\beta_2 - \alpha_2)^2}.$$
- Let $z_0, z_1, \ldots, z_n$ be mutually distinct points of the plane $\mathbb{R}^2$. By the length of the broken line $[z_0; z_1; \ldots; z_n] := \bigcup_{k=1}^{n} [z_{k-1}; z_k]$ we mean the number
$$\lambda([z_0; z_1; \ldots; z_n]) := \sum_{k=1}^{n} \lambda([z_{k-1}; z_k]).$$
- Let us return to the function f that is continuous on $[a, b]$. For every partition
$$D : a = x_0 < x_1 < \ldots < x_n = b$$
of the interval $[a, b]$, we consider points
$$z_0 = (x_0, f(x_0)),\ z_1 = (x_1, f(x_1)),\ \ldots,\ z_n = (x_n, f(x_n))$$
and define the number
$$L_D := \lambda([z_0; z_1; \ldots; z_n]).$$

32.2.2 Definition. Let f be a continuous function in an interval $[a, b]$, where $-\infty < a < b < +\infty$. By the length of the curve

$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge y = f(x)\}$$

we mean the number

$$\Lambda(f; a, b) := \sup \{L_D : D \text{ is a partition of } [a, b]\}.$$

32.2.3 Observation. If a function f is continuous on an interval $[a, b]$, then

$$0 < \Lambda(f; a, b) \le +\infty.$$

32.2.4 Examples.

1) Let the function f be given by $f(x) := 2x$. Then $L_D := \sqrt{5}$ for all partitions D of the interval $[0,\ 1]$. Therefore $\Lambda(f; 0, 1) = \sqrt{5}$. (In fact, we calculated the length of the line segment with the end points $(0, 0)$ and $(1, 2)$.)

2) Let the function f be given by

$$f(x) := \begin{cases} x^2 \sin \frac{\pi}{2x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is continuous in $\mathbb{R}$ (and so also in $[0,\ 1]$) and it can be proved that

$$\Lambda(f; 0, 1) = +\infty.$$

32.2.5 Theorem. *If a function f has its first derivative continuous in an interval $[a,\ b]$, then*

$$\Lambda(f; a, b) = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

SKETCH OF THE PROOF. If D is a partition of the interval $[a,\ b]$, then

$$L_D = \sum_k \lambda([z_{k-1}; z_k]) = \sum_k \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} =$$

$$= \sum_k \sqrt{(x_k - x_{k-1})^2 + (f'(\xi_k)(x_k - x_{k-1}))^2} = \sum_k \left(\sqrt{1 + (f'(\xi_k))^2}\, (x_k - x_{k-1})\right)$$

for suitable $\xi_k \in (x_{k-1},\ x_k)$ (see Theorem 14.2); the bright reader can already see the consequence with the second statement of Remark 28.5.

□

32.2.6 Examples. Let us calculate the length of the curve

1)

$$\left\{(x, y) \in \mathbb{R}^2 :\ x \in \left[-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right]\ \wedge\ y = \sqrt{1 - x^2}\right\}.$$

SOLUTION.

$$\Lambda\left(\sqrt{1 - x^2}; -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{1 + \left(\frac{-2x}{2\sqrt{1 - x^2}}\right)^2}\, \mathrm{d}x = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{\frac{1}{1 - x^2}}\, \mathrm{d}x =$$

$$= [\arcsin x]_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

(We actually calculated the length of the quarter-circle with radius 1.)

■

2)

$$\left\{(x,y)\in\mathbb{R}^2:\ x\in[-1,\ 1]\ \wedge\ y=\cosh x\right\}.$$

SOLUTION.

$$\Lambda(\cosh;-1,1)=\int_{-1}^{1}\sqrt{1+(\cosh' x)^2}\,\mathrm{d}x=\int_{-1}^{1}\sqrt{1+\sinh^2 x}\,\mathrm{d}x=$$

$$=\int_{-1}^{1}\cosh x\,\mathrm{d}x=\mathrm{e}-\frac{1}{\mathrm{e}}.$$

■

32.3 Volume of a Rotational Solid

32.3.1 Definition. Assume f is a continuous and non-negative function on an interval $[a,\ b]$. Let us consider a body Ω obtained by rotating a "curvilinear" trapezium

$$\left\{(x,y)\in\mathbb{R}^2:\ x\in[a,\ b]\ \wedge\ 0\le y\le f(x)\right\}$$

around the axis x. It can be shown that it is reasonable to calculate (define) the volume of the rotational solid Ω by

$$V(f;a,b):=\pi\int_a^b f^2(x)\,\mathrm{d}x.$$

32.3.2 Example. Let us calculate the volume of the ball with radius $r\in\mathbb{R}^+$.

SOLUTION.

$$V\left(\sqrt{r^2-x^2};-r,r\right)=\pi\int_{-r}^{r}\left(r^2-x^2\right)\,\mathrm{d}x=2\pi\int_{0}^{r}\left(r^2-x^2\right)\,\mathrm{d}x=$$

$$=2\pi\left(r^3-\left[\frac{x^3}{3}\right]_0^r\right)=\frac{4}{3}\pi r^3.$$

■

32.4 Area of a Rotational Surface

32.4.1 Definition. Area of a rotational surface created by rotating a curve

$$\left\{(x,y)\in\mathbb{R}^2:\ x\in[a,\ b]\ \wedge\ y=f(x)\right\},$$

where f is a non-negative function having its derivative continuous in $[a,\ b]$, around the axis x is given by

$$\sigma(f;a,b):=2\pi\int_a^b f(x)\,\sqrt{1+(f'(x))^2}\,\mathrm{d}x.$$

32.4.2 Example. Let us calculate the area of the rotational surface that is obtained by rotating the curve

$$\left\{(x, y) \in \mathbb{R}^2 : \; x \in [0,\, 1] \;\wedge\; y = x^3\right\}$$

around the axis x.

SOLUTION.

$$\sigma(x^3; 0, 1) = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\,\mathrm{d}x = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\,\mathrm{d}x =$$

$$= 2\pi\,\frac{1}{36} \int_1^{10} \sqrt{t}\,\mathrm{d}t = \frac{\pi}{18}\left[\frac{2\sqrt{t^3}}{3}\right]_1^{10} = \frac{\pi}{27}\left(10\sqrt{10} - 1\right).$$

■

32.4.3 Remark. There are many other applications of the definite integral. For example, in physics it is possible to calculate static moments and centres of gravity of a curve, curvilinear trapezium, rotational solids, etc., by using the definite integral.

33 IMPROPER INTEGRAL

33.1 Let us consider the following situation: we have a function f that is continuous and non-negative on an interval $[a, +\infty)$ ($a \in \mathbb{R}$) and we want to calculate (define) the area of the "unbounded surface"

$$\left\{(x, y) \in \mathbb{R}^2 : \; x \in [a,\, +\infty) \;\wedge\; 0 \le y \le f(x)\right\}.$$

It is certainly natural to express this area as the limit

$$\lim_{t \to +\infty} P(f; a, t) = \lim_{t \to +\infty} \int_a^t f(x)\,\mathrm{d}x.$$

Similarly,

$$\lim_{t \to b-} \int_a^t f(x)\,\mathrm{d}x$$

expresses the area of the surface

$$\left\{(x, y) \in \mathbb{R}^2 : \; x \in [a,\, b) \;\wedge\; 0 \le y \le f(x)\right\},$$

where the function f is continuous, non-negative (and eventually unbounded too) on an interval $[a,\, b)$ ($a,\, b \in \mathbb{R}$).

These ideas lead us to the following definitions.

33.2 Definitions. Let $-\infty < a < b \le +\infty$ and suppose f is such a function that $\int_a^t f(x)\,\mathrm{d}x$ exists for every $t \in [a,\, b)$. If $\lim\limits_{t \to b-} \int_a^t f(x)\,\mathrm{d}x$ exists (for $b = +\infty$, we understand the limit as $\lim\limits_{t \to +\infty} \int_a^t f(x)\,\mathrm{d}x$), we define

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t \to b-} \int_a^t f(x)\,\mathrm{d}x.$$

In case

- $\lim\limits_{t\to b-} \int_a^t f(x)\,\mathrm{d}x \in \mathbb{R}$, we say that the integral $\int_a^b f(x)\,\mathrm{d}x$ converges,
- $\lim\limits_{t\to b-} \int_a^t f(x)\,\mathrm{d}x$ does not exist or $\lim\limits_{t\to b-} \int_a^t f(x)\,\mathrm{d}x = \pm\infty$, we speak about the divergent integral.

Analogously we define $\int_a^b f(x)\,\mathrm{d}x$ and its convergence when $-\infty \le a < b < +\infty$ and $\int_t^b f(x)\,\mathrm{d}x$ exists for every $t \in (a, b]$.

33.3 Examples.

1)
$$\int_1^{+\infty} \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to+\infty} \int_1^t \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to+\infty} [\arctan x]_1^t = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

2)
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{t\to 1-} \int_0^t \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{t\to 1-} [\arcsin x]_0^t = \frac{\pi}{2}.$$

3)
$$\int_1^{\mathrm{e}} \frac{\mathrm{d}x}{x\log x} = \lim_{t\to 1+} \int_t^{\mathrm{e}} \frac{\mathrm{d}x}{x\log x} = \lim_{t\to 1+} [\log(\log x)]_t^{\mathrm{e}} = 0 - (-\infty) = +\infty.$$

33.4 Definition. Let $-\infty \le a < b \le +\infty$, f be a continuous function in an interval (a, b), and $c \in (a, b)$. Integral $\int_a^b f(x)\,\mathrm{d}x$ is said to be convergent if integrals $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ converge. In such case we moreover define

$$\int_a^b f(x)\,\mathrm{d}x := \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

It can be shown that this definition is independent of the choice of $c \in (a, b)$.

33.5 Theorem. *Assume f is a continuous function in an interval (a, b), where $-\infty \le a < b \le +\infty$, and F is an antiderivative of f on (a, b). Then $\int_a^b f(x)\,\mathrm{d}x$ converges if and only if*

$$\lim_{x\to b-} F(x) =: F(b-), \qquad \lim_{x\to a+} F(x) =: F(a+)$$

exist finite. Moreover, in such case

$$\int_a^b f(x)\,\mathrm{d}x = F(b-) - F(a+) =: [F(x)]_a^b.$$

$$\left(F(+\infty-) := \lim_{x\to+\infty} F(x),\ F(-\infty+) := \lim_{x\to-\infty} F(x).\right)$$

SKETCH OF THE PROOF.

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \lim_{t\to a+} \int_t^c f(x)\,\mathrm{d}x + \lim_{t\to b-} \int_c^t f(x)\,\mathrm{d}x =$$

$$= \lim_{t\to a+} [F(x)]_t^c + \lim_{t\to b-} [F(x)]_c^t = F(c) - F(a+) + F(b-) - F(c) = F(b-) - F(a+).$$

□

33.6 Example. Think over in detail that

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2+x+1} = \left[\frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right]_{-\infty}^{+\infty} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{2\pi}{\sqrt{3}}.$$

33.7 Theorem (Comparison Criterion). *Let*

i) $-\infty \le a < b \le +\infty$,

ii) functions f, g *and* h *be continuous in* $(a,\, b)$,

iii) $g(x) \le f(x) \le h(x)$ *hold for every* $x \in (a,\, b)$,

iv) integrals $\int_a^b g(x)\,\mathrm{d}x$ *and* $\int_a^b h(x)\,\mathrm{d}x$ *converge.*

Then the integral $\int_a^b f(x)\,\mathrm{d}x$ *converges too.*

33.8 Corollary. *If a function* f *is continuous in an interval* $(a,\, b)$ *and if* $\int_a^b |f(x)|\,\mathrm{d}x$ *converges, then* $\int_a^b f(x)\,\mathrm{d}x$ *also converges. Moreover, in such case*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x.$$

(Compare this statement with Theorem 28.7 iii).)

33.9 Example. Let us decide on the convergence of the integral

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos x}{x^2}\,\mathrm{d}x.$$

SOLUTION. Since

$$0 \le \left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2} \quad \text{for every } x \in \left(\frac{\pi}{2},\, +\infty\right),$$

and

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_{\frac{\pi}{2}}^{+\infty} = \frac{2}{\pi} \in \mathbb{R},$$

the examined integral converges.

■

33.10 Exercise. Let $\alpha \in \mathbb{R}$. Decide on the convergence of the integrals

1) $\int_0^1 \frac{\mathrm{d}x}{x^\alpha}$,

2) $\int_1^{+\infty} \frac{\mathrm{d}x}{x^\alpha}$,

3) $\int_0^{+\infty} \frac{\mathrm{d}x}{x^\alpha}$.

References

[1] Rudin, W.: Priciples of Mathematical Analysis, 3rd ed., McGraw-Hill, New York, 1976.

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