Příklad: Vyšetřete průběh funkce

1 $f(x) = \frac{x}{1+x^2}$

$1+x^2 \neq 0$

$Df = \mathbb{R}$

f je lichá $\forall x \in Df$:

$f(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} = -f(x)$

$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to\infty} \frac{x}{x^2(\frac{1}{x^2}+1)} = \lim_{x\to\infty} \frac{1}{x(\frac{1}{x^2}+1)}$

$= \boxed{\frac{1}{\infty\cdot(0+1)}} = \underline{0} \Rightarrow$ $\boxed{\text{osa } x}$ je asymptota pro $x \to +\infty$

$\lim_{x\to-\infty} \frac{x}{1+x^2} = \dots = \boxed{\frac{1}{-\infty\cdot(0+1)}} = \underline{0} \Rightarrow$ $\boxed{\text{osa } x}$ je asymptota pro $x \to -\infty$

NULOVÉ BODY

průs. s osou y: $x=0 \Rightarrow y = \frac{0}{1+0^2} = 0$

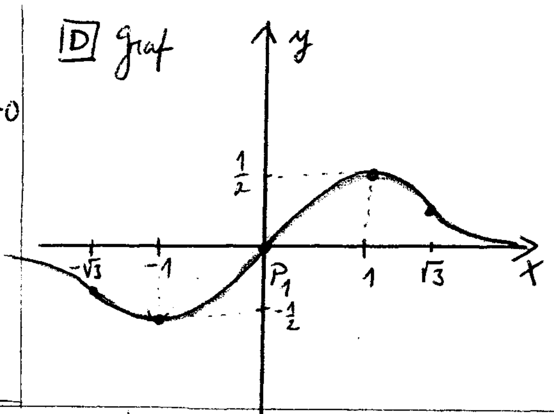
$\underline{P_1 = [0,0]}$

průs. s osou x: $f(x)=0 \Rightarrow 0 = \frac{x}{1+x^2}$

$\underline{P_1 = [0,0]}$ $\quad x=0$

	$(-\infty,0)$	0	$(0,\infty)$
f	−	0	+

D graf



B $f'(x) = \frac{1-x^2}{(1+x^2)^2}$

$Df' = \mathbb{R}$

stac. body: $f'(x) = 0$

$\frac{1-x^2}{1+x^2} = 0 \Leftrightarrow 1-x^2 = 0$

$\underline{x = \pm 1}$

	$(-\infty,-1)$	−1	$(-1,1)$	1	$(1,\infty)$
f'	−	0	+	0	−
f	↘ kles.	LOK MIN	↗ rost.	LOK MAX	↘ kles.

$f(-1) = -\frac{1}{2}$ $\quad f(1) = \frac{1}{2}$

C $f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}$

$Df'' = \mathbb{R}$

$f''(x) = 0$

$\frac{2x(x^2-3)}{(1+x^2)^3} = 0$

$2x = 0 \vee x^2-3 = 0$

$x = 0 \vee x = \pm\sqrt{3}$

	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},0)$	0	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
f''	−	0	+	0	−	0	+
f	∩ konkáv.	INFL. BOD	∪ konvex.	INFL. BOD	∩ konkáv.	INFL. BOD	∪ konvex.

$f(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$ $\quad f(0)=0$ $\quad f(\sqrt{3}) = \frac{\sqrt{3}}{4}$

2 $f(x) = \frac{x^2}{x+1}$

podm: $x+1 \neq 0$

$Df = \mathbb{R} \setminus \{-1\}$

f není ani sudá ani lichá:

např. pro $x=2$: $f(-x) \neq f(x) \wedge f(-x) \neq -f(x)$

$f(-2) \neq f(2) \wedge f(-2) \neq -f(2)$

$-4 \neq \frac{4}{3} \qquad -4 \neq -\frac{4}{3}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x+1} = \infty$

$\lim_{x\to-\infty} \frac{x^2}{x+1} = \lim_{x\to-\infty} \frac{x}{1+\frac{1}{x}} = -\infty$

$\lim_{x\to-1^+} \frac{x^2}{x+1} = \frac{1}{0^+} = +\infty$

$\lim_{x\to-1^-} \frac{x^2}{x+1} = \frac{1}{0^-} = -\infty$

$\boxed{x=-1}$ je asymptota bez směrnice

NULOVÉ BODY

průs. s osou y: $x=0 \Rightarrow$

$f(0) = y = \frac{0}{0+1} = 0$

$\underline{P_1 = [0,0]}$

průs. s osou x: $f(x)=0 \Rightarrow$

$0 = \frac{x^2}{x+1} \Leftrightarrow x=0$

$\underline{P_1 = [0,0]}$

		−1		0	
f	−	N	+	0	+

asymptota se směrnicí: $y = kx+q$

$k = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x}{x+1} = 1$

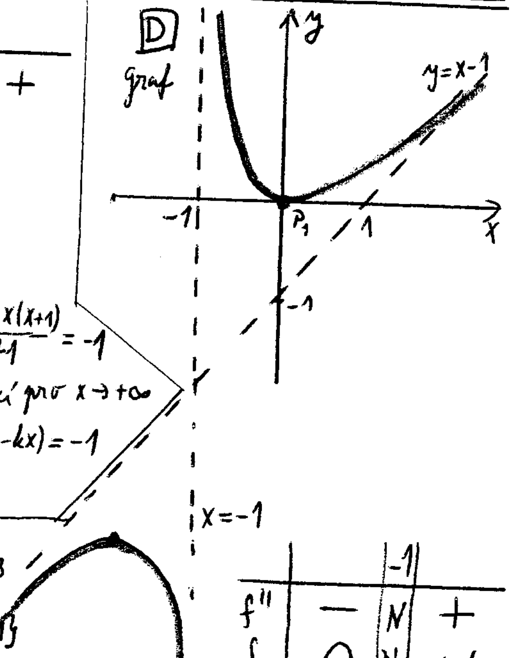
$q = \lim_{x\to\infty}(f(x)-k\cdot x) = \lim_{x\to\infty}\left(\frac{x^2}{x+1} - x\right) = \lim \frac{x^2-x(x+1)}{x+1} = -1$

$\boxed{y = x-1}$ asymptota se směrnicí pro $x \to +\infty$

$k = \lim_{x\to-\infty} \frac{f(x)}{x} = \dots = 1$; $\quad q = \lim_{x\to-\infty}(f(x)-kx) = -1$

$\boxed{y = x-1}$ pro $x \to -\infty$

D graf



B $f'(x) = \frac{x(x+2)}{(x+1)^2}$

$Df' = \mathbb{R} \setminus \{-1\}$

$f'(x)=0 \Leftrightarrow x=0 \vee x+2=0$

		−2		−1		0	
f'	+	0	−	N	−	0	+
f	↗	LOK MAX	↘	N	↘	LOK MIN	↗

$f(-2) = -4$ $\quad f(0) = 0$

C $f'' = \frac{2}{(x+1)^3}$

$Df'' = \mathbb{R} \setminus \{-1\}$

$f''(x) = 0 \Leftrightarrow \emptyset$

		−1	
f''	−	N	+
f	∩	N	∪

3 $f(x) = 2x^3 - 9x^2 + 12x - 5$ v intervalu $\langle 0,3 \rangle$

$\underline{Df = \langle 0,3 \rangle}$

f není ani sudá ani lichá

$\lim_{x\to0^+} f(x) = -5$; $\lim_{x\to3^-} f(x) = 54-81+36-5 = 4$

nemá asymptoty bez směrnice ani se směrnicí

NULOVÉ BODY

průs. s osou y: $x=0 \Rightarrow y=-5$ $\quad \underline{P_1 = [0,-5]}$

průs. s osou x: $f(x)=0 \Rightarrow 0 = 2x^3-9x^2+12x-5$

$0 = (x-1)\cdot(2x^2-7x+5)$

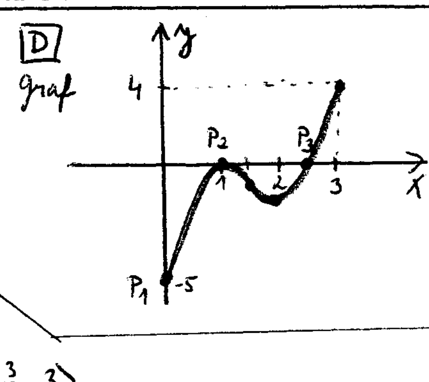
$0 = (x-1)^2(2x-5)$

$\underline{P_2 = [1,0]}$

$\underline{P_3 = [\frac{5}{2},0]}$

	$\langle 0,1)$	1	$(1,\frac{5}{2})$	$\frac{5}{2}$	$(\frac{5}{2},3\rangle$
f	−	0	−	0	+

D graf



B $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$

$Df' = \langle 0,3 \rangle$

stac. body

$f'(x) = 0$

$6(x-1)(x-2) = 0$

$x=1 \vee x=2$

	$\langle 0,1)$	1	$(1,2)$	2	$(2,3\rangle$
f'	+	0	−	0	+
f	↗	LOK MAX	↘	LOK MIN	↗

$f(1) = 0$ $\quad f(2) = -1$

C $f''(x) = 12x - 18 = 6\cdot(2x-3)$

$Df'' = \langle 0,3 \rangle$

$f''(x) = 0$

$6(2x-3) = 0$

$x = \frac{3}{2}$

	$\langle 0,\frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2},3\rangle$
f''	−	0	+
f	∩	INFL BOD	∪

$f(\frac{3}{2}) = -\frac{1}{2}$

4 $\boxed{f(x)=\dfrac{3}{x}-\dfrac{1}{x^3}}$

podm: $x \neq 0$

$Df = \mathbb{R} \setminus \{0\}$

NULOVÉ BODY

průs. s osou y: $x=0 \notin Df$

nemá průs. s osou y

průs. s osou x: $f(x)=0 \Rightarrow$

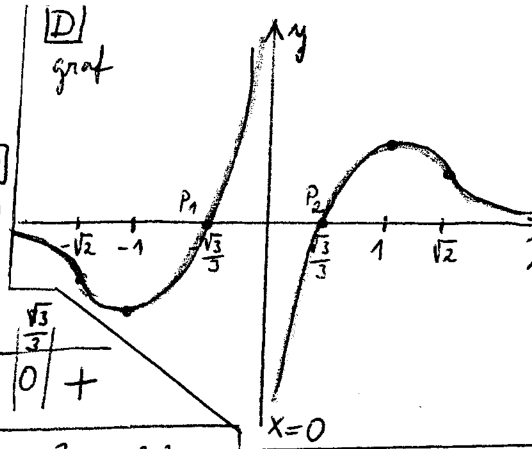
$\Rightarrow 0 = \dfrac{3}{x} - \dfrac{1}{x^3}$ $P_1 = \left[-\dfrac{\sqrt{3}}{3}, 0\right]$

$0 = \dfrac{3x^2-1}{x^3}$ $P_2 = \left[\dfrac{\sqrt{3}}{3}, 0\right]$

$3x^2-1=0$ $x = \pm\dfrac{\sqrt{3}}{3}$

		$-\frac{\sqrt{3}}{3}$		0		$\frac{\sqrt{3}}{3}$	
f	−	0	+	N	−	0	+

D graf



f je lichá: $\forall x \in Df$:

$f(-x) = \dfrac{3}{-x} - \dfrac{1}{(-x)^3} = -\dfrac{3}{x} + \dfrac{1}{x^3} =$

$= -\left(\dfrac{3}{x} - \dfrac{1}{x^3}\right) = -f(x)$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{3x^2-1}{x^3} = 0$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{3x^2-1}{x^3} = 0$

$\boxed{\text{osa } x}$ je asymptota se směrnicí pro $x \to \pm\infty$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{3x^2-1}{x^3} = \boxed{\dfrac{-1}{0^+}} = -\infty$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{3x^2-1}{x^3} = \boxed{\dfrac{-1}{0^-}} = +\infty$

$\boxed{X=0}$ je asymptota bez směrnice

B $f'(x) = -\dfrac{3}{x^2} + \dfrac{3}{x^4} = \dfrac{-3x^2+3}{x^4}$

$Df' = \mathbb{R} \setminus \{0\}$

$f'(x)=0$

$\dfrac{-3x^2+3}{x^4} = 0$

$-3x^2+3=0$

$x = \pm 1$

		-1		0		1	
f'	−	0	+	N	+	0	−
f	↘	LOK MIN	↗	N	↗	LOK MAX	↘

$f(-1) = -2$ $f(1) = 2$

C $f''(x) = \dfrac{6}{x^3} - \dfrac{12}{x^5} = \dfrac{6(x^2-2)}{x^5}$

$Df'' = \mathbb{R} \setminus \{0\}$

$f''(x) = 0$ $\dfrac{6(x^2-2)}{x^5} = 0$ $x = \pm\sqrt{2}$

		$-\sqrt{2}$		0		$\sqrt{2}$	
f''	−	0	+	N	−	0	+
f	∩	INFL BOD	∪	N	∩	INFL BOD	∪

$f(-\sqrt{2}) = -\dfrac{5\sqrt{2}}{4}$

$f(\sqrt{2}) = \dfrac{5\sqrt{2}}{4}$

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5 $\boxed{f(x) = (x^4-1)^{\frac{2}{3}}}$ $Df = \mathbb{R}$

f je sudá: $\forall x \in Df$:

$f(-x) = \left((-x)^4-1\right)^{\frac{2}{3}} = (x^4-1)^{\frac{2}{3}} = f(x)$

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (x^4-1)^{\frac{2}{3}} = +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x^4-1)^{\frac{2}{3}} = +\infty$

asymptota se směrnicí: $y = kx+q$

$k = \lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{(x^4-1)^{\frac{2}{3}}}{x} = \lim\limits_{x\to\infty} \dfrac{x^{\frac{8}{3}}\left(1-\frac{1}{x^4}\right)^{\frac{2}{3}}}{x} = \infty$

$k = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} \dfrac{x^{\frac{8}{3}}\left(1-\frac{1}{x^4}\right)^{\frac{2}{3}}}{x} = \lim\limits_{x\to-\infty} \dfrac{X^{\frac{5}{3}}\left(1-\frac{1}{x^4}\right)^{\frac{2}{3}}}{1} = -\infty$

$\Rightarrow$ nemá asymptoty se směrnicí

NULOVÉ BODY

průs. s osou y: $x=0 \Rightarrow y = (-1)^{\frac{2}{3}} = 1$

$P_1 = [0,1]$

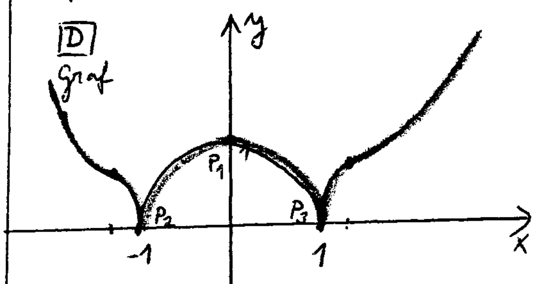
průs. s osou x: $f(x)=0 \Rightarrow 0 = (x^4-1)^{\frac{2}{3}}$

$x^4 = 1 \Rightarrow x = \pm 1$

$P_2 = [-1,0]$, $P_3 = [1,0]$

		-1		1	
f	+	0	+	0	+

D graf



B $f'(x) = \dfrac{8}{3}x^3(x^4-1)^{-\frac{1}{3}}$

podm: $x^4-1 \neq 0$

$Df' = \mathbb{R} \setminus \{-1; 1\}$

stac. body

$f'(x) = 0 \Leftrightarrow x = 0$

		-1		0		1	
f'	−	N	+	0	−	N	+
f	↘	LOK MIN	↗	LOK MAX	↘	LOK MIN	↗

$f(-1)=0$ $f(0)=1$ $f(1)=0$

C $f''(x) = \dfrac{8x^2(5x^4-9)}{9\sqrt[3]{(x^4-1)^4}} = \dfrac{8}{9}x^2(5x^4-9)\cdot(x^4-1)^{-\frac{4}{3}}$

$Df'' = \mathbb{R} \setminus \{-1; 1\}$

$f''(x) = 0 \Leftrightarrow x = 0 \vee x = \pm\sqrt[4]{\dfrac{9}{5}} \doteq \pm 1{,}158$

		$-\sqrt[4]{\frac{9}{5}}$		-1		0		1		$\sqrt[4]{\frac{9}{5}}$	
f''	+	0	−	N	−	0	−	N	−	0	+
f	∪	INFL BOD	∩		∩		∩		∩	INFL BOD	∪

$f\left(\sqrt[4]{\frac{9}{5}}\right) \doteq 0{,}862$ $f\left(\sqrt[4]{\frac{9}{5}}\right) = \left(\frac{4}{5}\right)^{\frac{2}{3}} \doteq 0{,}86.$

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6 $\boxed{f(x) = \dfrac{x+2}{x^2}}$

podm $x^2 \neq 0$

$Df = \mathbb{R} \setminus \{0\}$

f není ani sudá ani lichá:

např. pro $x=1$: $f(x) \neq f(x) \wedge f(-x) \neq -f(x)$

$f(-1) \neq f(1) \wedge f(-1) \neq -f(1)$

$1 \neq 3$ $1 \neq -3$

$\lim\limits_{x\to\pm\infty} f(x) = 0 \Rightarrow$ osa x je asymptota se směrnicí

$\lim\limits_{x\to 0^+} f(x) = \dfrac{2}{0^+} = +\infty$; $\lim\limits_{x\to 0^-} f(x) = \dfrac{2}{0^+} = +\infty \Rightarrow \boxed{x=0}$ je asymptota bez směrnice

NULOVÉ BODY

průs. s osou y: $x = 0 \notin Df$ nejsou

průs. s osou x: $f(x)=0 \Rightarrow 0 = \dfrac{x+2}{x^2}$

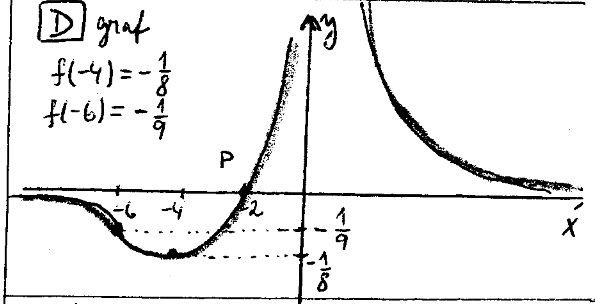
$x=-2$ $P = [-2,0]$

		-2		0	
f	−	0	+	N	+

D graf

$f(-4) = -\dfrac{1}{8}$

$f(-6) = -\dfrac{1}{9}$



B $f'(x) = \dfrac{-(x+4)}{x^3}$

$Df' = \mathbb{R} \setminus \{0\}$, $f'(x) = 0 \Leftrightarrow x = -4$

		-4		0	
f'	−	0	+	N	−
f	↘	LOK MIN	↗	N	↘

C $f''(x) = \dfrac{2(x+6)}{x^4}$

$Df'' = \mathbb{R} \setminus \{0\}$

$f''(x) = 0 \Leftrightarrow x = -6$

		-6		0	
f''	−	0	+	N	+
f	∩	INFL BOD	∪	N	∪