Příklad: $\begin{aligned} 4x - y \quad\;\; &= 0 \\ 3x + 2y + z &= 3 \\ 2x \quad\;\; + z &= 6 \end{aligned}$

Naleznéte všechna řešení soustavy rovnic různými způsoby a proveďte zkoušku.

1. způsob – pomocí GEM (Gaussova eliminační metoda)

Rozšířenou matici soustavy upravíme na schodovitý tvar

$$\left(\begin{array}{ccc|c} 4 & -1 & 0 & 0 \\ 3 & 2 & 1 & 3 \\ 2 & 0 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & 1 & 6 \\ 3 & 2 & 1 & 3 \\ 4 & -1 & 0 & 0 \end{array}\right) \begin{array}{l} \cdot(-3);\ \cdot(-2) \\ \cdot 2 \\ \end{array} \sim \left(\begin{array}{ccc|c} 2 & 0 & 1 & 6 \\ 0 & 4 & -1 & -12 \\ 0 & -1 & -2 & -12 \end{array}\right) \begin{array}{l} \\ \\ \cdot 4 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 1 & 6 \\ 0 & 4 & -1 & -12 \\ 0 & 0 & -9 & -60 \end{array}\right) \cdot\left(-\tfrac{1}{3}\right)$$

$h(A|b) = 3 = h(A) \Rightarrow$ soustava má řešení

$n = 3 = h(A|b) = h(A) \Rightarrow$ řešení je jediné

$-9z = -60 \quad /:(-9)$

$z = \dfrac{20}{3}$

$4y - z = -12$

$4y - \dfrac{20}{3} = -12 \quad / + \dfrac{20}{3}$

$4y = -\dfrac{16}{3} \quad /:4$

$y = -\dfrac{4}{3}$

$2x + z = 6$

$2x + \dfrac{20}{3} = 6 \quad / - \dfrac{20}{3}$

$2x = -\dfrac{2}{3} \quad /:2$

$x = -\dfrac{1}{3}$

nebo

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 1 & 6 \\ 0 & 4 & -1 & -12 \\ 0 & 0 & 3 & 20 \end{array}\right) \begin{array}{l} \cdot(-3) \\ \cdot 3 \\ \end{array} \sim \left(\begin{array}{ccc|c} -6 & 0 & 0 & 2 \\ 0 & 12 & 0 & -16 \\ 0 & 0 & 3 & 20 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{20}{3} \end{array}\right) \begin{array}{l} \Rightarrow x = -\frac{1}{3} \\ \Rightarrow y = -\frac{4}{3} \\ \Rightarrow z = \frac{20}{3} \end{array}$$

Zkouška:

$L_1 = 4x - y = 4\cdot\left(-\tfrac{1}{3}\right) - \left(-\tfrac{4}{3}\right) = -\tfrac{4}{3} + \tfrac{4}{3} = 0;$

$P_1 = 0; \quad L_1 = P_1$

$L_2 = 3x + 2y + z = 3\cdot\left(-\tfrac{1}{3}\right) + 2\cdot\left(-\tfrac{4}{3}\right) + \tfrac{20}{3} =$

$= -\tfrac{3}{3} - \tfrac{8}{3} + \tfrac{20}{3} = \tfrac{9}{3} = 3; \quad P_2 = 3; \quad L_2 = P_2$

$L_3 = 2x + z = 2\cdot\left(-\tfrac{1}{3}\right) + \tfrac{20}{3} = \tfrac{18}{3} = 6; \quad P_3 = 6; \quad L_3 = P_3$

2. způsob – Cramerovo pravidlo – pomocí determinantů

$$A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}; \quad x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad z = \frac{|A_3|}{|A|}$$

$$|A| = \begin{vmatrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 4\cdot2\cdot1 + 3\cdot0\cdot0 + 2\cdot(-1)\cdot1 - (2\cdot2\cdot0 + 4\cdot0\cdot1 + 3\cdot(-1)\cdot1) = 8 - 2 - (-3) = \underline{9}$$

$$|A_1| = \begin{vmatrix} 0 & -1 & 0 \\ 3 & 2 & 1 \\ 6 & 0 & 1 \end{vmatrix} = 1\cdot2\cdot0 + 3\cdot0\cdot0 + 6\cdot(-1)\cdot1 - (6\cdot2\cdot0 + 0\cdot0\cdot1 + 3\cdot(-1)\cdot1) = -6 - (-3) = \underline{-3}$$

$$|A_2| = \begin{vmatrix} 4 & 0 & 0 \\ 3 & 3 & 1 \\ 2 & 6 & 1 \end{vmatrix} = 4\cdot3\cdot1 + 3\cdot6\cdot0 + 2\cdot0\cdot1 - (2\cdot3\cdot0 + 4\cdot6\cdot1 + 3\cdot0\cdot1) = 12 - 24 = \underline{-12}$$

$$|A_3| = \begin{vmatrix} 4 & -1 & 0 \\ 3 & 2 & 3 \\ 2 & 0 & 6 \end{vmatrix} = 4\cdot2\cdot6 + 3\cdot0\cdot0 + 2\cdot(-1)\cdot3 - (2\cdot2\cdot0 + 4\cdot0\cdot3 + 3\cdot(-1)\cdot6) = 48 - 6 - (-18) = \underline{60}$$

$$x = \frac{|A_1|}{|A|} = \frac{-3}{9} = -\frac{1}{3}; \quad y = \frac{|A_2|}{|A|} = \frac{-12}{9} = -\frac{4}{3}; \quad z = \frac{|A_3|}{|A|} = \frac{60}{9} = \frac{20}{3}$$

$\left[-\dfrac{1}{3};\ -\dfrac{4}{3};\ \dfrac{20}{3}\right]$ + zkouška

3. způsob – pomocí inverzní matice $\quad A\cdot X = b \Rightarrow X = A^{-1}\cdot b$

$$\left(\underbrace{\begin{matrix} 4 & -1 & 0 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{matrix}}_{A}\middle|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\right)\begin{matrix}\cdot(-3) \\ \cdot 4 \\ \cdot(-2)\end{matrix} \sim \left(\begin{matrix} 4 & -1 & 0 \\ 0 & 11 & 4 \\ 0 & -1 & -2 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ 1 & 0 & -2 \end{matrix}\right) \sim$$

$$\sim \left(\begin{matrix} 4 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 11 & 4 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ -3 & 4 & 0 \end{matrix}\right)\begin{matrix} \\ \cdot(-1)\ \cdot 11 \\ \\ \end{matrix} \sim \left(\begin{matrix} 4 & 0 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & -18 \end{matrix}\middle|\begin{matrix} 0 & 0 & 2 \\ 1 & 0 & -2 \\ 8 & 4 & -22 \end{matrix}\right)\begin{matrix} \cdot 9 \\ \cdot(-9) \\ \\ \end{matrix} \sim$$

$$\sim \left(\begin{matrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -18 \end{matrix}\middle|\begin{matrix} 8 & 4 & -4 \\ -1 & 4 & -4 \\ 8 & 4 & -22 \end{matrix}\right)\begin{matrix} :36 \\ :9 \\ :(-18) \end{matrix} \sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E}\middle|\underbrace{\begin{matrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{11}{9} \end{matrix}}_{A^{-1}}\right)$$

$$X = A^{-1}\cdot b = \begin{pmatrix} \frac{2}{9} & \frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{11}{9} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{3-6}{9} \\ \frac{12-24}{9} \\ \frac{-6+66}{9} \end{pmatrix} = \begin{pmatrix} -\frac{3}{9} \\ -\frac{12}{9} \\ \frac{60}{9} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ \frac{20}{3} \end{pmatrix}$$

$$\boxed{\begin{matrix} x = -\frac{1}{3} \\ y = -\frac{4}{3} \\ z = \frac{20}{3} \end{matrix}}$$

+Zkouška