DVOJNÝ INTEGRÁL

- Fubiniova věta

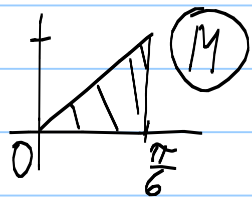
Př: $I = \iint\limits_M (2xy + 3y^2)\,dx\,dy$ kde $M = \langle 0,1\rangle \times \langle 1,2\rangle$.

$$I = \int_0^1 \left(\int_1^2 (2xy + 3y^2)\,dy \right) dx =$$

$$= \int_0^1 \left[xy^2 + y^3 \right]_{y=1}^{2} dx = \int_0^1 (3x + 7)\,dx = \left[\frac{3x^2}{2} + 7x \right]_0^1 =$$

$$= \frac{3}{2} + 7 = \boxed{\frac{17}{2}}.$$

Př: $I = \iint\limits_M \cos(x + 2y)\,dx\,dy$,



kde $M = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x,\ x \leq \frac{\pi}{6}\}$.

M: $0 \leq x \leq \frac{\pi}{6}$
$0 \leq y \leq x$

$$I = \int_0^{\frac{\pi}{6}} \left(\int_0^x \cos(x+2y)\,dy \right) dx = \left| \begin{array}{l} \text{subst.} \\ x+2y = u \\ 2dy = du \\ 0 \mapsto x,\ x \mapsto 3x \end{array} \right| =$$

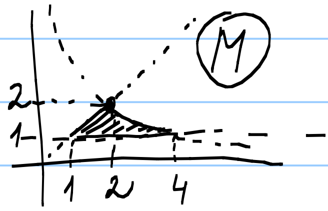
$$= \int_0^{\frac{\pi}{6}} \left(\int_x^{3x} \frac{1}{2} \cos u\,du \right) dx = \int_0^{\frac{\pi}{6}} \left[\frac{1}{2} \sin u \right]_{u=x}^{3x} dx =$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{6}} (\sin 3x - \sin x)\,dx = \frac{1}{2}\left[-\frac{1}{3}\cos 3x + \cos x\right]_0^{\frac{\pi}{6}} =$$

$$= \frac{1}{2}\cdot\left[\left(0+\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{3}+1\right)\right] = \frac{1}{2}\cdot\left(\frac{\sqrt{3}}{2}-\frac{2}{3}\right) =$$

$$= \boxed{\frac{\sqrt{3}}{4}-\frac{1}{3}}.$$

Př: $I = \iint\limits_M xy\,dx\,dy$, kde



$$M = \{(x,y)\in\mathbb{R}^2 : y \le x,\ xy \le 4,\ y \ge 1\}.$$

$$M: \quad \begin{aligned} 1 \le\ & y \le 2 \\ y \le\ & x \le \frac{4}{y} \end{aligned}$$

$$I = \int_1^2 \left(\int_y^{\frac{4}{y}} xy\,dx \right) dy = \int_1^2 y\cdot\left[\frac{x^2}{2}\right]_{x=y}^{\frac{4}{y}} dy =$$

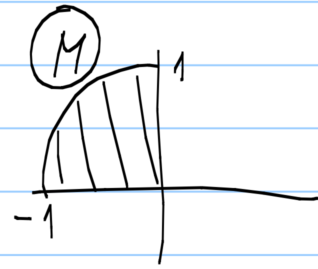
$$= \int_1^2 y\cdot\left(\frac{8}{y^2} - \frac{y^2}{2}\right) dy = \int_1^2 \left(\frac{8}{y} - \frac{y^3}{2}\right) dy =$$

$$= \left[8\ln|y| - \frac{1}{8}y^4\right]_1^2 = (8\ln 2 - 2) - \left(-\frac{1}{8}\right) =$$

$$= \boxed{8\ln 2 - \frac{15}{8}}.$$

Transformace dvojného integrálu.

Př: $$I = \iint_M \ln(x^2+y^2+1)\,dx\,dy$$



$$M = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \leq 1,\ x \leq 0,\ y \geq 0\}.$$

Pol. souřadnice $\begin{array}{l} x = r\cos t \\ y = r\sin t \end{array}$ $(J = r)$ $\left.\begin{array}{l} \frac{\pi}{2} \leq t \leq \pi \\ 0 \leq r \leq 1 \end{array}\right\} \Rightarrow$

$$\Rightarrow I = \int_{\frac{\pi}{2}}^{\pi}\left(\int_0^1 \ln(r^2+1)\cdot r\,dr\right)dt = \left|\begin{array}{l} \text{subst.} \\ r^2+1 = s \\ 2r\,dr = ds \\ 0 \mapsto 1,\ 1 \mapsto 2 \end{array}\right| =$$

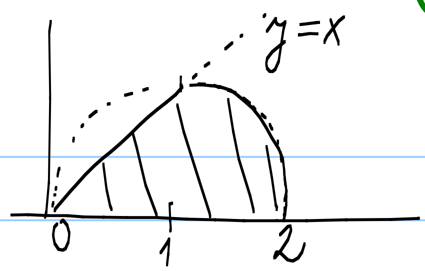
$$= \int_{\frac{\pi}{2}}^{\pi}\left(\int_1^2 \frac{1}{2}\ln s\,ds\right)dt = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\int_1^2 \ln s\,ds\right)dt =$$

$$= \left|\begin{array}{ll} \text{per partes} & \\ u = \ln s & v' = 1 \\ u' = \frac{1}{s} & v = s \end{array}\right| = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\left[\ln s\cdot s\right]_1^2 - \int_1^2 1\,ds\right)dt =$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(2\ln 2 - 1)\,dt = \frac{1}{2}\cdot(2\ln 2 - 1)\cdot\int_{\frac{\pi}{2}}^{\pi} 1\,dt =$$

$$= \frac{1}{2}\cdot(2\ln 2 - 1)\cdot\left[t\right]_{\frac{\pi}{2}}^{\pi} = \boxed{\frac{\pi}{4}\cdot(2\ln 2 - 1)}.$$

Př: $I = \iint_M xy\,dx\,dy$, kde



$$M = \left\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 2x,\ y\le x,\ y\ge 0\right\}.$$

Polární souřadnice $\Rightarrow$ $\left.\begin{array}{l} 0\le t\le \frac{\pi}{4} \\ 0\le r\le 2\cos t \end{array}\right\} \Rightarrow$

$x = r\cos t$ $\quad (J = r)$

$y = r\sin t$

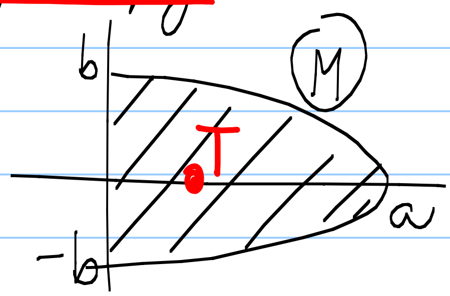
$$\Rightarrow I = \int_0^{\frac{\pi}{4}}\left(\int_0^{2\cos t} r^2\cdot\cos t\sin t\cdot r\,dr\right)dt =$$

$$= \int_0^{\frac{\pi}{4}}\cos t\sin t\cdot\left[\frac{r^4}{4}\right]_{r=0}^{2\cos t}dt = 4\cdot\int_0^{\frac{\pi}{4}}\cos^5 t\cdot\sin t\,dt =$$

$$= \left|\begin{array}{l} \text{subst.} \\ \cos t = s \\ -\sin t\,dt = ds \\ 0\mapsto 1,\ \frac{\pi}{4}\mapsto\frac{\sqrt2}{2} \end{array}\right| = 4\cdot\int_{\frac{\sqrt2}{2}}^{1} s^5\,ds = \frac{2}{3}\cdot\left[s^6\right]_{\frac{\sqrt2}{2}}^{1} =$$

$$= \frac{2}{3}\cdot\left(1-\frac{1}{8}\right) = \frac{2}{3}\cdot\frac{7}{8} = \boxed{\frac{7}{12}}.$$

Př: Vypočtěte souřadnice těžiště „půlelipsy"



Vzhledem k symetrii je

$T = (x_T, 0)$, kde

$$x_T = \frac{\iint_M x\,dx\,dy}{\iint_M 1\,dx\,dy}.$$

9

Zobecněné polární souřadnice $\longrightarrow$ $x = a \cdot r\cos t$, $y = b \cdot r \sin t$ $(J = abr)$

$$\longrightarrow \quad \left. \begin{array}{l} -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ 0 \leq r \leq 1 \end{array} \right\} \Longrightarrow$$

$$\Longrightarrow \iint_M 1 \, dx\, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\left(\int_0^1 abr\, dr \right)}_{\frac{1}{2}ab} dt =$$

$$= \boxed{\frac{1}{2}\pi ab.}$$

$$\iint_M x \, dx\, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^1 a r\cos t \cdot abr\, dr \right) dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 b \cos t \cdot \left[\frac{r^3}{3} \right]_0^1 dt = \frac{1}{3} a^2 b \underbrace{\left[\sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{2} =$$

$$= \boxed{\frac{2}{3} a^2 b.}$$

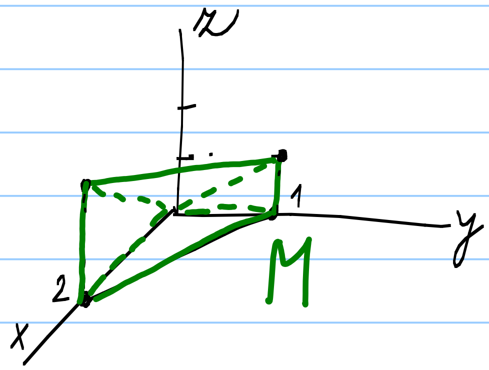
Odtud vidíme, že $x_T = \dfrac{\frac{2}{3}a^2 b}{\frac{1}{2}\pi ab} = \dfrac{4}{3\pi} \cdot a$

$$\Longrightarrow \boxed{T = \left(\frac{4}{3\pi} \cdot a,\ 0 \right)}.$$

TROJNÝ INTEGRÁL

- Fubiniova věta

Př: $I = \iiint_M 1 \, dx\,dy\,dz$, kde



$M = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x+2y \leq 2, 0 \leq z \leq x+y\}$.

$$M: \left.\begin{array}{l} 0 \leq y \leq 1 \\ 0 \leq x \leq 2-2y \\ 0 \leq z \leq x+y \end{array}\right\} \Rightarrow$$

$$\Rightarrow I = \int_0^1 \left(\int_0^{2-2y} \Big(\underbrace{\int_0^{x+y} 1\, dz}_{x+y} \Big) dx \right) dy =$$

$$= \int_0^1 \left(\int_0^{2-2y} (x+y)\, dx \right) dy = \int_0^1 \left[\frac{x^2}{2} + xy \right]_{x=0}^{2-2y} dy =$$

$$= \int_0^1 \left(2\cdot(1-y)^2 + (2-2y)y\right) dy = \int_0^1 \left(2 - 4y + \cancel{2y^2} + 2y - \cancel{2y^2}\right) dy =$$

$$= \int_0^1 (2-2y)\, dy = \left[2y - y^2\right]_0^1 = \boxed{1}.$$

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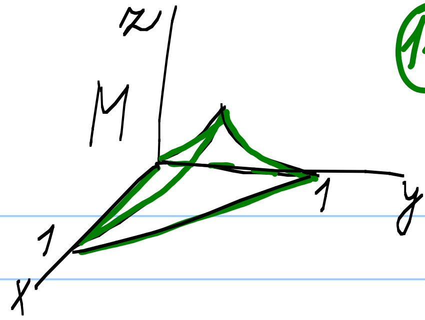


Př: $I = \iiint\limits_M xy \, dx\,dy\,dz$, kde

$$M = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x+y \leq 1, z \leq xy\}.$$

$$M: \left.\begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \\ 0 \leq z \leq xy \end{array}\right\} \Rightarrow$$

$$I = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{xy} xy \, dz \right) dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} xy \left[z \right]_{z=0}^{xy} dy \right) dx = \int_0^1 \left(\int_0^{1-x} x^2 y^2 \, dy \right) dx =$$

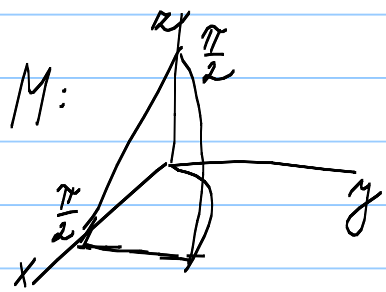
$$= \int_0^1 x^2 \cdot \left[\frac{y^3}{3} \right]_{y=0}^{1-x} dx = \frac{1}{3} \int_0^1 x^2 \cdot (1-x)^3 \, dx =$$

$$= \frac{1}{3} \int_0^1 x^2 (1 - 3x + 3x^2 - x^3) \, dx = \frac{1}{3} \int_0^1 (x^2 - 3x^3 + 3x^4 - x^5) \, dx =$$

$$= \frac{1}{3} \cdot \left[\frac{x^3}{3} - \frac{3x^4}{4} + \frac{3x^5}{5} - \frac{x^6}{6} \right]_0^1 = \frac{1}{3} \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) =$$

$$= \frac{1}{3} \cdot \frac{20 - 45 + 36 - 10}{60} = \frac{1}{3} \cdot \frac{1}{60} = \boxed{\frac{1}{180}}.$$

Př: $I = \iiint\limits_M y \cos(x+z) \, dx\,dy\,dz$, M:



kde $M = \{(x,y,z) \in \mathbb{R}^3 : y \geq 0, y \leq \sqrt{x}, z \geq 0, x \geq 0, x + z \leq \frac{\pi}{2}\}$.

(12)

$$M: \left.\begin{array}{l} 0 \le x \le \frac{\pi}{2} \\ 0 \le y \le \sqrt{x} \\ 0 \le z \le \frac{\pi}{2} - x \end{array}\right\} \Rightarrow$$

$$I = \int_0^{\frac{\pi}{2}} \left(\int_0^{\sqrt{x}} \left(\int_0^{\frac{\pi}{2}-x} y \cos(x+z)\, dz \right) dy \right) dx =$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^{\sqrt{x}} \Big[y \sin(x+z) \Big]_{z=0}^{\frac{\pi}{2}-x} dy \right) dx =$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^{\sqrt{x}} y \cdot (1 - \sin x)\, dy \right) dx =$$

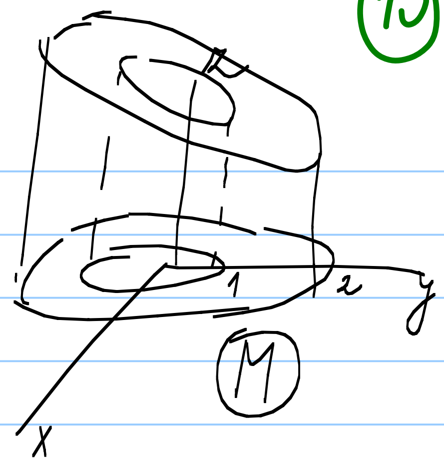
$$= \int_0^{\frac{\pi}{2}} (1-\sin x) \cdot \left[\frac{y^2}{2} \right]_0^{\sqrt{x}} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cdot (1 - \sin x)\, dx =$$

$$= \left| \begin{array}{ll} \text{per-partes} & \\ u = x & v' = 1 - \sin x \\ u' = 1 & v = x + \cos x \end{array} \right| = \frac{1}{2} \left(\Big[x(x+\cos x) \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x + \cos x)\, dx \right) =$$

$$= \frac{1}{2} \cdot \left(\frac{\pi^2}{4} - \left[\frac{x^2}{2} + \sin x \right]_0^{\frac{\pi}{2}} \right) =$$

$$= \frac{1}{2} \cdot \left(\frac{\pi^2}{4} - \left(\frac{\pi^2}{8} + 1 \right) \right) = \frac{1}{2} \cdot \left(\frac{\pi^2}{8} - 1 \right) = \boxed{\frac{\pi^2}{16} - \frac{1}{2}}.$$

Transformace trojného integrálu
(cylindrické souřadnice)



Př: $I = \iiint\limits_M x^2 y \, dx\, dy\, dz$, kde

$M = \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x^2+y^2 \leq 4,\ 0 \leq z \leq 3-y\}$.

Cylindrické souř. $\begin{array}{l} x = r\cos t \\ y = r\sin t \\ z = z \end{array}$ $(J = r)$ $\Rightarrow$

$$\Rightarrow M: \left.\begin{array}{l} 0 \leq t \leq 2\pi \\ 1 \leq r \leq 2 \\ 0 \leq z \leq 3 - r\sin t \end{array}\right\} \Rightarrow$$

$$\Rightarrow I = \int_0^{2\pi} \left(\int_1^2 \left(\int_0^{3-r\sin t} \underbrace{r^2\cos^2 t \cdot r\sin t \cdot r}_{\text{neobsahuje } z} \, dz \right) dr \right) dt =$$

$$= \int_0^{2\pi} \left(\int_1^2 (3 - r\sin t) \cdot r^4 \sin t \cos^2 t \, dr \right) dt =$$

$$= \int_0^{2\pi} \left(\sin t \cos^2 t \cdot \int_1^2 (3r^4 - r^5 \sin t)\, dr \right) dt =$$

$$= \int_0^{2\pi} \sin t \cos^2 t \cdot \left[\frac{3r^5}{5} - \frac{r^6}{6} \cdot \sin t \right]_{r=1}^{2} dt =$$

$$= \int_0^{2\pi} \sin t \cos^2 t \cdot \left(\frac{93}{5} - \frac{21}{2} \sin t \right) dt =$$

$$= \frac{93}{5} \cdot \underbrace{\int_0^{2\pi} \sin t \cos^2 t \, dt}_{0} - \frac{21}{2} \int_0^{2\pi} \sin^2 t \cos^2 t \, dt =$$

subst.

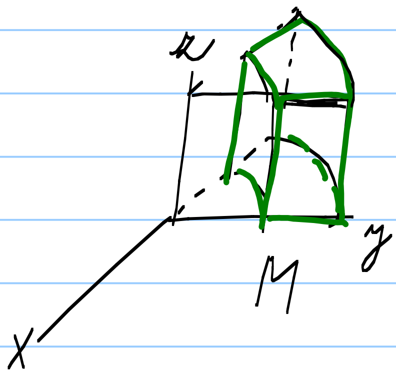
$$= \left| \begin{array}{l} \cos t = s \\ -\sin t \, dt = ds \\ 0 \mapsto 1, \; 2\pi \mapsto 1 \\ \Rightarrow \int_1^1 \text{(nulový)} \end{array} \right| = -\frac{21}{2} \cdot \int_0^{2\pi} \frac{1}{4} \sin^2 2t \, dt =$$

$$= -\frac{21}{16} \int_0^{2\pi} (1 - \cos 4t) \, dt = -\frac{21}{16} \cdot \left[t - \frac{1}{4} \sin 4t \right]_0^{2\pi} =$$

$$= -\frac{21}{16} \cdot 2\pi = \boxed{-\frac{21}{8} \pi}.$$

Př:

$$\iiint_M xz \cdot \sqrt{x^2 + y^2} \, dx\,dy\,dz, \quad \text{kde}$$



$$M = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 \le 4, \; 0 \le z \le 3, \; x \le 0, \; y \ge 0\}.$$

Cylindrické souř.

$$\begin{array}{l} x = r \cos t \\ y = r \sin t \\ z = z \end{array} \quad (J = r) \Rightarrow$$

$$\Rightarrow M: \quad \left. \begin{array}{l} \frac{\pi}{2} \le t \le \pi \\ 1 \le r \le 2 \\ 0 \le z \le 3 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \quad I = \int_{\frac{\pi}{2}}^{\pi} \left(\int_1^2 \left(\int_0^3 r \cos t \cdot z \cdot r \cdot r \, dz \right) dr \right) dt =$$

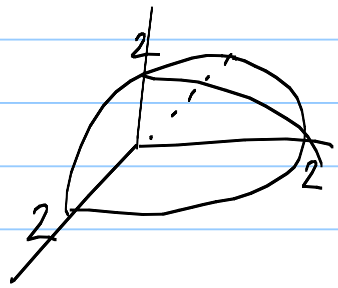
$$= \int_{\frac{\pi}{2}}^{\pi} \left(\int_1^2 r^3 \cos t \cdot \left[\frac{z^2}{2}\right]_0^3 dr \right) dt =$$

$$= \frac{9}{2} \int_{\frac{\pi}{2}}^{\pi} \cos t \cdot \left[\frac{r^4}{4}\right]_1^2 dt = \frac{9}{2}\cdot\frac{15}{4}\cdot \underbrace{\left[\sin t\right]_{\frac{\pi}{2}}^{\pi}}_{-1} =$$

$$= \boxed{-\frac{135}{8}}.$$

Transformace trojného integrálu (sférické souřadnice).

Př: $I = \iiint_M (x+y+z)\,dx\,dy\,dz$,



kde $M = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 4,\ y \ge 0,\ z \ge 0\}$.

Sférické souřadnice
$$x = \rho\cos\varphi\cos\vartheta$$
$$y = \rho\sin\varphi\cos\vartheta \qquad (J = \rho^2\cdot\cos\vartheta) \implies$$
$$z = \rho\sin\vartheta$$

$$\implies M: \left.\begin{array}{l} 0 \le \varphi \le \pi \\ 0 \le \vartheta \le \frac{\pi}{2} \\ 0 \le \rho \le 2 \end{array}\right\} \implies$$

$$\implies I = \int_0^{\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^2 \underbrace{\rho\cdot(\cos\varphi\cos\vartheta+\sin\varphi\cos\vartheta+\sin\vartheta)\cdot\rho^2\cos\vartheta}_{\rho^3\cdot\left(\cos\varphi\cos^2\vartheta+\sin\varphi\cos^2\vartheta+\sin\vartheta\cos\vartheta\right)}\,d\rho\right)d\vartheta\right)d\varphi$$

16

$$= \int_0^{\pi} \left(\int_0^{\frac{\pi}{2}} (\cos\varphi \cos^2\vartheta + \sin\varphi \cos^2\vartheta + \sin\vartheta\cos\vartheta) \cdot \left[\frac{\rho^4}{4} \right]_0^2 d\vartheta \right) d\varphi =$$

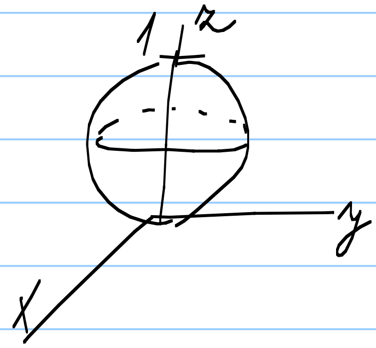
$$4 \cdot \int_0^{\pi} \left(\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\vartheta}{2} \cdot (\cos\varphi + \sin\varphi) + \frac{1}{2}\sin 2\vartheta \right) d\vartheta \right) d\varphi =$$

$$= 2 \cdot \int_0^{\pi} \left[\left(\vartheta + \frac{1}{2}\sin 2\vartheta\right)(\cos\varphi + \sin\varphi) - \frac{1}{2}\cos 2\vartheta \right]_{\vartheta=0}^{\frac{\pi}{2}} d\varphi =$$

$$= 2 \cdot \int_0^{\pi} \left(\frac{\pi}{2}(\cos\varphi + \sin\varphi) + 1 \right) d\varphi =$$

$$= \int_0^{\pi} (\pi \cdot (\cos\varphi + \sin\varphi) + 2)\, d\varphi = \left[\pi(\sin\varphi - \cos\varphi) + 2\varphi \right]_0^{\pi} =$$

$$= 3\pi + \pi = \boxed{4\pi.}$$



Př: $I = \iiint_M \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz,$

kde $M = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq z \}.$

Sférické souřadnice $\Rightarrow$

$\Rightarrow M:$ $0 \leq \varphi \leq 2\pi$ $\quad (J = \rho^2 \cdot \cos\vartheta)$

$0 \leq \vartheta \leq \frac{\pi}{2}$

$0 \leq \rho \leq$?? $\sin\vartheta$

$x^2+y^2+z^2 \leq z \iff \rho^2 \leq \rho \cdot \sin\vartheta$ $\Rightarrow$

$\rho \cdot (\rho - \sin\vartheta) \leq 0$

$$\Longrightarrow I = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^{\sin\vartheta} \rho\cdot\rho^2\cos\vartheta \, d\rho\right)d\vartheta\right)d\varphi =$$

$$= \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\cos\vartheta\left[\frac{\rho^4}{4}\right]_0^{\sin\vartheta}d\vartheta\right)d\varphi =$$

$$= \frac{1}{4}\cdot\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\cos\vartheta\cdot\sin^4\vartheta \, d\vartheta\right)d\varphi = \left|\begin{array}{l}\text{subst.}\\ \sin\vartheta = \alpha\\ \cos\vartheta \, d\vartheta = d\alpha\\ 0\mapsto 0,\ \frac{\pi}{2}\mapsto 1\end{array}\right| =$$

$$= \frac{1}{4}\int_0^{2\pi}\left(\int_0^1 \alpha^4 d\alpha\right)d\varphi = \frac{1}{4}\int_0^{2\pi}\left[\frac{\alpha^5}{5}\right]_0^1 d\varphi =$$

$$= \frac{1}{20}\int_0^{2\pi} 1 \, d\varphi = \boxed{\frac{\pi}{10}}.$$