Příklad 2.9. Drábek, Holušová: Parciální diferenciální rovnice, MI21

Klasifikujte rovnici

$$u_{xx} - 6u_{xy} + 12u_{yy} = 0.$$

Nalezněte transformaci nezávisle proměnných, která ji převede na Laplaceovu rovnici.

$$\left(\partial_x^2 - 6\partial_x\partial_y + 12\partial_y^2\right) u = \left[(\partial_x - 3\partial_y)^2 + \left(\sqrt{3}\partial_y\right)^2\right] u = 0$$

Toto je rovnost mezi funkcemi. Bude platit v každém bodě (x, y), tedy také v $(x(s,t),\, x(s,t))$:

$$\begin{aligned}
&\left[\left(\partial_x^2 - 6\partial_x\partial_y + 12\partial_y^2\right) u\right](x(s,t),\, x(s,t)) = \\
&\quad = \left[\left((\partial_x - 3\partial_y)^2 + \left(\sqrt{3}\partial_y\right)^2\right) u\right](x(s,t),\, x(s,t)) \\
&\quad = \left[\partial_s^2 + \partial_t^2 u^\star\right](s,t) = 0
\end{aligned}$$

Máme tedy rovnosti

$$\begin{aligned}
\partial_s u^\star &= \partial_x u \cdot 1 + \partial_y u \cdot (-3) \\
\partial_t u^\star &= \qquad\quad\;\; \partial_y u \cdot \sqrt{3}
\end{aligned} \tag{1}$$

Vidíme, že nám pomůže lineární substituce. Připomeňme (odvoďme) si obecné vztahy pro lineární substituci

$$\Psi : \mathbb{R}^2 \mapsto \mathbb{R}^2,\ (s,t) \mapsto (x,y) \qquad \begin{aligned} x &= d_{11}s + d_{12}t \\ y &= d_{21}s + d_{22}t \end{aligned}$$

Pro inverzní vztah platí, že je rovněž lineární a koeficienty pro přepočet lze získat inverzí matice:

$$\Psi^{-1} : \mathbb{R}^2 \mapsto \mathbb{R}^2,\ (x,y) \mapsto (s,t) \qquad \begin{aligned} s &= c_{11}x + c_{12}y \\ t &= c_{21}x + c_{22}y \end{aligned}$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^{-1} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

Pokud provedeme substituci, pracujeme vlastně se složenou funkcí

$$\begin{aligned}
u^\star &= u \circ \Psi \\
u^\star(s,t) &= [u \circ \Psi](s,t) = u\left(\Psi_1(s,t), \Psi_2(s,t)\right) = \\
&= u\left(x(s,t), y(s,t)\right)
\end{aligned}$$

$$
\begin{aligned}
\partial_s u^\star &= \partial_x u \cdot \partial_s \Psi_1 + \partial_y u \cdot \partial_s \Psi_2 = \partial_x u \cdot \partial_s x + \partial_y u \cdot \partial_s y \\
&= \partial_x u \cdot d_{11} + \partial_y u \cdot d_{21} \\
\partial_t u^\star &= \partial_x u \cdot \partial_t \Psi_1 + \partial_y u \cdot \partial_t \Psi_2 = \partial_x u \cdot \partial_t x + \partial_y u \cdot \partial_t y \\
&= \partial_x u \cdot d_{12} + \partial_y u \cdot d_{22}
\end{aligned} \tag{2}
$$

Porovnáme-li (1) s (2), máme $d_{11} = 1$, $d_{12} = 0$, $d_{21} = -3$ a $d_{22} = \sqrt{3}$. Hledaný vztah je tedy

$$
\begin{aligned}
x &= s \\
y &= -3s + \sqrt{3}t
\end{aligned}
$$

a jelikož

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 & 0 \\ -3 & \sqrt{3} & 0 & 1 \end{bmatrix} \ \mathrm{r}_2 + 3\mathrm{r}_1 &\mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \sqrt{3} & 3 & 1 \end{bmatrix} \ \sqrt{3}^{-1}\mathrm{r}_2 \\
&\mapsto \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \sqrt{3} & \sqrt{3}^{-1} \end{bmatrix}
\end{aligned}
$$

platí

$$
\begin{aligned}
s &= x \\
t &= \sqrt{3}s + \frac{1}{\sqrt{3}}t
\end{aligned}
$$

Jiné řešení:

$$
\begin{aligned}
u &= u^\star \circ \Psi^{-1} \\
u(x,y) &= \left[u^\star \circ \Psi^{-1}\right](x,y) = u^\star\left(s(x,y), t(x,y)\right) = u^\star\left(\left(\Psi^{-1}\right)_1(x,y), \left(\Psi^{-1}\right)_2(x,y)\right)
\end{aligned}
$$

$$
\begin{aligned}
\partial_x u &= \partial_s u^\star \cdot \partial_x \left(\Psi^{-1}\right)_1 + \partial_t u^\star \cdot \partial_x \left(\Psi^{-1}\right)_2 = \partial_s u^\star \cdot \partial_x s + \partial_t u^\star \cdot \partial_x t \\
&= \partial_s u^\star \cdot c_{11} + \partial_t u^\star \cdot c_{21} \\
\partial_y u &= \partial_s u^\star \cdot \partial_y \left(\Psi^{-1}\right)_1 + \partial_t u^\star \cdot \partial_y \left(\Psi^{-1}\right)_2 = \partial_s u^\star \cdot \partial_y s + \partial_t u^\star \cdot \partial_y t \\
&= \partial_s u^\star \cdot c_{12} + \partial_t u^\star \cdot c_{22} \\
\partial^2_{xx} u &= \left(\partial^2_{ss} u^\star \cdot \partial_x s + \partial^2_{st} u^\star \cdot \partial_x t\right) \partial_x s + \partial_s u^\star \cdot \partial^2_{xx} s + \left(\partial^2_{st} u^\star \cdot \partial_x s + \partial^2_{tt} u^\star \cdot \partial_x t\right) \partial_x t + \partial_t u^\star \cdot \partial^2_{xx} t \\
&= \partial^2_{ss} u^\star \cdot c_{11}^2 + \partial^2_{st} u^\star \cdot 2c_{11}c_{21} + \partial^2_{tt} u^\star \cdot c_{21}^2 \\
\partial^2_{xy} u &= \partial^2_{ss} u^\star \cdot c_{11}c_{12} + \partial^2_{st} u^\star \cdot (c_{11}c_{22} + c_{12}c_{21}) + \partial^2_{tt} u^\star \cdot c_{21}c_{22} \\
\partial^2_{yy} u &= \partial^2_{ss} u^\star \cdot c_{12}^2 + \partial^2_{st} u^\star \cdot 2c_{12}c_{22} + \partial^2_{tt} u^\star \cdot c_{22}^2
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_{xx}u - 6\partial^2_{xy}u + 12\partial^2_{yy} &= \partial^2_{ss}u^\star \cdot c_{11}^2 + \partial^2_{st}u^\star \cdot 2c_{11}c_{21} + \partial^2_{tt}u^\star \cdot c_{21}^2 + \\
&- 6\left(\partial^2_{ss}u^\star \cdot c_{11}c_{12} + \partial^2_{st}u^\star \cdot (c_{11}c_{22} + c_{12}c_{21}) + \partial^2_{tt}u^\star \cdot c_{21}c_{22}\right) + \\
&+ 12\left(\partial^2_{ss}u^\star \cdot c_{12}^2 + \partial^2_{st}u^\star \cdot 2c_{12}c_{22} + \partial^2_{tt}u^\star \cdot c_{22}^2\right) = \\
&= \partial^2_{ss}u^\star\left(c_{11}^2 - 6c_{11}c_{12} + 12c_{12}^2\right) + \\
&+ \partial^2_{st}u^\star \cdot 2\left(c_{11}c_{21} - 3\left(c_{11}c_{22} + c_{12}c_{21}\right) + 12c_{12}c_{22}\right) + \\
&+ \partial^2_{tt}u^\star\left(c_{21}^2 - 6c_{21}c_{22} + 12 \cdot c_{22}^2\right)
\end{aligned}
$$

Z převodu na Laplaceovu rovnici tedy máme vztahy

$$
\begin{aligned}
c_{11}^2 - 6c_{11}c_{12} + 12c_{12}^2 = \left(c_{11} - 3c_{12}\right)^2 + 3c_{12}^2 &= 1 \\
c_{11}c_{21} - 3\left(c_{11}c_{22} + c_{12}c_{21}\right) + 12c_{12}c_{22} &= 0 \\
c_{21}^2 - 6c_{21}c_{22} + 12 \cdot c_{22}^2 = \left(c_{21} - 3c_{22}\right)^2 + 3c_{22}^2 &= 1
\end{aligned}
$$

což jsou tři kvadratické rovnice o čtyřech neznámých. Zajímá nás jakékoli řešení.

- Zkusme tedy položit $c_{12} = 0$

  $$
  \begin{aligned}
  c_{11} &= 1 \\
  c_{11}\left(c_{21} - 3c_{22}\right) &= 0 \\
  \left(c_{21} - 3c_{22}\right)^2 + 3c_{22}^2 &= 1
  \end{aligned}
  $$

  Z druhé rovnice pak plyne $c_{21} = 3c_{22}$ a tento vztah dosadíme do třetího vztahu

  $$c_{22} = \frac{1}{\sqrt{3}},\ c_{21} = \sqrt{3},\ c_{11} = 1,\ c_{12} = 0$$

  Vidíme, že jsme obdrželi stejný vztah jako předchozím postupem.

- Zkusme naopak položit $c_{21} = 0$

  $$
  \begin{aligned}
  c_{11}^2 - 6c_{11}c_{12} + 12c_{12}^2 &= 1 \\
  -3\left(c_{11} - 4c_{12}\right)c_{22} &= 0 \\
  12 \cdot c_{22}^2 &= 1
  \end{aligned}
  $$

  Z druhé rovnice pak plyne $c_{11} = 4c_{12}$ a tento vztah dosadíme do prvního vztahu

  $$1 = 16c_{12}^2 - 24c_{12}^2 + 12c_{12}^2 = 4c_{12}^2$$

  V tomto případě vyšlo

  $$c_{22} = \frac{1}{\sqrt{12}},\ c_{21} = 0,\ c_{12} = \frac{1}{2},\ c_{11} = 2$$

  Vidíme, že možných substitucí, kterak upravit zadanou PDR na Laplaceovu rovnici je vícero.