CALCULATION OF SHOCK RESPONSE SPECTRUM

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Motivation

- Launch Vehicle Stage Separation Test

http://www.vibrationdata.com/SRS.htm
Outline

- Motivation
  - Research work for the Visteon Company
- Calculation Shock Response Spectrum
  - Single-Degree-of-Freedom Model
  - Transfer Functions for Relative Velocity and Displacement Response
  - Approximation of Continuous Transfer Function by Discrete Transfer Function
  - Approximation of the transfer function $G(s) = 1/(s + a)$ by $G(z) = (\beta_0 + \beta_1 z^{-1})/(1 + \alpha_1 z^{-1})$
  - Assumptions about the function of $x_1(t)$ in the time interval $[nT_S, (n+1)T_S]$
  - Ramp invariant method
  - Calculation SRS
  - SRS of A Half-Sine Impulse
  - Examples
  - Conclusion
Shock Response Spectrum

A set of SDOF systems - substructures

Natural frequencies for SDOF system

\[ f_{n1} < f_{n2} < f_{n3} < f_{n4} \quad \cdots \cdots \]

Amplitude of vibration

\[ \cdots \cdots \quad f_{nK} \]

Resonance gain \( Q = 10 \)

A base structure exposed by vibration transient
Single-Degree-of-Freedom Model

Equation of motion

\[ m \ddot{x}_2 = -c(\ddot{x}_2 - \dot{x}_1) - k(x_2 - x_1) \]

Laplace transform

\[
\begin{align*}
L\{x_1(t)\} &= X_1(s), & L\{v_1(t)\} &= V_1(s), & L\{a_1(t)\} &= A_1(s) \\
L\{x_2(t)\} &= X_2(s), & L\{v_2(t)\} &= V_2(s), & L\{a_2(t)\} &= A_2(s)
\end{align*}
\]

Transfer functions with parameters: mass, damping and stiffness

\[
G(s) = \frac{X_2(s)}{X_1(s)} = \frac{cs + k}{ms^2 + cs + k}, \quad G(j\omega) = \frac{X_2(j\omega)}{X_1(j\omega)} = \frac{j\omega c + k}{j\omega c + k - m\omega^2}
\]

\[
G(s) = \frac{V_2(s)}{V_1(s)} = \frac{cs + k}{ms^2 + cs + k}, \quad G(s) = \frac{A_2(s)}{A_1(s)} = \frac{cs + k}{ms^2 + cs + k}
\]

Transfer functions with parameters: natural frequency and resonance gain

\[
f_n = \frac{1}{2\pi} \sqrt{\frac{k}{m}}, \quad Q = \frac{\sqrt{km}}{c}, \quad \xi = \frac{1}{2Q} = \frac{c}{2\sqrt{km}} \quad \frac{A_2(s)}{A_1(s)} = \frac{\omega_n s + \omega_n^2}{Qs^2 + \frac{\omega_n s}{Q} + \omega_n^2}
\]
## Transfer Functions for Relative Velocity and Displacement Response

<table>
<thead>
<tr>
<th>Relative velocity response</th>
<th>Relative displacement response spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{V_2 - V_1}{A_1} = -\frac{ms}{ms^2 + cs + k} )</td>
<td>( \frac{X_2 - X_1}{A_1} = -\frac{m}{ms^2 + cs + k} )</td>
</tr>
</tbody>
</table>

Pseudo velocity response spectrum

<table>
<thead>
<tr>
<th>Relative displacement response spectrum expressed as equivalent static acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{X_2 - X_1}{A_1} \frac{\omega_n}{\omega_n} = -\frac{m\omega_n}{ms^2 + cs + k} )</td>
</tr>
</tbody>
</table>
Approximation of the Continuous Transfer Function by the Discrete Transfer Function

Continuous time signals
\[ x_2(t), v_2(t), a_2(t), x_1(t), v_1(t), a_1(t) \]
t ... Continuous time

Sampled time signals
\[ x_2[k], v_2[k], a_2[k], x_1[k], v_1[k], a_1[k] \]
k ... Index of a sample
\[ x_1[k] = x_1(k T_s) \]
\[ x_2[k] = x_2(k T_s) \]

Differential equation of motion
\[ m \ddot{x}_2 = -c(\dot{x}_2 - \dot{x}_1) - k(x_2 - x_1) \]

Difference equation
\[ x_2[k] = \beta_0 x_1[k] + \beta_1 x_1[k - 1] + \beta_2 x_1[k - 2] - \alpha_1 x_2[k - 1] - \alpha_2 x_2[k - 2] \]
\[ \beta_0, \beta_1, \beta_2, \alpha_1, \alpha_2 \ldots \text{parameters} \]

Decomposition into partial fractions
\[ \frac{A_2(s)}{A_1(s)} = \frac{\omega_n s^2 + \omega_n^2}{s^2 + \frac{\omega_n s}{Q} + \omega_n^2} \]
\[ H(z) = \frac{A_2(z)}{A_1(z)} = \frac{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \]
Approximation of the transfer function
\[ G(s) = \frac{1}{s + a} \] by
\[ G(z) = \frac{\beta_0 + \beta_1 z^{-1}}{1 + \alpha_1 z^{-1}} \]

The input signal is an unknown function except for a sequence of samples \( x_i[0], x_i[1], x_i[2], \ldots \)

Convolution integral
\[ x_2(t) = \int_0^t g(t - \tau)x_1(\tau) d\tau \]

Impulse response
\[ g(t) = L^{-1}\{G(s)\} = \exp(-at) \]

The output signal for \( t = nT_S \)
\[ x_2(nT_S + T_S) = \int_0^{nT_S + T_S} \exp(-a(nT_S + T_S - \tau))x_1(\tau) d\tau = \exp(-aT_S) \left( \int_0^{nT_S} \exp(-a(nT_S - \tau))x_1(\tau) d\tau + \int_{nT_S}^{nT_S + T_S} \ldots d\tau \right) \]
\[ = \exp(-aT_S)x_2(nT_S) + \exp(-aT_S) \int_0^{T_S} \exp(au)x_1(u + nT_S) du \]

The result depends on the assumption dealing with \( x_1(t) \) in between \( nT_S \) and \( (n+1)T_S \)
\[ x_2(nT_S + T_S) = \exp(-aT_S)x_2(nT_S) + \beta_0 x_1(nT_S + T_S) + \beta_1 x_1(nT_S) \]

\[ G(z) = \frac{X_2(z)}{X_1(z)} = \frac{\beta_0 + \beta_1 z^{-1}}{1 - \exp(-aT_S)z^{-1}} \]

Assumptions about the function of $x_1(t)$ in the time interval $[nT_S, (n+1)T_S]$.

### Impulse invariant

$$x_1(t) = x_1((n+1)T_S)\delta(t - (n+1)T_S)$$

$$\int_0^{T_S} \exp(a u)x_1(u + nT_S) du = \int_0^{T_S} \exp(a u)x_1(u + nT_S) \, du = x_1((n+1)T_S)\exp(a T_S)$$

### Step invariant

$$x_1(t) = x_1(nT_S), \quad nT_S \leq t < (n+1)T_S$$

$$\int_0^{T_S} \exp(a u)x_1(u + nT_S) du = \int_0^{T_S} \exp(a u)x_1(u + nT_S) \, du = x_1(nT_S)(\exp(a T_S) - 1)/a$$

### Centered step invariant

$$x_1(t) = \begin{cases} x_1(nT_S), & nT_S \leq t < (n+0.5)T_S \\ x_1((n+1)T_S), & (n+0.5)T_S \leq t < (n+1)T_S \end{cases}$$

### Ramp invariant

$$x_1(t) = x_1(nT_S) + \left(x_1((n+1)T_S) - x_1(nT_S)\right)(t - nT_S) / T_S, \quad nT_S \leq t < (n+1)T_S$$
Ramp Invariant Method

Digital filter
\[ H(z) = \frac{A_2(z)}{A_1(z)} = \frac{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \]

Coefficients
\[ \beta_0 = 1 - \exp(-A) \sin(B) / B \]
\[ \beta_1 = 2 \exp(-A) \left[ \sin(B) / B - \cos(B) \right] \]
\[ \beta_2 = \exp(-2A) - \exp(-A) \sin(B) / B \]
\[ \alpha_1 = -2 \exp(-A) \cos(B) \]
\[ \alpha_2 = \exp(-2A) \]
where
\[ A = \frac{\omega_n T_s}{2Q}, \quad B = \omega_n T_s \sqrt{1 - \frac{1}{4Q^2}} \]

Accuracy

Continuous system
\[ G(j\omega) = \frac{\omega_n}{j\omega / Q + \omega_n^2} \]

Bilinear transform with taking on account prewarping frequency
\[ f_s / 2 \]
Calculation SRS

Shock Response Spectrum

- Peak values of responses as a function of the natural frequency.

- A set of responses to the input signal.

- Substructures:
  - $f_{n1} = 10$ Hz
  - $f_{n2} = 100$ Hz
  - $f_{n3} = 1000$ Hz
  - $f_{n4} = 10000$ Hz

- Resonance gain $Q = 10$

- Input signal
SRS of A Half-Sine Impulse

Input impulse

Shock Response Spectrum

Peak values

Absolute value of SDOF-system responses

Peak values as a function of $f_n$
Signal Analyzer Script

‘Shock Response Spectrum’;
‘CrLf’;
ymax = []; ff = []; fmin = 1; fmax = 1000; n = 90;
qv = (fmax/fmin)^(1/n); T = 1/get(input1,'freq'); Q = 10;
for(i=0;i<n;i=i+1)
{
    fn = fmin*qv^i; ff = [ff,fn]; wn = 2*pi*fn;
    A = wn*T/2/Q;
    B = wn*T*sqr(1-1/4/Q/Q);
    b0 = 1-exp(-A)*sin(B)/B;
    b1 = 2*exp(-A)*(sin(B)/B-cos(B));
    b2 = exp((-2)*A)-exp(-A)*sin(B)/B;
    a1 = (-2)*exp((-1)*A)*cos(B);
    a2 = exp((-2)*A);
    BB = [b0,b1,b2];
    AA = [-a1,-a2];
    y = filter(input1,BB,AA);
    yy = max(y);
    ymax = [ymax,yy];
};
save(ff);
save(ymax);
Metal hammer and plastic shield
Examples: impact of metal to plastic

Acceleration during impact of a car light system and corresponding SRS

Acceleration during impact of a steel hammer to the plastic shield and corresponding SRS
Conclusion

- The paper presents the method of calculation the shock response spectrum, which is corresponding to an acceleration signal exciting the resonance vibration of substructures. SRS determines the maximum or minimum of the substructure acceleration response as a function of the natural frequencies of a set of the single-degree-of-freedom systems modeling the mentioned substructures.

- The vibration (or shock) is recorded in digital form, commonly as acceleration signal. The single-degree-of-freedom systems are approximated by an IIR digital filter and the filter response to the sampled acceleration signal may be easily calculated. This shock response spectrum shows how the individual component of the impulse signal excites the mechanical structure to resonate.