Semismooth Newton method for solving contact problems

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Outline

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Definition

Let Y, Z be Banach spaces (e.g., $Y = Z = \mathbb{R}^p$).
A function $F : Y \mapsto Z$ is said to be slantly differentiable at $y \in Y$, if

- $\exists F^o : Y \mapsto L(Y, Z)$:

$$\lim_{h \to 0} \frac{\|F(y+h) - F(y) - F^o(y+h)h\|}{\|h\|} = 0.$$

- F^o is uniformly bounded in an open neighbourhood of y.

F^o is said to be a slanting function for F at y.

Example

$F : \mathbb{R} \mapsto \mathbb{R}, y \mapsto \max\{0, y\}$

$$F^o(y) = \begin{cases} 0, & \text{if } y < 0 \\ 1, & \text{if } y > 0 \\ \sigma \in \mathbb{R}, & \text{if } y = 0 \end{cases}$$

Remarks

- A slanting function is a single valued function.
- A slantly differentiable function F has infinitely many slanting functions.
- A continuous function is not necessarily slantly differentiable ($F(x) = \operatorname{sign} x \sqrt{|x|}$).
- For locally Lipschitzian and semi-smooth function in finite dimensional spaces, the single valued selection of the Clark Jacobian serves a slanting function.

References

- R. Mifflin: Semismooth and semiconvex functions in constrained optimization. SIAM Control Optim. 15 (1977), pp. 959-972.
- X. Chen, Z. Nashed, L. Qi: Smoothing methods and semismooth methods for non-differentiable operator equations. SIAM Numer. Anal. 38 (2000), pp. 1200-1216.
- M. Hintermüller, K. Ito, K. Kunisch: The primal-dual active set strategy as a semismooth Newton method. SIAM Optim. 13 (2003), pp. 865-888.

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Problem

$$F(y) = 0, \quad \text{solution: } y^* \in U \subset Y$$

Newton iterations

$$y^{k+1} = y^k - \left[F^o(y^k)\right]^{-1} F(y^k)$$

Theorem

Let F be slantly differentiable in U with a slanting function F^o. If $\{\|F^o(y)^{-1}\| : y \in U\}$ is bounded by a constant $M \geq 0$, then the Newton iterations converge superlinearly to y^*, provided that $\|y^0 - y^*\|$ is sufficiently small.

Proof idea

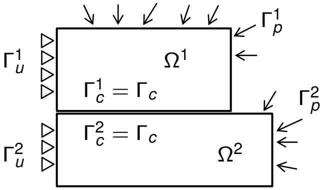
$$\begin{aligned}
y^{k+1} &= y^k - F^o(y^k)^{-1}F(y^k) \\
y^{k+1} - y^* &= y^k - y^* - F^o(y^k)^{-1}(F(y^k) - F(y^*)), \qquad h = y^k - y^* \\
y^{k+1} - y^* &= F^o(y^* + h)^{-1}\left(F^o(y^* + h)h - F(y^* + h) + F(y^*)\right) \\
\|y^{k+1} - y^*\| &\leq M\,\|F^o(y^* + h)h - F(y^* + h) + F(y^*)\| \\
\frac{\|y^{k+1} - y^*\|}{\|y^k - y^*\|} &\leq M\,\frac{\|F(y^* + h) - F(y^*) - F^o(y^* + h)h\|}{\|h\|}
\end{aligned}$$

- if it converges, then it converges super-linearly
- it converges, provided that $\|y^0 - y^*\|$ is sufficiently small

$$y^0 \in \mathcal{B}(y^*, r), \qquad r : \max_{\|h\| \leq r} \frac{M\,\|F(y^* + h) - F(y^*) - F^o(y^* + h)h\|}{\|h\|} = \eta < 1$$

Outline

PDEs



- two bodies Ω^1, Ω^2 in $\mathbb{R}^2$

$$\partial\Omega^k = \overline{\Gamma}_u^k \cup \overline{\Gamma}_p^k \cup \overline{\Gamma}_c^k$$

- zero displacement on Γ_u^k
- surface traction on Γ_p^k
- contact conditions on Γ_c

- Lamé PDEs in Ω^k, $k = 1, 2$:

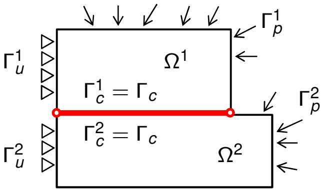
$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}^k) &= \boldsymbol{f}^k \\ \boldsymbol{\sigma}(\boldsymbol{u}^k) &= \lambda^k \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}^k))\boldsymbol{I} + 2\mu^k \boldsymbol{\varepsilon}(\boldsymbol{u}^k) \\ \boldsymbol{\varepsilon}(\boldsymbol{u}^k) &= \tfrac{1}{2}(\nabla \boldsymbol{u}^k + \nabla^\top \boldsymbol{u}^k) \end{aligned}$$

- Dirichlet and Neumann b.c.:

$$\left.\begin{aligned} \boldsymbol{u}^k &= \boldsymbol{0} \quad \text{on } \Gamma_u^k \\ \boldsymbol{\sigma}(\boldsymbol{u}^k)\boldsymbol{n}^k &= \boldsymbol{p}^k \quad \text{on } \Gamma_p^k \end{aligned}\right\} \quad k = 1, 2$$

$\boldsymbol{u}^k$ ... displacement, $\boldsymbol{\sigma}(\boldsymbol{u}^k)$ ... stress,
$\boldsymbol{f}^k$ ... volume force, $\boldsymbol{p}^k$ ... surface traction,
$\boldsymbol{n}^k$ ... outer normal vector, $\boldsymbol{n} = \boldsymbol{n}^1$,
$\lambda^k, \mu^k > 0$ ... material parameters

Contact conditions



- unilateral contact law on Γ_c:

$$u_n \leq 0,\ \sigma_n \leq 0,\ \sigma_n u_n = 0,$$

where $u_n = (\boldsymbol{u}^1 - \boldsymbol{u}^2)^\top \boldsymbol{n}$ and $\sigma_n = \boldsymbol{n}^\top \boldsymbol{\sigma}(\boldsymbol{u}^1)\boldsymbol{n}$.

- transmission of contact stresses:

$$\boldsymbol{\sigma}(\boldsymbol{u}^1)\boldsymbol{n} = \boldsymbol{\sigma}(\boldsymbol{u}^2)\boldsymbol{n} \quad \text{on} \quad \Gamma_c$$

- Coulomb friction law on Γ_c:

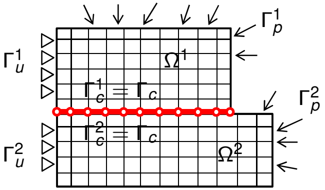
$$\begin{array}{lll} |\sigma_t| \leq -\mathcal{F}\sigma_n & & \\ |\sigma_t| < -\mathcal{F}\sigma_n & \Rightarrow & u_t = 0 \\ |\sigma_t| = -\mathcal{F}\sigma_n & \Rightarrow & \exists c_t \geq 0 :\ u_t = -c_t\sigma_t \end{array}$$

where $u_t = (\boldsymbol{u}^1 - \boldsymbol{u}^2)^\top \boldsymbol{t}$, $\sigma_t = \boldsymbol{t}^\top \boldsymbol{\sigma}(\boldsymbol{u}^1)\boldsymbol{n}$, $\boldsymbol{t}$ is orthogonal to $\boldsymbol{n}$, and $\mathcal{F} > 0$ is coefficient of friction.

- Tresca friction law on Γ_c:

$-\mathcal{F}\sigma_n$ is replaced by a-priori given $g \geq 0$

Discrete formulation



- FEM approximation with:

$(\mathbf{u}, \boldsymbol{\lambda}_n, \boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$

$\boldsymbol{\lambda}_n \approx -\sigma_n, \ \boldsymbol{\lambda}_t \approx -\sigma_t$

n dofs, m contact nodes,
$\mathbf{K} \in \mathbb{R}^{n\times n}$ positive definite,
$\mathbf{N}, \mathbf{T} \in \mathbb{R}^{m\times n}$ full row-rank,
$\mathbf{f} \in \mathbb{R}^n$ nodal forces.

- primal-dual discrete problem:

$$\mathbf{K}\mathbf{u} + \mathbf{N}^\top \boldsymbol{\lambda}_n + \mathbf{T}^\top \boldsymbol{\lambda}_t = \mathbf{f}$$

$$\mathbf{N}\mathbf{u} \le 0, \quad \boldsymbol{\lambda}_n \ge 0, \quad \boldsymbol{\lambda}_n^\top \mathbf{N}\mathbf{u} = 0$$

$$\begin{cases} |\lambda_{t,i}| \le \mathcal{F}\lambda_{n,i} \\ |\lambda_{t,i}| < \mathcal{F}\lambda_{n,i} \Rightarrow (\mathbf{T}\mathbf{u})_i = 0 \\ |\lambda_{t,i}| = \mathcal{F}\lambda_{n,i} \Rightarrow \exists c_i \ge 0 : \ (\mathbf{T}\mathbf{u})_i = c_{t,i}\lambda_{t,i} \end{cases}$$

for $i \in \mathcal{M} = \{1, \dots, m\}$

- existence and uniqueness:

Coulomb: The solution exists and is unique, if $\mathcal{F}$ sufficiently small (Haslinger '83).

Tresca: The solution exists and is unique.

Outline

NCP functions

- unilateral contact law: $\boxed{\mathbf{Nu} \leq 0,\ \boldsymbol{\lambda}_n \geq 0,\ \boldsymbol{\lambda}_n^\top \mathbf{Nu} = 0}$
- NCP function: $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$, continuous PC^1-function

$$\boxed{a \leq 0,\ b \geq 0,\ ab = 0 \iff \phi(a,b) = 0}$$

$$\phi(a,b) := b - \max\{0, b + \rho a\},\ \rho > 0 \text{ fixed}$$

by projection: $\boxed{\phi(a,b) := b - P_{\mathbb{R}_+}(b + \rho a)}$

- $\boxed{\boldsymbol{\lambda}_n - P_{\mathbb{R}_+^m}(\boldsymbol{\lambda}_n + \rho \boldsymbol{Nu}) = 0}$
- active set: $\mathcal{A}_n = \{i \in \mathcal{M} : \lambda_{n,i} + \rho(\mathbf{Nu})_i \geq 0\}, \quad \mathcal{I}_n = \mathcal{M} \setminus \mathcal{A}_n$
- indicator matrix for $\mathcal{S} \subseteq \mathcal{M}$: $\mathbf{D}_{\mathcal{S}} = diag(s_1, \ldots, s_m), \quad s_i = \begin{cases} 1 & i \in \mathcal{S} \\ 0 & i \notin \mathcal{S} \end{cases}$
- $\boxed{\boldsymbol{\lambda}_n - \mathbf{D}_{\mathcal{A}_n}(\boldsymbol{\lambda}_n + \rho \mathbf{Nu}) = 0}$

NCP functions

- NCP function for friction law: $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$, continuous PC^1-function

$$\left. \begin{array}{l} |b| \le g \\ |b| < g \Rightarrow a = 0 \\ |b| = g \Rightarrow \exists c \ge 0 : \; a = cb \end{array} \right\} \iff \psi(a,b) = 0$$

by projection: $\boxed{\psi(a,b) := b - P_{[-g,g]}(b + \rho a)}$ $\rho > 0$ fixed

- $\boxed{\boldsymbol{\lambda}_t - P_{[-\mathbf{g},\mathbf{g}]}(\boldsymbol{\lambda}_t + \rho \mathbf{T}\mathbf{u}) = 0}$
- active set: $\mathcal{A}_t = \mathcal{M} \setminus (\mathcal{I}_t^+ \cup \mathcal{I}_t^-)$

$$\mathcal{I}_t^+ = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i > g_i\}, \quad \mathcal{I}_t^- = \{i \in \mathcal{M} : \lambda_{t,i} + \rho(\mathbf{T}\mathbf{u})_i < -g_i\}$$

- $\boxed{\boldsymbol{\lambda}_t - \mathbf{D}_{\mathcal{A}_t}(\boldsymbol{\lambda}_t + \rho \mathbf{T}\mathbf{u}) - \mathbf{D}_{\mathcal{I}_t^+}\mathbf{g} + \mathbf{D}_{\mathcal{I}_t^-}\mathbf{g} = 0}$

NCP functions

- NCP functions for unilateral law:

$$\phi(a,b) := b - \max\{0, b + \rho a\}$$

$$\phi(a,b) := \max\{a, -b\}$$

$$\phi(a,b) := \min\{-a, b\}$$

$$\phi(a,b) := \sqrt{a^2 + b^2} - (b - a) \qquad \text{(Fischer-Burmeister)}$$

$$\phi(a,b) := ab + \frac{1}{2}\min{}^2\{0, b - a\} \qquad \text{(differentiable)}$$

D. Sun, L. Qi: On NCP-Functions. COA 13 (1999), pp. 201-220.

- NCP function for friction law:

$$\psi(a,b) := b - P_{[-g,g]}(b + \rho a)$$

$$\psi(a,b) := b - g\,\frac{b + \rho a}{\max\{g, |b + \rho a|\}} \qquad \text{(Christensen)}$$

$$\psi(a,b) := \max\{g, |b + \rho a|\} b - g(b + \rho a) \qquad \text{(Wolmuth)}$$

Outline

Projective formulations

- pd-formulation by NCP functions:

$$F : \mathbb{R}^{n+2m} \mapsto \mathbb{R}^{n+2m}$$

$$F(\mathbf{y}) = \begin{pmatrix} \mathbf{Ku} + \mathbf{N}^\top \boldsymbol{\lambda}_n + \mathbf{T}^\top \boldsymbol{\lambda}_t - \mathbf{f} \\ \boldsymbol{\lambda}_n - P_{\mathbb{R}^m_+}(\boldsymbol{\lambda}_n + \rho \mathbf{Nu}) \\ \boldsymbol{\lambda}_t - P_{[-\mathbf{g},\mathbf{g}]}(\boldsymbol{\lambda}_t + \rho \mathbf{Tu}) \end{pmatrix},$$

where $\mathbf{y} = (\mathbf{u}^\top, \boldsymbol{\lambda}_n^\top, \boldsymbol{\lambda}_t^\top)^\top$.

- i.e., formulation as one equation:

$$F(\mathbf{y}) = 0$$

F is non-differentiable PC^1-function.

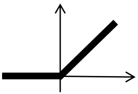
Tresca: $\mathbf{g} \in \mathbb{R}^m_+$ is constant

Coulomb: $\mathbf{g} := \mathcal{F} P_{\mathbb{R}^m_+}(\boldsymbol{\lambda}_n + \rho \mathbf{Nu})$

Alart-Curnier (1991)

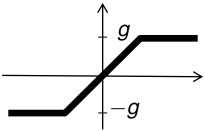
- projection $P_{\mathbb{R}_+} : \mathbb{R} \mapsto \mathbb{R}_+$

$$P_{\mathbb{R}_+}(x) = \max\{0, x\}$$



- projection $P_{[-g,g]} : \mathbb{R} \mapsto [-g, g]$

$$P_{[-g,g]}(x) = \max\{0, x+g\} - \max\{0, x-g\} - g$$



Active-set formulation

- inactive/active sets at $\boldsymbol{y} \in \mathbb{R}^{n+2m}$

$$\mathcal{A}_n,\ \mathcal{I}_n,\ \mathcal{I}_t^+,\ \mathcal{I}_t^-,\ \mathcal{A}_t$$

$$\mathbf{D}_{\mathcal{A}_n}, \mathbf{D}_{\mathcal{I}_n}, \mathbf{D}_{\mathcal{A}_t}, \mathbf{D}_{\mathcal{I}_t^+}, \mathbf{D}_{\mathcal{I}_t^-} \in \mathbb{R}^{m\times m}$$

$$\mathbf{D}_{\mathcal{A}_n} + \mathbf{D}_{\mathcal{I}_n} = \mathbf{I}$$

$$\mathbf{D}_{\mathcal{A}_t} + \mathbf{D}_{\mathcal{I}_t^+} + \mathbf{D}_{\mathcal{I}_t^-} = \mathbf{I}$$

- $F(\boldsymbol{y})$ by indicator matrices

$$F(\mathbf{y}) = \begin{pmatrix} \mathbf{K}\mathbf{u} + \mathbf{N}^\top \boldsymbol{\lambda}_n + \mathbf{T}^\top \boldsymbol{\lambda}_t - \mathbf{f} \\ \boldsymbol{\lambda}_n - \mathbf{D}_{\mathcal{A}_n}(\boldsymbol{\lambda}_n + \rho\mathbf{N}\mathbf{u}) \\ \boldsymbol{\lambda}_t - \mathbf{D}_{\mathcal{A}_t}(\boldsymbol{\lambda}_t + \rho\mathbf{T}\mathbf{u}) - \dots \\ \dots - \mathbf{D}_{\mathcal{I}_t^+}\mathbf{g} + \mathbf{D}_{\mathcal{I}_t^-}\mathbf{g} \end{pmatrix}$$

$\mathbf{g}$ constant or $\mathbf{g} := \mathcal{F}\mathbf{D}_{\mathcal{A}_n}(\boldsymbol{\lambda}_n + \rho\mathbf{N}\mathbf{u})$

- slanting function for Tresca friction

$$F^o(\mathbf{y}) = \left(\begin{array}{c|c|c} \mathbf{K} & \mathbf{N}^\top & \mathbf{T}^\top \\ \hline -\rho\mathbf{D}_{\mathcal{A}_n}\mathbf{N} & \mathbf{D}_{\mathcal{I}_n} & \mathbf{0} \\ \hline -\rho\mathbf{D}_{\mathcal{A}_t}\mathbf{T} & \mathbf{0} & \mathbf{D}_{\mathcal{I}_t^+\cup\mathcal{I}_t^-} \end{array}\right)$$

- slanting function for Coulomb friction

$$F^o(\mathbf{y}) = \left(\begin{array}{c|c|c} \mathbf{K} & \mathbf{N}^\top & \mathbf{T}^\top \\ \hline -\rho\mathbf{D}_{\mathcal{A}_n}\mathbf{N} & \mathbf{D}_{\mathcal{I}_n} & \mathbf{0} \\ \hline -\rho\mathbf{D}_{\mathcal{A}_t}\mathbf{T} & \mathcal{F}\mathbf{D}_{\mathcal{A}_n}(\mathbf{D}_{\mathcal{I}_t^-} - \mathbf{D}_{\mathcal{I}_t^+}) & \mathbf{D}_{\mathcal{I}_t^+\cup\mathcal{I}_t^-} \end{array}\right)$$

Algorithms (primal-dual)

- Newton iterations: $\mathbf{y}^{(0)}$ given, for $k = 1, 2, \ldots$

$$\begin{aligned} \mathbf{y}^{(k)} &= \mathbf{y}^{(k-1)} - \left[F^o(\mathbf{y}^{(k-1)})\right]^{-1} F(\mathbf{y}^{(k-1)}) \\ F^o(\mathbf{y}^{(k-1)})\mathbf{y}^{(k)} &= F^o(\mathbf{y}^{(k-1)})\mathbf{y}^{(k-1)} - F(\mathbf{y}^{(k-1)}) \end{aligned}$$

- Algorithm SSNM (Coulomb): Let $\mathbf{y}^{(0)} \in \mathbb{R}^{n+2m}$, $\rho > 0$. For $k = 1, 2, \ldots$:

  (i) Define the active/inactive sets at $\mathbf{y}^{(k-1)}$ for $\mathbf{g} = \mathcal{F}(\boldsymbol{\lambda}_n^{(k-1)} + \rho \mathbf{N}\mathbf{u}^{(k-1)})_+$

  (ii) Compute $\mathbf{y}^{(k)}$ by solving:

$$\left(\begin{array}{c|c|c} \mathbf{K} & \mathbf{N}^\top & \mathbf{T}^\top \\ \hline -\rho \mathbf{D}_{\mathcal{A}_n}\mathbf{N} & \mathbf{D}_{\mathcal{I}_n} & \mathbf{0} \\ \hline -\rho \mathbf{D}_{\mathcal{A}_t}\mathbf{T} & \mathcal{F}\mathbf{D}_{\mathcal{A}_n}(\mathbf{D}_{\mathcal{I}_t^-} - \mathbf{D}_{\mathcal{I}_t^+}) & \mathbf{D}_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} \end{array}\right) \begin{pmatrix} \mathbf{u}^{(k)} \\ \boldsymbol{\lambda}_n^{(k)} \\ \boldsymbol{\lambda}_t^{(k)} \end{pmatrix} = \left(\begin{array}{c} \mathbf{f} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \end{array}\right)$$

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Size of the problem n/m: 24960/384
Tolerance/iteration: 1e-6/50
------------------------------------------------
Iter - err -- resid -- An:In/Itp:Itm:At
------------------------------------------------
01 000000001 6.40e-013 0:192/0:0:192
02 2.91e+000 3.52e-012 192:0/88:104:0
03 8.77e-001 1.45e-013 160:32/91:29:101
04 5.01e-001 2.61e-013 145:47/107:30:73
05 3.57e-001 4.14e-013 130:62/121:29:57
06 2.59e-001 5.80e-013 117:75/133:27:45
07 1.92e-001 8.33e-013 105:87/143:23:38
08 1.14e-001 1.14e-012 96:96/151:19:31
09 7.52e-002 1.48e-012 88:104/158:16:26
10 3.32e-002 1.72e-012 83:109/163:12:22
11 1.73e-002 1.93e-012 79:113/166:10:20
12 5.52e-003 1.99e-012 77:115/168:8:18
13 3.13e-003 2.05e-012 75:117/169:8:17
14 7.34e-004 2.03e-012 74:118/170:7:16
15 1.11e-004 2.07e-012 73:119/171:7:15

16 000000000 2.07e-012 73:119/171:6:15

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Convergence rate

- linear rate

$$\exists \eta,\ 0 < \eta < 1: \quad \|\mathbf{y}^{(k+1)} - \mathbf{y}^*\| \leq \eta \|\mathbf{y}^{(k)} - \mathbf{y}^*\|$$

- superlinear rate

$$\lim_{k \to \infty} \frac{\|\mathbf{y}^{(k+1)} - \mathbf{y}^*\|}{\|\mathbf{y}^{(k)} - \mathbf{y}^*\|} = 0$$

$$(\forall \varepsilon > 0)\ (\exists k_0 \in \mathbb{N}):\ k \geq k_0: \ \|\mathbf{y}^{(k+1)} - \mathbf{y}^*\| \leq \varepsilon \|\mathbf{y}^{(k)} - \mathbf{y}^*\|$$

- qudratic rate

$$\exists c > 0: \quad \|\mathbf{y}^{(k+1)} - \mathbf{y}^*\| \leq c \|\mathbf{y}^{(k)} - \mathbf{y}^*\|^2$$

Algorithms (primal-dual)

$$\left(\begin{array}{c|c|c} \mathbf{K} & \mathbf{N}^\top & \mathbf{T}^\top \\ \hline \mathbf{D}_{\mathcal{A}_n}\mathbf{N} & \mathbf{D}_{\mathcal{I}_n} & \mathbf{0} \\ \hline \mathbf{D}_{\mathcal{A}_t}\mathbf{T} & \mathbf{0} & \mathbf{D}_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} \end{array}\right) \begin{pmatrix} \mathbf{u}^{(k)} \\ \boldsymbol{\lambda}_n^{(k)} \\ \boldsymbol{\lambda}_t^{(k)} \end{pmatrix} = \left(\begin{array}{c} \mathbf{f} \\ \hline \mathbf{0} \\ \hline (\mathbf{D}_{\mathcal{I}_t^+} - \mathbf{D}_{\mathcal{I}_t^-})\mathbf{g} \end{array}\right)$$

- Algorithm SSNM (Tresca): Let $\mathbf{y}^{(0)} \in \mathbb{R}^{n+2m}$, $\rho > 0$. For $k = 1, 2, \ldots$:
  - (i) Define the active/inactive sets at $\mathbf{y}^{(k-1)}$ for $\mathbf{g}$ constant.
  - (ii) Compute $\mathbf{y}^{(k)}$ by solving:

$$\left(\begin{array}{c|c|c} \mathbf{K} & \mathbf{N}_{\mathcal{A}_n}^\top & \mathbf{T}_{\mathcal{A}_t}^\top \\ \hline \mathbf{N}_{\mathcal{A}_n} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{T}_{\mathcal{A}_t} & \mathbf{0} & \mathbf{0} \end{array}\right) \begin{pmatrix} \mathbf{u}^{(k)} \\ \boldsymbol{\lambda}_{n,\mathcal{A}_n}^{(k)} \\ \boldsymbol{\lambda}_{t,\mathcal{A}_t}^{(k)} \end{pmatrix} = \left(\begin{array}{c} \widetilde{\mathbf{f}} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \end{array}\right)$$

and setting: $\boldsymbol{\lambda}_{n,\mathcal{I}_n}^{(k)} = \mathbf{0}$, $\boldsymbol{\lambda}_{t,\mathcal{I}_t{}^+}^{(k)} = \mathbf{g}_{\mathcal{I}_t{}^+}$, $\boldsymbol{\lambda}_{t,\mathcal{I}_t{}^-}^{(k)} = -\mathbf{g}_{\mathcal{I}_t{}^-}$,

Algorithms (dual)

$$\mathbf{u}^{(k)} = \mathbf{K}^{-1}(\widetilde{\mathbf{f}} - \mathbf{B}_{\mathcal{A}}\boldsymbol{\lambda}_{\mathcal{A}}^{(k)}), \quad \mathbf{B}_{\mathcal{A}} = \begin{pmatrix} \mathbf{N}_{\mathcal{A}_n} \\ \mathbf{T}_{\mathcal{A}_t} \end{pmatrix}, \quad \boldsymbol{\lambda}_{\mathcal{A}}^{(k)} = \begin{pmatrix} \boldsymbol{\lambda}_{n,\mathcal{A}_n}^{(k)} \\ \boldsymbol{\lambda}_{t,\mathcal{A}_t}^{(k)} \end{pmatrix}$$

$$q(\boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{\lambda}^\top \mathbf{F}\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top \mathbf{d}, \quad \mathbf{F} = \mathbf{B}\mathbf{K}^{-1}\mathbf{B}^\top, \quad \mathbf{d} = \mathbf{B}\mathbf{K}^{-1}\widetilde{\mathbf{f}}, \quad \text{with} \quad \mathcal{A}_n = \mathcal{A}_t = \mathcal{M}$$

- Algorithm SSNM (Tresca): Let $\boldsymbol{\lambda}^{(0)} \in \mathbb{R}^{2m}$, $\rho > 0$. For $k = 1, 2, \ldots$:
  - (i) Define the active/inactive sets at $\boldsymbol{\lambda}^{(k-1)}$ for $\mathbf{g}$ constant.
  - (ii) $\boldsymbol{\lambda}^{(k)} := \text{argmin}\ q(\boldsymbol{\lambda})$, s.t. $\boldsymbol{\lambda}_{n,\mathcal{I}_n}^{(k)} = \mathbf{0}$, $\boldsymbol{\lambda}_{t,\mathcal{I}_t^+}^{(k)} = \mathbf{g}_{\mathcal{I}_t^+}$, $\boldsymbol{\lambda}_{t,\mathcal{I}_t^-}^{(k)} = -\mathbf{g}_{\mathcal{I}_t^-}$
- active/inactive sets without $\mathbf{u}$: $\mathbf{r} = \mathbf{F}\boldsymbol{\lambda} - \mathbf{d} = \mathbf{B}\mathbf{K}^{-1}(\mathbf{B}^\top\boldsymbol{\lambda} - \mathbf{f}) = -\mathbf{B}\mathbf{u}$

$$\mathcal{A}_n = \{i \in \mathcal{M} : \lambda_{n,i} - \rho r_i \geq 0\}$$

$$\mathcal{I}_t^+ = \{i \in \mathcal{M} : \lambda_{t,i} - \rho r_{i+m} > g_i\}, \quad \mathcal{I}_t^- = \{i \in \mathcal{M} : \lambda_{t,i} - \rho r_{i+m} < -g_i\}$$

Remarks

- sequence $\mathbf{y} = \mathbf{y}^{(k)}$ of the Dirichlet-Neumann type problems on Γ_c:

$$(\mathbf{Nu})_i = 0,\ i \in \mathcal{A}_n, \quad (\mathbf{Tu})_i = 0,\ i \in \mathcal{A}_t,$$

$$\lambda_{n,i} = 0,\ i \in \mathcal{I}_n, \quad \lambda_{t,i} = g_i,\ i \in \mathcal{I}_t^+, \quad \lambda_{t,i} = -g_i,\ i \in \mathcal{I}_t^-$$

- ρ is (may be) discarded from the inner linear systems
- if the inner linear systems are solved exactly, ρ is discarded also from $\mathcal{A}_n$, $\mathcal{I}_n$ and partially from $\mathcal{A}_t$, $\mathcal{I}_t^+$, $\mathcal{I}_t^-$
- if the inner linear systems are solved inexactly, then ρ plays a significant role

$$\rho \approx \alpha_{\max}(\mathbf{F})^{-1}$$

Globalization strategy: Tresca, dual

- constrained minimization: $\min q(\boldsymbol{\lambda})$, s.t. $\boldsymbol{\lambda}_n \geq 0$, $|\boldsymbol{\lambda}_t| \leq \mathbf{g}$
- dual Algorithm SSNM (Tresca): restarted CGM
- globalization idea: decreasing sequence $\{q(\boldsymbol{\lambda}^{(k,j)})\}$ for all iterations
- natural restart: $\boldsymbol{\lambda}^{(k+1,0)}_{\mathcal{A}_n} = \boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n}$ and $\boldsymbol{\lambda}^{(k+1,0)}_{\mathcal{I}_n} = 0$

$$\begin{aligned} q(\boldsymbol{\lambda}_n^{(k)}) - q(\boldsymbol{\lambda}_n^{(k+1,0)}) &= (\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n})^\top \mathbf{r}^{(k)}_{\mathcal{A}_n} - \frac{1}{2}(\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n})^\top \mathbf{F}_{\mathcal{A}_n\mathcal{A}_n}\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n} \\ &\geq (\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n})^\top \mathbf{r}^{(k)}_{\mathcal{A}_n} - \frac{1}{2}\alpha_{\max}(\mathbf{F})\|\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n}\|^2 \\ &> \left(\rho^{-1} - \frac{1}{2}\alpha_{\max}(\mathbf{F})\right)\|\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n}\|^2 \;> 0 \implies \boxed{\rho < 2\alpha_{\max}(\mathbf{F})^{-1}} \end{aligned}$$

since e.g.: $0 \leq \boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n} < \rho\mathbf{r}^{(k)}_{\mathcal{A}_n} \implies 0 \leq \|\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n}\|^2 < \rho(\boldsymbol{\lambda}^{(k)}_{\mathcal{A}_n})^\top \mathbf{r}^{(k)}_{\mathcal{A}_n}$

Globalization strategy: Tresca, dual

- two requirements for the globalization strategy:

$$\boxed{\textit{(i)}\ \text{feasible iterations,}\quad \textit{(ii)}\ \rho \in (0, 2\|\mathbf{F}\|^{-1})}$$

- interpretation of the natural restart: $\boldsymbol{\lambda}_{\mathcal{A}_n}^{(k+1,0)} = \boldsymbol{\lambda}_{\mathcal{A}_n}^{(k)}$ and $\boldsymbol{\lambda}_{\mathcal{I}_n}^{(k+1,0)} = 0$

$$\lambda_{\mathcal{A}_n}^{(k+1,0)} = 0 = P_{\mathbb{R}_+^m}(\lambda_{\mathcal{A}_n}^{(k)} - \rho r_{\mathcal{A}_n}^{(k)}) = \lambda_{\mathcal{A}_n}^{(k)} - \rho \widetilde{r}_{\mathcal{A}_n}^{(k)}$$

  i.e., it is the step by the projected gradient $\widetilde{r}_{\mathcal{A}_n}^{(k)}$ with respect $\mathcal{A}_n$ (proportioning)
- Dostál, Z., Kučera, R.: An optimal algorithm for minimization of quadratic functions with bounded spectrum subject to separable convex inequality and linear equality constraints. SIAM Optim., 20(2010), pp. 2913-2938.
- Kučera, R.: Convergence rate of an optimization algorithm for minimizing quadratic functions with separable convex constraints. SIAM Optim., 19(2008), 2, pp. 846-862.

Outline

Terminating criterions

- outer terminating criterion:

$$\mathrm{err}^{(k)} = \frac{\|\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^{(k-1)}\|}{\|\boldsymbol{\lambda}^{(k)}\|} \leq \varepsilon$$

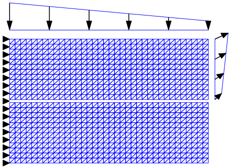
- inner linear systems are solved by CGM or BiCGSTAB with the adaptive inner terminating tolerance:

$$\mathrm{tol}^{(k)} = \min\{r_{tol} \times \mathrm{err}^{(k-1)}, c_{fact} \times \mathrm{tol}^{(k-1)}\},$$

where $0 < r_{tol} < 1$, $0 < c_{fact} < 1$

Experiments

- problem with the Tresca friction



- efficiency

	MPRGP	adaptive SSNM	globalization
n/m	n_A	$iter/n_A$	$iter/n_A$
30600/150	181	7/19	56/95
59640/210	213	11/35	72/119
109440/285	246	11/35	74/127
146520/330	255	13/46	93/156
204360/390	281	13/49	87/151
271800/450	302	14/55	85/149

Outline

SSNM+TFETI: projective formulations

- gluing matrix $\mathbf{B}_e \in \mathbb{R}^{m_e \times n}$

$$F : \mathbb{R}^{n+2m+m_e} \mapsto \mathbb{R}^{n+2m+m_e}, \quad \mathbf{y} = \left(\mathbf{u}^\top, \boldsymbol{\lambda}_n^\top, \boldsymbol{\lambda}_t^\top, \boldsymbol{\lambda}_e^\top\right)^\top$$

$$F(\mathbf{y}) := \begin{pmatrix} \mathbf{K}\mathbf{u} + \mathbf{N}^\top \boldsymbol{\lambda}_n + \mathbf{T}^\top \boldsymbol{\lambda}_t + \mathbf{B}_e^\top \boldsymbol{\lambda}_e - \mathbf{f} \\ \boldsymbol{\lambda}_n - P_{\mathbb{R}_+^m}(\boldsymbol{\lambda}_n + \rho \mathbf{N}\mathbf{u}) \\ \boldsymbol{\lambda}_t - P_{[-\mathbf{g},\mathbf{g}]}(\boldsymbol{\lambda}_t + \rho \mathbf{T}\mathbf{u}) \\ \mathbf{B}_e \mathbf{u} \end{pmatrix} = 0$$

- $\mathbf{K}$ is block-diagonal, symmetric positive semidefinite and singular
- generalized inverse $\mathbf{K}^+ \in \mathbb{R}^{n \times n}$ and the kernel matrix $\mathbf{R} \in \mathbb{R}^{n \times l}$ are needed

SSNM+TFETI: slanting function

- slanting function for Coulomb friction

$$F^o(\mathbf{y}) = \left(\begin{array}{c|c|c|c} \mathbf{K} & \mathbf{N}^\top & \mathbf{T}^\top & \mathbf{B}_e^\top \\ \hline -\rho \mathbf{D}_{\mathcal{A}_n}\mathbf{N} & \mathbf{D}_{\mathcal{I}_n} & \mathbf{0} & \mathbf{0} \\ \hline -\rho \mathbf{D}_{\mathcal{A}_t}\mathbf{T} & \mathcal{F}\mathbf{D}_{\mathcal{A}_n}(\mathbf{D}_{\mathcal{I}_t^-} - \mathbf{D}_{\mathcal{I}_t^+}) & \mathbf{D}_{\mathcal{I}_t^+ \cup \mathcal{I}_t^-} & \mathbf{0} \\ \hline \mathbf{B}_e & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right)$$

- symplified notation:

$$F^o(\mathbf{y}) = \left(\begin{array}{c|c} \mathbf{K} & \mathbf{B}^\top \\ \hline \mathbf{B}_{\mathcal{A}} & -\mathbf{D} \end{array}\right) \implies \mathcal{S} = \left(\begin{array}{c|c} \mathbf{F} & \mathbf{G}_{\mathcal{A}}^\top \\ \hline \mathbf{G} & \mathbf{0} \end{array}\right) \qquad \begin{array}{l} \mathbf{F} = \mathbf{B}_{\mathcal{A}}\mathbf{K}^+\mathbf{B} + \mathbf{D} \\ \mathbf{G} = -\mathbf{R}^\top\mathbf{B}^\top,\ \mathbf{G}_{\mathcal{A}} = -\mathbf{R}^\top\mathbf{B}_{\mathcal{A}}^\top \end{array}$$

- two projectors: $\mathbf{P}_{\mathcal{A}} = \mathbf{I} - \mathbf{G}_{\mathcal{A}}^\top(\mathbf{G}_{\mathcal{A}}\mathbf{G}_{\mathcal{A}}^\top)^{-1}\mathbf{G}_{\mathcal{A}}$, $\quad \mathbf{P} = \mathbf{I} - \mathbf{G}^\top(\mathbf{G}\mathbf{G}^\top)^{-1}\mathbf{G}$
- projected equation: $\mathbf{P}_{\mathcal{A}}\mathbf{F}\boldsymbol{\lambda} = \mathbf{P}_{\mathcal{A}}\mathbf{d}$, $\quad \boldsymbol{\lambda} \in Ker\,\mathbf{G}$ (ProjBiCGTAB, e.g.)

SSNM+TFETI: experiments

- Tresca friction

H/h	2	4	6	8	10	12
N_x	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$
10	7/55	13/361	16/710	8/62	15/1559	20/802
15	7/61	14/408	14/360	8/72	13/135	8/98
20	15/181	19/483	20/244	8/80	20/2098	19/2261
25	11/139	12/90	17/869	8/92	15/1271	13/155

- Coulomb friction

H/h	2	4	6	8	10	12
N_x	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$	$iter/n_A$
10	13/159	14/248	15/406	20/597	17/891	8/356
15	12/178	14/318	16/615	17/811	16/1837	19/1792
20	12/185	17/457	18/829	15/1294	15/1717	19/2127
25	14/211	13/526	19/1353	14/1161	16/1859	17/2734