Topological Complexity III

Piotr Oprocha

AGH University of Science and Technology
Faculty of Applied Mathematics
Kraków, Poland

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Disjointness and weak disjointness

1. (X, f), (Y, g) are weakly disjoint if $(X \times Y, f \times g)$ is transitive
2. (X, f), (Y, g) are disjoint if the only joining,
    - i.e. $J \subset X \times Y$ invariant for $f \times g$
    - with projections X and Y on respective coordinates

   is $J = X \times Y$.
3. (X, f) can be weakly disjoint with itself (e.g. weak mixing) but not disjoint (Δ is a joining).
4. if (X, f) and (Y, g) are disjoint then one of them is minimal.

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Disjointness, distality and minimality

Theorem (Furstenberg, 1967)

If (X, f) is weakly mixing and (Y, g) is minimal and distal then they are disjoint.

Theorem (Petersen, 1970)

If (X, f) disjoint with all distal systems if and only if it is minimal and weakly mixing.

Theorem (Blanchard, Host, Mass, 2000)

If (X, f) is scattering and (Y, g) is minimal and distal then they are disjoint.

Question

Can the above be proved without Furstenberg's structure theorem of distal systems?

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Related question

Theorem

If transitive (X, f) disjoint with all minimal systems then it is weakly mixing.

Theorem

If (X, f) is weakly mixing and has dense distal points then it is disjoint with all minimal systems.

Question (Furstenberg, 1967)

What is exactly the class of (transitive) systems disjoint with all minimal systems?

Local aspects of complexity

1. a standard cover $\mathcal{C} = \{C, D\}$ separates points $x, y \in X$ if $x \in \operatorname{Int} C^c$ and $y \in \operatorname{Int} D^c$, where $A^c = X \setminus A$.
2. points $x \neq y$ are a complexity pair if $c(\mathcal{C}, \cdot)$ is unbounded for any standard cover $\mathcal{C}$ separating x, y.
3. $\mathrm{Com}(X, f)$ – set of complexity pairs
4. $\mathrm{Com}(X, f) \cup \Delta$ is closed and $f \times f$ invariant.
5. if $\mathcal{U} = \{U, V\}$ is a standard cover with unbounded complexity, then $(U^c \times V^c) \cap \mathrm{Com}(X, f) \neq \emptyset$
6. if $\pi\colon (X, f) \to (Y, g)$ is a factor map then:
    1. if $x, y \in \mathrm{Com}(X, f)$ with $\pi(x) \neq \pi(y)$ then $(\pi(x), \pi(y)) \in \mathrm{Com}(Y, g)$,
    2. if $x, y \in \mathrm{Com}(Y, g)$ then $\pi^{-1}(\{x\}) \times \pi^{-1}(\{y\}) \cap \mathrm{Com}(X, f) \neq \emptyset$.

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Complexity pairs and maximal equicontinuous factor

1. Every dynamical system (X, f) posses an equicontinuous factor (Y, g) (and factor map $\pi\colon (X, f) \to (Y, g)$) (called maximal equicontinous factor), such that
   - for any equicontinuous (Z, h) and factor map $\phi\colon (X, f) \to (Z, h)$
   - there is factor $\psi : (Y, g) \to (Z, h)$ such that $\phi = \psi \circ \pi$.
2. maximal equicontinuous factor is unique up to conjugacy.

Theorem

Let R be the smallest ICER (invariant, closed equivalence relation) such that $\mathrm{Com}(X, f) \subset R$*. Then* $\pi\colon (X, f) \to (X/R, f/R)$ *is maximal equicontinuous factor.*

Regionally proximal relation

1. $x, y \in \mathrm{RP}(X, f)$ if for every
   - open neighborhoods $U \ni x$, $V \ni y$
   - $\varepsilon > 0$

   there are $n > 0$ and $u \in U$, $v \in V$ such that $d(f^n(u), f^n(v)) < \varepsilon$.
2. $\mathrm{RP}(X, f)$ is closed and invariant. If (X, f) is minimal then $\mathrm{RP}(X, f)$ is an equivalence relation.

Theorem

If (X, f) is invertible then $\mathrm{Com}(X, f) \cup \Delta \subset \mathrm{RP}(X, f)$ and if it is additionally minimal then $\mathrm{Com}(X, f) \cup \Delta = \mathrm{RP}(X, f)$.

Complexity along sequences

1. For an infinite set $A = \{a_1 < a_2 < \dots\}$ and cover $\mathcal{C}$ define
$$C_A(\mathcal{C}) = \lim_{n\to\infty} r(\bigvee_{j=1}^{n} f^{-a_i}(\mathcal{C})).$$
2. A is thick if for every n there is i such that $\{i, i+1, \dots, i+n\} \subset A$.
3. A is syndetic is $\mathbb{N} \setminus A$ is not thick
4. A is piecewise syndetic is $A = S \cap T$ for some syndetic S and thick T

Theorem

Let (X, f) be a dynamical system. The following conditions are equivalent:

1. *(X, f) is scattering*
2. *$C_A(\mathcal{U}) = \infty$ for any standard open cover $\mathcal{U}$ of X and any syndetic (or piecewise syndetic) set A.*
3. *$C_A(\mathcal{U}) = \infty$ for any nontrivial open cover $\mathcal{U}$ of X and any syndetic (or piecewise syndetic) set A.*

Mild mixing

1. (X, f) is mild mixing if $(X \times Y, f \times g)$ is transitive for every transitive (Y, g) (i.e. mild mixing $\equiv$ weakly disjoint from all transitive systems).
2. mild mixing $\implies$ weak mixing
3. $A \subset \mathbb{N}$ is IP-set if $A = \{p_{i_1} + \ldots + p_{i_k} : i_1 < i_2 < \ldots < i_k\}$ for some sequence $p_1, p_2, \ldots \subset \mathbb{N}$.

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3. *$C_A(\mathcal{U}) = \infty$ for any nontrivial open cover $\mathcal{U}$ of X and any IP-set A.*

Uniform rigidity

1. (X, f) is uniformly rigid if for every $\varepsilon > 0$ there is $n > 0$ such that $d(x, f^n(x)) < \varepsilon$ for every $x \in X$.

Theorem

If a nontrivial (X, f) is mild mixing then

- *it is not uniformly rigid,*
- *it is disjoint with minimal uniformly rigid systems.*

Example (Glasner & Maon)

There exists minimal, weakly mixing and uniformly rigid dynamical system.

mild mixing $\Longrightarrow$ weak mixing $\Longrightarrow$ scattering $\Longrightarrow$ total transitivity

Entropy pairs (Blanchard, 1993)

1. points $x \neq y$ are a entropy pair if $h_{\text{top}}(f, \mathcal{U}) > 0$ for any standard open cover $\mathcal{U} = \{U, V\}$ separating x, y (i.e. $x \in \text{Int}\, U^c$, $y \in \text{Int}\, V^c$).
2. $E_2(X, f)$ – set of entropy pairs.
3. $E_2(X, f) \cup \Delta$ is closed and $f \times f$ invariant.
4. $E_2(X, f) \neq \emptyset$ if and only if $h_{\text{top}}(f) > 0$.
5. if $\pi\colon (X, f) \to (Y, g)$ is a factor map then:
   1. if $x, y \in E_2(X, f)$ with $\pi(x) \neq \pi(y)$ then $(\pi(x), \pi(y)) \in E_2(Y, g)$,
   2. if $x, y \in E_2(Y, g)$ then $\pi^{-1}(\{x\}) \times \pi^{-1}(\{y\}) \cap E_2(X, f) \neq \emptyset$.

   In other words $E_2(Y, g) \subset (\pi \times \pi)(E_2(X, f)) \subset E_2(Y, g) \cup \Delta_Y$.

Remark

A dynamical system (X, f) has uniformly positive entropy iff $E_2(X, f) \cup \Delta = X \times X$.

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Remark

A dynamical system (X, f) has uniformly positive entropy iff $E_2(X, f) \cup \Delta = X \times X$.

u.p.e. and disjointness

Theorem

If (X, f) has u.p.e then it is disjoint with all minimal systems with zero entropy.

Entropy pairs and topological Pinsker factor (Blanchard & Lacroix, 1993)

1. Every dynamical system (X, f) posses a factor (Y, g) with zero entropy (and factor map $\pi\colon (X, f) \to (Y, g)$) (called topological Pinsker factor), such that
   - for any zero entropy system (Z, h), i.e. $h_{\text{top}}(h) = 0$, and factor map $\phi\colon (X, f) \to (Z, h)$
   - there is factor $\psi : (Y, g) \to (Z, h)$ such that $\phi = \psi \circ \pi$.
2. the topological Pinsker factor (or maximal zero entropy factor) is unique up to conjugacy.

Theorem

Let R be the smallest ICER (invariant, closed equivalence relation) such that $E_2(X, f) \subset R$. Then $\pi\colon (X, f) \to (X/R, f/R)$ is topological Pinsker factor.

u.p.e. and mixing

1. if $h_{\text{top}}(\mathcal{U}, f) > 0$ then $c(\mathcal{U}, \cdot)$ grows exponentially for every standard cover, hence there is a chance for
   - u.p.e. $\Longrightarrow$ weak mixing

   But in fact, more can be proved.

Theorem (Huang, Shao, Ye, 2005)

If (X, f) is transitive and $\{(x, f(x)) : x \in X\} \subset \overline{E_2(X, f)}$ (so-called diagonal flow) then it is mild mixing. In particular u.p.e. implies mild mixing.

Theorem (Huang, Shao, Ye, 2005)

If (X, f) is a minimal topological K system then it is mixing.

Question

Is every minimal u.p.e. system mixing?

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Asymtpotic pairs

- points x, y are asymptotic if $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$
- the set of asymptotic pairs is denoted $\mathrm{Asy}(X, f)$.

Theorem (Blanchard, Host and Ruette, 2004)

For every dynamical system $\overline{\mathrm{Asy}(X, f)} \supset E_2(X, f)$.

Corollary

If $\pi\colon (X, f) \to (Y, g)$ is such that $x, y \in \mathrm{Asy}(X, f)$ implies $\pi(x) = \pi(y)$ then $h_{\mathrm{top}}(g) = 0$.

Theorem (Huang, Li, Ye, 2013)

For every dynamical system $\mathrm{Asy}(X, f) \cap E_2(X, f)$ *is dense in* $E_2(X, f)$.

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Completely recurrent systems

1. $x \in \mathrm{Rec}(X, f)$ if $\liminf_{k\to\infty} d(f^k(x), x) = 0$.

Remark

If $\mathrm{Rec}(X \times X, f \times f) = X \times X$ then $\mathrm{Asy}(X, f) = \Delta$, in particular $E_2(X, f) = 0$ and hence $h_{\mathrm{top}}(X, f) = 0$.

1. in particular, uniformly rigid systems have entropy 0.

Question

Let (X, f) be invertible and suppose that for every $(x, y) \in X \times X$ there is a sequence $\lim_{k\to\infty} |n_k| = \infty$ such that $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = (x, y)$.

Is it true that $h_{\mathrm{top}}(f) = 0$?

Further reading

How all that started....

H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, **1** (1967), 1-49.

F. Blanchard, *A disjointness theorem involving topological entropy*, Bull. de Math. Soc. France, **121** (1993), 565-578.

F. Blanchard, B. Host and A. Maass, *Topological complexity*, Ergod. Th. and Dynam. Sys., **20** (2000), 641-662.

Two surveys.

E. Glasner and X. Ye, *Local entropy theory*, Ergodic Theory Dynam. Systems, **29** (2009), no. 2, 321–356.

P. Oprocha, G. Zhang, *Topological aspects of dynamics of pairs, tuples and sets*, in Recent Progress in Topology **III**, 2014