Topological Complexity – II

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Recall: our setting

1. (X, d) - compact metric space
2. $f : X \to X$ - continuous
3. f is (topologically) weakly mixing for any open sets $U_1, U_2, V_1, V_2 \neq \emptyset$ the following condition holds:

$$\exists\, n > 0 \qquad f^n(U_1) \cap U_2 \neq \emptyset, \quad f^n(V_1) \cap V_2 \neq \emptyset.$$

4. f is (topologically) mixing if for any open sets $U_1, U_2 \neq \emptyset$ the following condition is satisfied:

$$\exists\, N > 0 \qquad \forall\, n > N \qquad f^n(U_1) \cap U_2 \neq \emptyset.$$

Symbolic dynamics and spacing shifts

1. full shift- set $\{0,1\}^{\mathbb{N}}$ with
   - metric d:
   $$d(x,y)=\begin{cases}0 & x=y\\ 2^{-k} & x\neq y,\ k=\min\{i : x_i\neq y_i\}\end{cases}$$
   - map σ given by $\sigma(x)_i = x_{i+1}$
2. P – set of nonnegative integers. Define spacing shift
   $$\Sigma_P = \{x : x_i = x_j = 1 \implies |i-j| \in P\cup\{0\}\}$$
3. write $\sigma_P = \sigma|_{\Sigma_P}$ or simply σ

Example

$$\text{weak mixing} \not\Downarrow \text{mixing}$$

Theorem

1. *σ is weakly mixing (on Σ_P) iff P is thick*
2. *σ is mixing (on Σ_P) iff P is co-finite*

(Sketch of proof)

Recall: complexity function and scattering

1. Let $\mathcal{C} = (C_1, \ldots, C_n)$. If k is the minimal number of elements of $\mathcal{C}$ covering X (i.e. $X = C_{i_1} \cup \ldots \cup C_{i_k}$ for some $1 \leqslant i_1, \ldots, i_k \leqslant n$), then we put $r(\mathcal{C}) = k$.

2. $c(\mathcal{C}, n) = r(\mathcal{C}_0^{n-1})$ – complexity function.

Definition (Blanchard, Host, Mass, 2000)

A dynamical system (X, f) is scattering (resp. 2-scattering) if any nontrivial (resp. standard) open cover $\mathcal{U}$ has unbounded complexity function $c(\mathcal{U}, \cdot)$.

Scattering, weak mixing and transitivity

Theorem

If (X, f) is weakly mixing then it is scattering.

Proof (sketch)

For $\mathcal{C}$ with $r(\mathcal{C}) = k$ denote $I(\mathcal{C}) = \left\{ C_{i_1}^c \cap \ldots \cap C_{i_{k-1}}^c : C_{i_j} \in \mathcal{C} \right\}$

Lemma

If $\mathcal{C}, \mathcal{D}$ are covers with $r(\mathcal{C}) = k$, $r(\mathcal{D}) = \ell$ and $A \cap B \neq \emptyset$ for every $A \in I(\mathcal{C})$, $B \in I(\mathcal{D})$ then $r(\mathcal{C} \vee \mathcal{D}) \geqslant k + \ell - 1$.

Theorem

If for any open cover $\mathcal{U} \in \mathscr{C}_X$ there is $n > 0$ such that $c(\mathcal{U}, n) > n + 1$ then (X, f) is weakly mixing.

Scattering, weak mixing and transitivity

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Theorem

If (X, f) is 2-scattering then (X, f^n) is 2-scattering and transitive for $n = 1, 2, 3, \dots$

weak mixing $\Longrightarrow$ scattering $\Longrightarrow$ 2-scattering $\Longrightarrow$ total transitivity

A characterization of scattering

Theorem

For transitive (X, f) the following conditions are equivalent:

1. (X, f) *is not scattering,*
2. *there is a minimal (and invertible)* (Y, g) *such that* $(X \times Y, f \times g)$ *is not transitive*
3. *there exists a minimal (and invertible)* (Y, g), $N > 0$ *and a proper invariant closed set* $J \subset X \times Y$ *for* $f \times g$ *such that*

$$X \times Y = \bigcup_{n=0}^{N} (Id \times g^n)(J).$$

A characterization of scattering

Corollary

For transitive (X, f) the following conditions are equivalent:

1. (X, f) is scattering,
2. for every minimal (and invertible) (Y, g) the product $(X \times Y, f \times g)$ is transitive.

Equicontinuous systems

1. x is equicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for every $n > 0$.
2. (X, f) is equicontinuous if every $x \in X$ is equicontinuous.
3. (X, f) is almost equicontinuous if every $x \in Z$ is equicontinuous for some residual $Z \subset X$.

4. if (X, f) is transitive and some $x \in X$ is equicontinuous then (X, f) is almost equicontinuous.

5. if nontrivial (X, f) is weakly mixing then there is no equicontinuous point in X (i.e. (X, f) cannot be almost equicontinuous).

Theorem

Let (X, f) be a dynamical system. The following conditions are equivalent:

1. *(X, f) is equicontinuous,*
2. *$c(\mathcal{C}, \cdot)$ is bounded for every finite open cover $\mathcal{C}$ of X.*

Example of Akin and Glasner

Example

There exists almost equicontinuous scattering system.

scattering $\not\Longrightarrow$ weak mixing

Theorem

If (X, f) is minimal then the following properties are equivalent:

1. *2-scattering*
2. *scattering*
3. *weak mixing*

Later in this talk it will be more clear why...

Result of Huang and Ye

Theorem

Let (X, f) be a dynamical system. The following conditions are equivalent:

1. (X, f) *is 2-scattering*
2. (X, f) *is scattering*

$$\begin{array}{ccc} \text{weakly mixing} & & \\ \Downarrow \not\Uparrow & & \\ 2-\text{scattering} & \Longleftrightarrow & \text{scattering} \\ \Downarrow \not\Uparrow & & \\ \text{totally transitive} & & \end{array}$$

Distality

1. $x, y \in X$ are proximal if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.
2. x is distal if x, y are not proximal for every $y \in \overline{\mathrm{Orb}^+(x, f)} \setminus \{x\}$.
3. (X, f) is distal if every $x \in X$ is distal.
4. if (X, f) is distal then it is union of minimal systems.

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Theorem (Glasner, Weiss; Akin et. al)

If f is transitive and almost equicontinuous then:

1. f is a homeomorphism,
2. a point x is transitive iff it is equicontinuous
3. f is uniformly rigid, i.e. for every $\varepsilon > 0$ there is $n > 0$ such that $d(x, f^n(x)) < \varepsilon$ for every $x \in X$.

Corollary

Every transitive equicontinuous system is distal.

Distality

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Example

Dynamical system T on tori defined by

$$T(x,y) = (x + \alpha \pmod 1, x + y \pmod 1)$$

is distal but not equicontinuous.

Corollary

Every transitive equicontinuous system is distal.