Topological Complexity – I

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Standing assumptions

1. (X, d) - compact metric space
2. $f : X \to X$ - continuous
3. $f^n = f \circ \ldots \circ f$.
4. (X, f) – **dynamical system**
5. $p \in X$ is periodic if $f^n(p) = p$ for some $n > 0$.
6. $\omega_f(x) = \{y \in X \ : \ \lim_{k\to\infty} f^{n_k}(x) = y\}$.
7. $\mathrm{Orb}^+(x) = \{x, f(x), f^2(x), \ldots\}$.

Hierarchy of mixing (standard)

1. f is **transitive** if for any open sets $U_1, U_2 \neq \emptyset$ it holds that
$$\exists\, n > 0 \qquad f^n(U_1) \cap U_2 \neq \emptyset.$$
2. f is **totally transitive** if for every integer m the map f^m is transitive.
3. f is (topologically) **weakly mixing** for any open sets $U_1, U_2, V_1, V_2 \neq \emptyset$ the following condition holds:
$$\exists\, n > 0 \qquad f^n(U_1) \cap U_2 \neq \emptyset, \quad f^n(V_1) \cap V_2 \neq \emptyset.$$
4. f is (topologically) **mixing** if for any open sets $U_1, U_2 \neq \emptyset$ the following condition is satisfied:
$$\exists\, N > 0 \qquad \forall\, n > N \qquad f^n(U_1) \cap U_2 \neq \emptyset.$$

Remark

If f is transitive then is onto.

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Relations between definitions

$$\begin{array}{c}\text{mixing}\\ \Downarrow \\ \text{weakly mixing}\\ \Downarrow \\ \text{totally transitive}\\ \Downarrow \\ \text{transitive}\end{array}$$

Theorem (Furstenberg)

A map f is weakly mixing iff its n-times Cartesian product

$$f \times \ldots \times f : X \times \ldots \times X \to X \times \ldots \times X$$

(denoted $f^{(n)}$) is transitive for every $n \geqslant 2$.

Relations between definitions

mixing

$\Downarrow \not\Uparrow$

weakly mixing

$\Downarrow \not\Uparrow$

totally transitive

$\Downarrow \not\Uparrow$

transitive

Relations between definitions

mixing

$\Downarrow$

???

$\Downarrow$

weakly mixing

$\Downarrow$

???

$\Downarrow$

totally transitive

$\Downarrow$

transitive

Covers

1. Covers in this talk are not necessarily open, but are finite.
2. $\mathscr{C}_X$ – the set of all finite open covers of X
3. $\mathcal{C} = (C_1, \ldots, C_n)$ is nontrivial if C_i is not dense for every i.
4. a nontrivial cover $\mathcal{C}$ is standard if $\#\mathcal{C} = 2$.
5. $\mathcal{C} \prec \mathcal{D}$ if for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $D \subset C$.
6. $\mathcal{C} \vee \mathcal{D} = \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$
7. $\mathcal{C}_n^m = \bigvee_{i=n}^m f^{-i}(\mathcal{C})$ where
8. $f^{-i}(\mathcal{C}) = \left\{f^{-i}(C) : C \in \mathcal{C}\right\}$.

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Complexity function

1. Let $\mathcal{C} = (C_1, \dots, C_n)$. If k is the minimal number of elements of $\mathcal{C}$ covering X (i.e. $X = C_{i_1} \cup \dots \cup C_{i_k}$ for some $1 \leqslant i_1, \dots, i_k \leqslant n$), then we put $r(\mathcal{C}) = k$.
2. $c(\mathcal{C}, n) = r(\mathcal{C}_0^{n-1})$ – complexity function.
3. if $\mathcal{C} \prec \mathcal{D}$ then $r(\mathcal{C}) \leqslant r(\mathcal{D})$
4. $r(\mathcal{C} \vee \mathcal{D}) \leqslant r(\mathcal{C}) \cdot r(\mathcal{D})$, in particular $c(\mathcal{C}, \cdot)$ is nondecreasing.
5. $r(f^{-i}(\mathcal{C})) \leqslant r(\mathcal{C})$

Topological entropy

1. Every subadditive sequence of non-negative real numbers, i.e.

$$0 \leqslant a_{n+m} \leqslant a_n + a_m \quad \forall\ m, n$$

satisfies $\inf_{n\in\mathbb{N}} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n}$.

2. For any open cover $\mathcal{U} \in \mathscr{C}_X$ we define its (topological) entropy by

$$\begin{aligned} h_{\text{top}}(\mathcal{U}, f) &= \limsup_{n\to\infty} \frac{1}{n} \log r(\mathcal{U}_0^{n-1}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log r(\mathcal{U}_0^{n-1}) \end{aligned}$$

3. Topological entropy

$$h_{\text{top}}(f) = \sup_{\mathcal{U}\in\mathscr{C}_X} h_{\text{top}}(U, f)$$

Complexity function and dynamics

Definition (Blanchard, Host, Mass, 2000)

A dynamical system (X, f) is scattering (resp. 2-scattering) if any nontrivial (resp. standard) open cover $\mathcal{U}$ has unbounded complexity function $c(\mathcal{U}, \cdot)$.

Definition (Blanchard, 1993)

A dynamical system (X, f) has uniformly positive entropy (u.p.e. for short) if any standard open cover $\mathcal{U}$ has positive entropy.

Remark

Similarly, we can define u.p.e. of order n and of all orders (called also topological K) (nontrivial cover by at most n open sets, and all covers $\mathcal{U} \in \mathscr{C}_X$, respectively)

Scattering, weak mixing and transitivity

Theorem

If (X, f) is weakly mixing then it is scattering.

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If for any open cover $\mathcal{U} \in \mathscr{C}_X$ there is $n > 0$ such that $c(\mathcal{U}, n) > n + 1$ then (X, f) is weakly mixing.

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Theorem

If (X, f) is 2-scattering then (X, f^n) is 2-scattering and transitive for $n = 1, 2, 3, \dots$

weak mixing $\Longrightarrow$ scattering $\Longrightarrow$ 2-scattering $\Longrightarrow$ total transitivity