

Topological Complexity – I

Piotr Oprocha

AGH University of Science and Technology
Faculty of Applied Mathematics
Kraków, Poland

VSB, IT4Innovations, Ostrava, Jan 28, 2015

Standing assumptions

- 1 (X, d) - compact metric space
- 2 $f: X \rightarrow X$ - continuous
- 3 $f^n = f \circ \dots \circ f$.
- 4 (X, f) – **dynamical system**
- 5 $p \in X$ is **periodic** if $f^n(p) = p$ for some $n > 0$.
- 6 $\omega_f(x) = \{y \in X : \lim_{k \rightarrow \infty} f^{n_k}(x) = y\}$.
- 7 $\text{Orb}^+(x) = \{x, f(x), f^2(x), \dots\}$.

Hierarchy of mixing (standard)

- ① f is **transitive** if for any **open sets** $U_1, U_2 \neq \emptyset$ it holds that

$$\exists n > 0 \quad f^n(U_1) \cap U_2 \neq \emptyset.$$

- ② f is **totally transitive** if for every integer m the map f^m is transitive.

- ③ f is (topologically) **weakly mixing** for any **open sets** $U_1, U_2, V_1, V_2 \neq \emptyset$ the following condition holds:

$$\exists n > 0 \quad f^n(U_1) \cap U_2 \neq \emptyset, \quad f^n(V_1) \cap V_2 \neq \emptyset.$$

- ④ f is (topologically) **mixing** if for any open sets $U_1, U_2 \neq \emptyset$ the following condition is satisfied:

$$\exists N > 0 \quad \forall n > N \quad f^n(U_1) \cap U_2 \neq \emptyset.$$

Remark

If f is transitive then is onto.

Hierarchy of mixing (standard)

- ① f is **transitive** if for any open sets $U_1, U_2 \neq \emptyset$ it holds that

$$\exists n > 0 \quad f^n(U_1) \cap U_2 \neq \emptyset.$$

- ② f is **totally transitive** if for every integer m the map f^m is transitive.

- ③ f is (topologically) **weakly mixing** for any open sets $U_1, U_2, V_1, V_2 \neq \emptyset$ the following condition holds:

$$\exists n > 0 \quad f^n(U_1) \cap U_2 \neq \emptyset, \quad f^n(V_1) \cap V_2 \neq \emptyset.$$

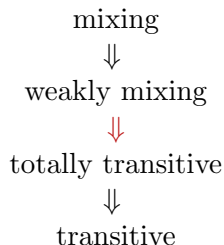
- ④ f is (topologically) **mixing** if for any open sets $U_1, U_2 \neq \emptyset$ the following condition is satisfied:

$$\exists N > 0 \quad \forall n > N \quad f^n(U_1) \cap U_2 \neq \emptyset.$$

Remark

If f is transitive then is onto.

Relations between definitions



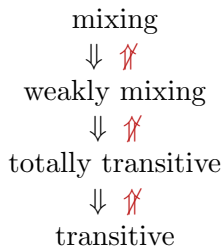
Theorem (Furstenberg)

A map f is weakly mixing iff its n -times Cartesian product

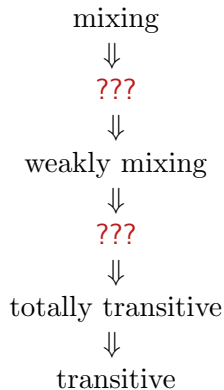
$$f \times \dots \times f : X \times \dots \times X \rightarrow X \times \dots \times X$$

(denoted $f^{(n)}$) is transitive for every $n \geq 2$.

Relations between definitions



Relations between definitions



Covers

- 1 Covers in this talk are **not** necessarily **open**, but are **finite**.
- 2 \mathcal{C}_X – the set of all finite **open** covers of X
- 3 $\mathcal{C} = (C_1, \dots, C_n)$ is nontrivial if C_i is not dense for every i .
- 4 a nontrivial cover \mathcal{C} is standard if $\#\mathcal{C} = 2$.
- 5 $\mathcal{C} \prec \mathcal{D}$ if for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $D \subset C$.
- 6 $\mathcal{C} \vee \mathcal{D} = \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$
- 7 $\mathcal{C}_n^m = \bigvee_{i=n}^m f^{-i}(\mathcal{C})$ where
- 8 $f^{-i}(\mathcal{C}) = \{f^{-i}(C) : C \in \mathcal{C}\}$.

Covers

- 1 Covers in this talk are not necessarily open, but are finite.
- 2 \mathcal{C}_X – the set of all finite open covers of X
- 3 $\mathcal{C} = (C_1, \dots, C_n)$ is **nontrivial** if C_i is **not dense** for every i .
- 4 a nontrivial cover \mathcal{C} is **standard** if $\#\mathcal{C} = 2$.
- 5 $\mathcal{C} \prec \mathcal{D}$ if for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $D \subset C$.
- 6 $\mathcal{C} \vee \mathcal{D} = \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$
- 7 $\mathcal{C}_n^m = \bigvee_{i=n}^m f^{-i}(\mathcal{C})$ where
- 8 $f^{-i}(\mathcal{C}) = \{f^{-i}(C) : C \in \mathcal{C}\}$.

Covers

- 1 Covers in this talk are not necessarily open, but are finite.
- 2 \mathcal{C}_X – the set of all finite open covers of X
- 3 $\mathcal{C} = (C_1, \dots, C_n)$ is nontrivial if C_i is not dense for every i .
- 4 a nontrivial cover \mathcal{C} is standard if $\#\mathcal{C} = 2$.
- 5 $\mathcal{C} \prec \mathcal{D}$ if for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $D \subset C$.
- 6 $\mathcal{C} \vee \mathcal{D} = \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$
- 7 $\mathcal{C}_n^m = \bigvee_{i=n}^m f^{-i}(\mathcal{C})$ where
- 8 $f^{-i}(\mathcal{C}) = \{f^{-i}(C) : C \in \mathcal{C}\}$.

Covers

- 1 Covers in this talk are not necessarily open, but are finite.
- 2 \mathcal{C}_X – the set of all finite open covers of X
- 3 $\mathcal{C} = (C_1, \dots, C_n)$ is nontrivial if C_i is not dense for every i .
- 4 a nontrivial cover \mathcal{C} is standard if $\#\mathcal{C} = 2$.
- 5 $\mathcal{C} \prec \mathcal{D}$ if for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $D \subset C$.
- 6 $\mathcal{C} \vee \mathcal{D} = \{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$
- 7 $\mathcal{C}_n^m = \bigvee_{i=n}^m f^{-i}(\mathcal{C})$ where
- 8 $f^{-i}(\mathcal{C}) = \{f^{-i}(C) : C \in \mathcal{C}\}$.

Complexity function

- 1 Let $\mathcal{C} = (C_1, \dots, C_n)$. If k is the **minimal** number of elements of \mathcal{C} covering X (i.e. $X = C_{i_1} \cup \dots \cup C_{i_k}$ for some $1 \leq i_1, \dots, i_k \leq n$), then we put $r(\mathcal{C}) = k$.
- 2 $c(\mathcal{C}, n) = r(\mathcal{C}_0^{n-1})$ – **complexity function**.
- 3 if $\mathcal{C} \prec \mathcal{D}$ then $r(\mathcal{C}) \leq r(\mathcal{D})$
- 4 $r(\mathcal{C} \vee \mathcal{D}) \leq r(\mathcal{C}) \cdot r(\mathcal{D})$, in particular $c(\mathcal{C}, \cdot)$ is **nondecreasing**.
- 5 $r(f^{-i}(\mathcal{C})) \leq r(\mathcal{C})$

Topological entropy

- ① Every **subadditive sequence** of non-negative real numbers, i.e.

$$0 \leq a_{n+m} \leq a_n + a_m \quad \forall m, n$$

satisfies $\inf_{n \in \mathbb{N}} \frac{a_n}{n} = \lim_{n \rightarrow \infty} \frac{a_n}{n}$.

- ② For any **open cover** $\mathcal{U} \in \mathcal{C}_X$ we define its **(topological) entropy** by

$$\begin{aligned} h_{\text{top}}(\mathcal{U}, f) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log r(\mathcal{U}_0^{n-1}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log r(\mathcal{U}_0^{n-1}) \end{aligned}$$

- ③ **Topological entropy**

$$h_{\text{top}}(f) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_{\text{top}}(\mathcal{U}, f)$$

Complexity function and dynamics

Definition (Blanchard, Host, Mass, 2000)

A dynamical system (X, f) is **scattering** (resp. **2-scattering**) if any **nontrivial** (resp. **standard**) open cover \mathcal{U} has unbounded complexity function $c(\mathcal{U}, \cdot)$.

Definition (Blanchard, 1993)

A dynamical system (X, f) has **uniformly positive entropy** (u.p.e. for short) if any **standard** open cover \mathcal{U} has positive entropy.

Remark

Similarly, we can define u.p.e. of **order n** and of **all orders** (called also **topological K**) (nontrivial cover by at most n open sets, and all covers $\mathcal{U} \in \mathcal{C}_X$, respectively)

Scattering, weak mixing and transitivity

Theorem

If (X, f) is *weakly mixing* then it is *scattering*.

Theorem

If for any open cover $\mathcal{U} \in \mathcal{C}_X$ there is $n > 0$ such that $c(\mathcal{U}, n) > n + 1$ then (X, f) is *weakly mixing*.

Scattering, weak mixing and transitivity

Theorem

If (X, f) is weakly mixing then it is scattering.

Theorem

If for any open cover $\mathcal{U} \in \mathcal{C}_X$ there is $n > 0$ such that $c(\mathcal{U}, n) > n + 1$ then (X, f) is weakly mixing.

Scattering, weak mixing and transitivity

Theorem

If (X, f) is weakly mixing then it is scattering.

Theorem

If for any open cover $\mathcal{U} \in \mathcal{C}_X$ there is $n > 0$ such that $c(\mathcal{U}, n) > n + 1$ then (X, f) is weakly mixing.

Theorem

If (X, f) is *2-scattering* then (X, f^n) is 2-scattering and *transitive* for $n = 1, 2, 3, \dots$

weak mixing \implies scattering \implies 2-scattering \implies total transitivity