Functions of Complex Analysis and Integral Transforms

Examples of two-dimensional electric fields

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September 29, 2021

The use of conformal mapping in electric field theory could be understood as an examination of the electric field given by a complex function of a complex variable

$$f = u + iv,$$

which satisfies the boundary conditions of the field, i.e. such a function, the real and imaginary part of which expresses the levels and forces of the electric field. Finding such a function is a difficult task. Therefore, let's choose the opposite procedure, i.e. we find boundary conditions for the given function, i.e. the image of the respective field (levels and forces).

The First Example

Let us consider

$$f(z) = z^2$$

and find the boundary conditions.

Solution:

Obviously

$$f(z) = f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy,$$

so

$$u(x, y) = x^2 - y^2, \;\; v(x, y) = 2xy.$$

The curves $u = c$, $c \in \mathbb{R}$ are equiaxed hyperbolas with asymptotes in the lines of axes I. and II. quadrant. The curves $v = c$, $c \in \mathbb{R}$ are equiaxed hyperbolas with asymptotes in the coordinate axes. The function w thus gives an image of the electric field between two semi-infinite electrodes perpendicular to each other, where $v = c$ are levels and $u = c$ are forces of the field, see Figure 1.

[1] I would like to thank to my wife RNDr. Alžběta Lampartová and prof. Ing. Jaroslav Zapoměl, DrSc. for proofreading and valuable comments that led to a substantial improvement in the text. This text was not supported by any grant.

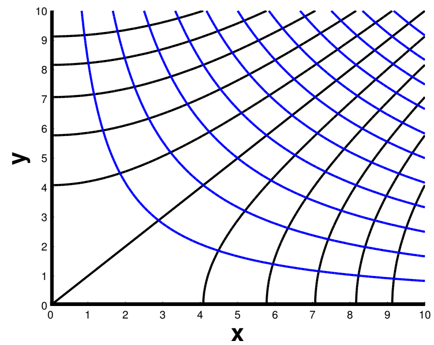

Figure 1: Levels $v = c$ (blue) and forces $u = c$ (blue) of the electric field of the firs example.

The Second Example

As in the previous example, consider

$$f(z) = \sqrt{z}$$

and find the boundary conditions.

Solution:

Firstly,

$$f(z) = f(x + iy) = \sqrt{x + iy},$$

so

$$\sqrt{x + iy} = u + iv,$$

$$x + iy = u^2 - v^2 + i2uv$$

and

$$x = u^2 - v^2, \;\; y = 2uv.$$

Now, put $v = y/2u$ into $x = u^2 - v^2$, we get

$$y^2 = -4u^2(x - u^2). \tag{1}$$

Analogously put $u = y/2v$ into $x = u^2 - v^2$, we get

$$y^2 = 4v^2(x + v^2). \tag{2}$$

For real u and v denotes (1), resp. (2) the system of parabolas with the focus at the point $(0,0)$, with the axis of symmetry x and open in direction of negative, resp. positive axis x.

The function f thus providing an image of the electric field generated by a semi-infinite plane bounded on one side by a line perpendicular to the drawing. The curves $v = c$, $c \in \mathbb{R}$ are levels and $u = c$, $c \in \mathbb{R}$ are forces, see Figure 2. The image is the same in each plane parallel to the drawing.

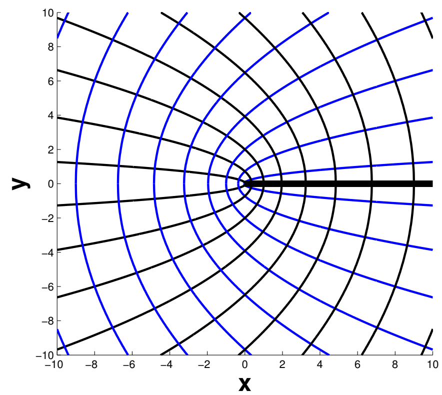

Figure 2: Levels $v = c$ (blue) and forces $u = c$ (black) of the electric field of the second example.

References

[1] Z. Pírko, *Komplexní čísla. Representace periodických dějů. Konformné zobrazení.* Skriptum ČVUT Praha, SPN Praha, 1952.