Solution of the Typical Final Exam Examples

Functions of Complex Analysis and Integral Transforms

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General rules of the final exam

- time: 90 minutes
- number of examples to solve: 7
- evaluation: each solved example has a maximum value of 10 points
- materials: official table of Laplace transforms, simple calculator

Example no. 1: Complex numbers

1.1. Compute $\operatorname{Re} z$ and $\operatorname{Im} z$ if:

$$z = \operatorname{Ln}(\cos(i)).$$

Solution: Let us recall that:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \operatorname{Ln}(z) = \ln|z| + i\operatorname{Arg}(z).$$

Compute

$$\cos(i) = \frac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \frac{e^{-1} + e^{1}}{2} = \cosh(1).$$

Next

$$\begin{aligned}\operatorname{Ln}(\cos(i)) = \operatorname{Ln}(\cosh(1)) &= \ln|\cosh(1)| + i\operatorname{Arg}(\cosh(1)) = \\ &= \ln(\cosh(1)) + i(\arg(\cosh(1)) + 2k\pi) = \\ &= \ln(\cosh(1)) + 2k\pi i, \quad k \in \mathbb{Z}.\end{aligned}$$

Here, $\arg(\cosh(1)) = 0$, since $\operatorname{Im}(\cosh(1)) = 0$ and $\operatorname{Re}(\cosh(1)) > 0$.

Then

$$\operatorname{Re}(z) = \ln(\cosh(1)) \quad \text{a} \quad \operatorname{Im}(z) = 2k\pi, \quad k \in \mathbb{Z}.$$

1.2. Find $\operatorname{Re} z_0$ and $\operatorname{Im} z_0$ if:

$$z_0 = (-\sqrt{2} + \sqrt{2}\,i)^4.$$

Solution: For solution De Moivre's theorem will be used:

$$z^n = |z|^n(\cos(n\phi) + i\sin(n\phi)).$$

So $z = -\sqrt{2} + \sqrt{2}\,i$, $|z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2+2} = 2$, $n = 4$ a $\arg(z) = \phi = 3/4\,\pi$. Let us substitute into foregoing theorem:

Here, $\operatorname{Re}(z) = -\sqrt{2}$ and $\operatorname{Im}(z) = \sqrt{2}$, so it holds $\arg(z) = \pi/2 + \arctan(\sqrt{2}/\sqrt{2}) = = \pi/2 + \pi/4 = 3/4\,\pi$.

$$(-\sqrt{2} + \sqrt{2}\,i)^4 = 2^4(\cos(4\cdot 3/4\,\pi) + i\sin(4\cdot 3/4\,\pi)) = 2^4(\cos(3\pi) + i\sin(3\pi)) = -16.$$

Consequently,

$$\operatorname{Re}(z_0) = -16 \quad \text{a} \quad \operatorname{Im}(z_0) = 0.$$

[1] I would like to thank to my wife RNDr. A. Lampartová and prof. RNDr. R. Kalus, Ph.D. for helpful comments that improved this text. I would like to express my thanks to students that pointed out many numerical typos and to Bc. R. Plouharová for proofreading. This text was not supported by any grant.

Example no. 2: Derivative of a complex function

2.1. Find out $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic on $\mathbb{C}$ and $f(1) = -2 + i$ if:

$$u(x, y) = x^4 + y^4 - 6x^2y^2 - 3.$$

Solution: Firstly,

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2, \quad \frac{\partial^2 u}{\partial y^2} = 12y^2 - 12x^2,$$

then

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (12x^2 - 12y^2) + (12y^2 - 12x^2) = 0,$$

hence, general solution of a given problem exists. For the solution, Cauchy–Riemann formulas will be used. Let us use the first of them:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad 4x^3 - 12xy^2 = \frac{\partial v}{\partial y},$$

Here it is $\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$.

so

$$v(x, y) = \int (4x^3 - 12xy^2)\mathrm{d}y = 4x^3y - 4xy^3 + \phi(x). \tag{1}$$

Now, substitute (1) into the second Cauchy–Riemann formula, and we get the equation for unknown function $\phi(x)$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad 4y^3 - 12yx^2 = -\frac{\partial v}{\partial x} = -(12x^2y - 4y^3 + \phi'(x)),$$

Here it is $\frac{\partial u}{\partial y} = 4y^3 - 12yx^2$.

$$\phi'(x) = 0, \quad \phi(x) = c, \quad c \in \mathbb{R}.$$

The general solution takes the following form:

$$v(x, y) = 4x^3y - 4xy^3 + c.$$

Now, find particular solution fulfilling condition $f(1) = f(1 + 0 \cdot i) = -2 + i$:

$$-2 + i = 1 - 3 + i \cdot c, \quad c = 1.$$

To u and v substitute $x = 1$ and $y = 0$.

The solution of a given problem fulfilling the initial condition takes the form

$$v(x, y) = 4x^3y - 4xy^3 + 1.$$

2.1. Find points in which the function is analytical:

$$f(z) = z \cdot \overline{z}.$$

Solution: Firstly, decompose f into its real and imaginary part:

$$f(x + i \cdot y) = (x + i \cdot y) \cdot (x - i \cdot y) = x^2 + y^2.$$

So,

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0.$$

Now, let us apply the following Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Longrightarrow 2x = 0 \Longrightarrow x = 0,$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \Longrightarrow 2y = 0 \Longrightarrow y = 0.$$

Cauchy–Riemann equations are fulfilled if and only if $z = 0$, hence given function has derivative in a point $z = 0$ and $f'(0) = 0$. This function is not analytic at point $z = 0$, since it has no derivative on its neighbourhood, therefore it is not analytic anywhere in $\mathbb{C}$.

Really, $f'(0) = \frac{\partial u}{\partial x}(0, 0) + i\frac{\partial v}{\partial x}(0, 0) = 0 + 0 \cdot i = 0$.

Example no. 3: Conformal functions

3.1. Illustrate sets Ω and $f(\Omega) = \{f(z) : z \in \Omega\}$ if $\Omega = U(1+i, 1)$ and

$$f(z) = \frac{i}{z-i} + 1.$$

Solution: Given function is conformal, linear fractional. Hence, it maps generalized circles into generalized circles (see the textbook for students). To solve this problem, it is enough to compute images of appropriately chosen points of this circle:

$$\begin{aligned} f(1) &= 1/2 + i/2, \\ f(1+2i) &= 3/2 + i/2, \\ f(2+i) &= 1 + i/2. \end{aligned}$$

Hence, image of region border Ω is a straight line parallel to the axis x. Now do interior test, that is search where interior of a region is mapped Ω:

Since it is a line parallel to the axis x, it is enough to consider $\mathrm{Im}\,(f(1)) = \mathrm{Im}\,(f(1+2i)) = \mathrm{Im}\,(f(2+i)) = 1/2$.

$$f(1+i) = 1 + i,$$

so $f(\Omega) = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 1/2\}$, see Figure 1.

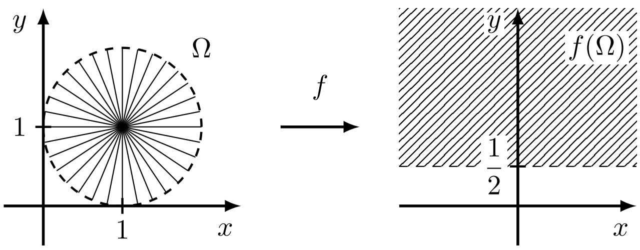

Figure 1: Regions Ω and $f(\Omega)$.

3.2. Find out linear fractional function f such that:

$$f(0) = i, \quad f(i) = 0, \quad f(1) = 1.$$

Solution: We are looking for coefficients a, b, c, and d such that the following conditions are fulfilled:

$$f(z) = \frac{az+b}{cz+d}, \quad ad \neq bc.$$

Let us substitute given values:

$$\begin{aligned} f(0) &= b/d = i, \\ f(i) &= (ai+b)/(ci+d) = 0, \\ f(1) &= (a+b)/(c+d) = 1. \end{aligned}$$

We get a system of three equations of four unknowns where we will pick an appropriate value instead of one of those and hence the system will be solved. Put $b = 1$, from the first equation we get $d = -i$. Substituting into the second equation we get $a = i$. Finally, substituting into the third equation we get $c = 2i + 1$. Consequently, the function we were looking for takes the form:

$$f(z) = \frac{iz+1}{(1+2i)z - i}.$$

Let us note that the solution of a given problem exists and is unique (see the textbook for students).

Example no. 4: Taylor and Laurent series

4.1. Write Taylor series of a function f with a centre $1+i$ where

$$f(z) = \frac{1}{1-z}.$$

Illustrate the region of convergency.

Solution: Firstly, notice that this function is holomorphic in a region $U(1+i,1)$ (see Figure 2). To evolve Taylor series, the theorem about convergence of a geometric series will be applied. Let us perform the following adjustment getting a series with the given centre:

$$\begin{aligned} f(z) &= \frac{1}{1-z} = \frac{1}{1-(1+i)-(z-(1+i))} = \frac{1}{-i-(z-(1+i))} = -\frac{1}{i}\frac{1}{1-\frac{z-(1+i)}{-i}} = \\ &= -\frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z-(1+i)}{-i}\right)^n = \sum_{n=0}^{\infty}\frac{(z-(1+i))^n}{-i^{n+1}}. \end{aligned}$$

Recall the geometric series $\sum_{n=0}^{\infty} q^n = 1/(1-q)$ has a sum if $|q|<1$, and here we put $q = (z-(1+i))/(-i)$.

Since a geometric series is convergent for $|q|<1$, the given series is convergent in the region $|z-(1+i)|<1$ (see Figure 2).

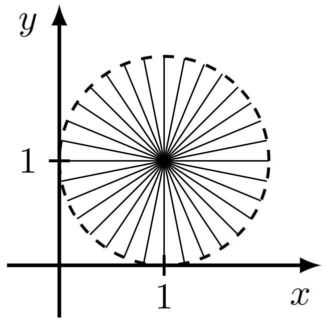

Figure 2: The region $U(1+i,1)$.

4.2. Write down Laurent series of the function

$$f(z) = \frac{1}{z^2-2z-3}$$

on the ring $\{z \in C :\ 1<|z|<3\}$. Illustrate the region of convergence.

Solution: Note that $z^2-2z-3=(z-3)(z+1)$. The point 3 ans -1 does not belong into the given ring and hence given function f in a given region is possible to derive, it is holomorphic, hence it makes sense to look for Laurent series. Firstly, we modify the given function using decomposition into the partial fractions:

$$f(z) = \frac{1}{z^2-2z-3} = \frac{1}{(z-3)(z+1)} = \frac{1}{4}\frac{1}{z-3} - \frac{1}{4}\frac{1}{z+1}.$$

Hint: how to do decomposition into the partial fractions: $\frac{1}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}$, calculate A and B.

Now, evolve both parts of the given function: if $|z|>1$, then

$$-\frac{1}{4}\frac{1}{z+1} = -\frac{1}{4z}\frac{1}{1-(-1/z)} = -\frac{1}{4z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{z^n},$$

Again, recall the geometric series $\sum_{n=0}^{\infty} q^n = 1/(1-q)$ has a sum for $|q|<1$. Here we pick $q=-1/z$, hence the region of convergence is $|z|>1$.

if $|z|<3$, then

$$\frac{1}{4}\frac{1}{z-3} = -\frac{1}{12}\frac{1}{1-(z/3)} = -\frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.$$

Consequently, Laurent series of a given function takes in the given ring (see Figure3) the form:

$$f(z) = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{z^n}.$$

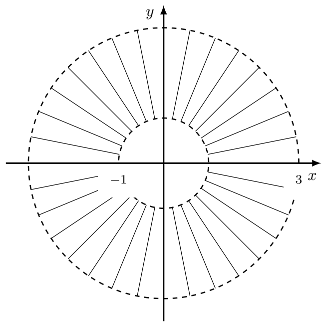

Figure 3: The ring $P(0,1,3)$.

Example no. 5: Integral of a complex function

5.1. Calculate

$$\int_\gamma z|z|\,\mathrm{d}z,$$

where $\gamma(t) = e^{it}$ and $t \in [0, \pi]$.

Solution: Firstly, note that the curve we are integrating through is not closed (see Figure 4). For the determination we use the following formula:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

In our case it holds:

$$\gamma(t) = e^{it}, \quad \gamma'(t) = ie^{it}, \quad a = 0, \quad b = \pi$$

and

$$f(\gamma(t)) = e^{it} \cdot |e^{it}| = e^{it} \cdot 1.$$

Here it is $|e^{it}| = |\cos(t) + i\sin(t)| = \sqrt{\cos^2(t) + \sin^2(t)} = 1.$

After substituting we get:

$$\int_\gamma z|z|\,\mathrm{d}z = \int_0^\pi e^{it} \cdot 1 \cdot ie^{it}\,\mathrm{d}t = i\int_0^\pi e^{2it}\,\mathrm{d}t = i\left[\frac{1}{2i}e^{2it}\right]_0^\pi = \frac{1}{2}\left[e^{2\pi i} - e^0\right] = 0.$$

It is possible to calculate the integral $\int_0^\pi e^{2it}$ dtcorrectly by linear substitution $w = 2it$.

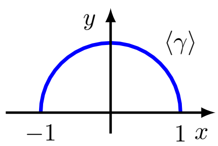

Figure 4: The graph $\langle\gamma\rangle$ from Example 5.1.

5.2. Calculate

$$\int_\gamma \frac{1}{z^2(z-1)^2}\,\mathrm{d}z,$$

where $\gamma(t) = 1/2\,e^{-it} + 1$ and $t \in [0, 6\,\pi]$.

Solution: The curve $\langle\gamma\rangle$ is closed and smooth, inside of the region bounded by $\langle\gamma\rangle$ there is only one singularity $z_0 = 1$ (the second singularity $z_1 = 0$ is outside of the region bounded by $\langle\gamma\rangle$, hence we do not care about it) see Figure 5. For the computation, we use Cauchy integral formula:

$$g^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{g(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

In this example

$$z_0 = 1, \quad g(z) = 1/z^2, \quad n+1 = 2.$$

Note that the curve is negatively oriented and the parameter is running about it three times. Now we can substitute

$$\int_\gamma \frac{1}{z^2(z-1)^2}\,\mathrm{d}z = \int_\gamma \frac{1/z^2}{(z-1)^2}\,\mathrm{d}z = -3 \cdot \frac{2\pi i}{1!}\left[\left(\frac{1}{z^2}\right)'\right]_{z=1} = -6\pi i[-2z^{-3}]_{z=1} = 12\pi i.$$

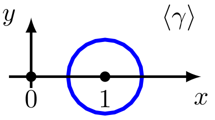

Figure 5: The graph of $\langle\gamma\rangle$ and singularities from Examples 5.2.

5.3. Calculate

$$\int_\gamma \frac{\sin(z)}{(z-i)(z+1)^2}\,\mathrm{d}z,$$

where $\gamma(t) = 100e^{2it} + 3$ a $t \in [0, \pi]$.

Solution: For the solution of this example, we will apply the Residue theorem:

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{i=1}^{n} \operatorname{Res} f(z_i).$$

The given function has in the region bounded by curve γ two singularities $z_1 = i$ and $z_2 = -1$. The point z_1 is clearly single pole and the point z_2 is the pole of order 2. Now, calculate residues at these points:

$$\operatorname{Res}_{z=z_1} f(z) = \lim_{z\to i} \frac{1}{0!}\left[f(z)(z-i)\right] = \lim_{z\to i} \frac{\sin(z)}{(z+1)^2} = \frac{\sin(i)}{(i+1)^2},$$

$$\operatorname{Res}_{z=z_2} f(z) = \lim_{z\to -1} \frac{1}{1!}\left[f(z)(z+1)^2\right]' = \lim_{z\to -1}\left[\frac{\sin(z)}{z-i}\right]' = \frac{\cos(-1)(-1-i)-\sin(-1)}{(i+1)^2}.$$

Here it is $\left[\frac{\sin(z)}{z-i}\right]' = \frac{\cos(z)(z-i)-\sin(z)}{(z-i)^2}$.

Now, let us substitute into the formula:

$$\int_\gamma \frac{\sin(z)}{(z-i)(z+1)^2}\,\mathrm{d}z = 2\pi i \sum_{i=1}^{2} \operatorname{Res} f(z_i) = \frac{2\pi i}{(1+i)^2}\left[\sin(i) - \cos(1)(1+i) + \sin(1)\right].$$

It is possible to simplify the result that can be done by the reader.

Example no. 6: Fourier series

6.1. Find out Fourier series of the periodic extension of the function and draw the graph of a sum of this Fourier series:

$$f(t) = \begin{cases} 1, & \text{for } t \in [0,1), \\ 0, & \text{for } t \in [1,2). \end{cases}$$

Solution: The period of a periodic extension is $T = 2$. Next $\omega = 2\pi/T = \pi$. Now calculate Fourier coefficients:

$$a_0 = \frac{2}{T}\int_0^T f(t)\mathrm{d}t = \frac{2}{2}\left[\int_0^1 1\mathrm{d}t + \int_1^2 0\mathrm{d}t\right] = \int_0^1 1\mathrm{d}t = 1,$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\mathrm{d}t = \int_0^1 \cos(n\pi t)\mathrm{d}t = \left[\frac{1}{n\pi}\sin(n\pi t)\right]_0^1 = 0,$$

Hint: $\int_1^2 0\cdot\cos(n\pi t)\mathrm{d}t = 0$ and for the computation $\int_0^1 \cos(n\pi t)\mathrm{d}t$ use the linear substitution $w = n\pi t$.

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\mathrm{d}t = \int_0^1 \sin(n\pi t)\mathrm{d}t = -\left[\frac{1}{n\pi}\cos(n\pi t)\right]_0^1 = -\frac{1}{n\pi}[(-1)^n - 1].$$

The Fourier series of a given function has the following form:

$$f(t) \approx \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}[1-(-1)^n]\sin(n\pi t). \tag{2}$$

The sum of series we get by Dirichlet's theorem, hence the value in each its point equals to the arithmetic mean of one-sided limits with the sum being given in Figure 6.

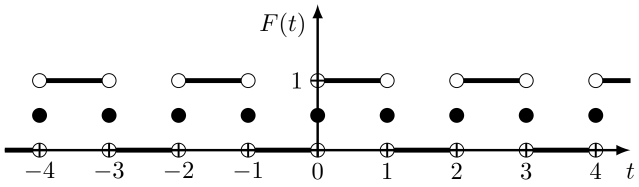

Figure 6: The graph is the Fourier series sum (2).

6.2. Find out period, angular speed, and the first four members of the one-sided amplitude spectrum and the first three members of the one-sided phase spectrum of the Fourier series:

$$-\sqrt{2} - \cos(4\pi t) + \sin(4\pi t) - \sqrt{5}\cos(8\pi t) - 4\sqrt{3}\cos(12\pi t) + 4\sin(12\pi t) \pm \dots$$

Write down formulas for its determination.

Solution: Firstly, from the arguments of functions sine and cosine derive $\omega = 4\pi$ and the period $T = 2\pi/\omega = 2\pi/4\pi = 1/2$. For the computation, we use formulas

$$A_0 = \left|\frac{a_0}{2}\right|, \quad A_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = -\arg c_n.$$

Here $c_n = 1/2(a_n - ib_n)$. Clearly $-\arg c_n = \arg c_{-n}$ where $c_{-n} = 1/2(a_n + ib_n)$. If $b_n = 0$ and $a_n < 0$ the solution is ϕ_n acceptable as $-\pi$ as well as π.

Substituting values given in Table. By substituting, we get values given in Table 1.

n	a_n	b_n	A_n	ϕ_n
0	$-2\sqrt{2}$	ndf	$\sqrt{2}$	ndf
1	-1	1	$\sqrt{2}$	$3\pi/4$
2	$-\sqrt{5}$	0	$\sqrt{5}$	π
3	$-4\sqrt{3}$	4	8	$5\pi/6$

Table 1: Table of values of amplitude and phase spectrum of the given series.

Example no. 7: Laplace transforms

7.1. Using Laplace transform, find the solution of the following equation with initial conditions:

$$y'' - 2y' - 3y = e^{-4t}, \quad y(0+) = y'(0+) = 0.$$

Solution:

Denote $\mathcal{L}(y(t)) = Y(p)$. Then

$$\mathcal{L}(y'(t)) = pY(p) - y(0) = pY(p)$$

and

$$\mathcal{L}(y''(t)) = p^2Y(p) - py(0) - y'(0) = p^2Y(p).$$

Then for the right-hand side of the equation we have

$$\mathcal{L}(e^{-4t}) = 1/(p+4).$$

Use table of the Laplace transforms.

Now, after substitution we get operator equation:

$$p^2Y(p) - 2pY(p) - 3Y(p) = \frac{1}{p+4},$$

$$Y(p) = \frac{1}{(p+4)(p+1)(p-3)}.$$

The function $Y(p)$ has three simple poles -4, -1, and 3. Applying Theorem 16, we are looking for the solution in the form $y(t) = \sum_{i=1}^{n} \operatorname{Res}[Y(p)e^{pt}]_{p=z_i}$. Let us calculate residua:

$$\operatorname{Res}[Y(p)e^{pt}]_{p=-4} = \lim_{p\to -4} \frac{1}{(p+1)(p-3)}e^{pt} = \frac{1}{21}e^{-4t},$$

$$\mathrm{Res}[Y(p)e^{pt}]_{p=-1} = \lim_{p\to-1}\frac{1}{(p+4)(p-3)}e^{pt} = -\frac{1}{12}e^{-t},$$

$$\mathrm{Res}[Y(p)e^{pt}]_{p=3} = \lim_{p\to3}\frac{1}{(p+4)(p+1)}e^{pt} = \frac{1}{28}e^{3t}.$$

The solution is given by the sum of the foregoing residua:

$$y(t) = \frac{1}{21}e^{-4t} - \frac{1}{12}e^{-t} + \frac{1}{28}e^{3t}.$$

The reader is asked to do a proof of the solution, which is optional at the final exam.

7.2. Using Laplace transform, find the solution of the following equation with the initial conditions:

$$y'' - 3y' + 2y = te^{3t}, \quad y(1+) = y'(1+) = 1.$$

Solution:

Since the initial conditions are not given in the point $t_0 = 0$, we have to do a substitution $t = \tau + 1$ and $y(t) = y(\tau+1) = z(\tau)$. The new equation takes the form

$$z'' - 3z' + 2z = (\tau+1)e^{3(\tau+1)}, \quad z(0_+) = z'(0_+) = 1.$$

Now, put $\mathcal{L}(z(\tau)) = Z(p)$, then

$$\mathcal{L}(z'(\tau)) = pZ(p) - 1,$$

$$\mathcal{L}(z''(\tau)) = p^2Z(p) - p - 1.$$

Next,

$$\mathcal{L}((\tau+1)e^{3(\tau+1)}) = e^3\,\mathcal{L}(\tau e^{3\tau} + e^{3\tau}) = e^3\left[\frac{1}{(p-3)^2} + \frac{1}{p-3}\right].$$

Use the table of the Laplace transforms.

Express $Z(p)$ after the decomposition into the partial fractions

$$Z(p) = \frac{1}{4}e^3\left[\frac{1}{p-1} + \frac{2}{(p-3)^2} - \frac{1}{p-3}\right] + \frac{1}{p-1}.$$

By the inverse Laplace transform for $\tau \geq 0$ we get the solution

$$z(\tau) = \frac{1}{4}e^3\left[e^{\tau} + 2\tau e^{3\tau} - e^{3\tau}\right] + e^{\tau}.$$

By the inverse Laplace transform $\tau = t-1$ and $z(\tau) = y(t)$ after the simplifications we get for $t \geq 1$ the solution

$$y(t) = \frac{2t-3}{4}e^{3t} + \frac{e^3+4}{4e}e^t.$$

The reader is asked to do a proof of the solution, which is optional at the final exam.

7.3. Using Laplace transform find the solution of the following equation with the initial conditions:

$$y' - y = f(t), \quad y(0+) = 0, \quad f(t) = \begin{cases} 1, & \text{for } 0 < t < 1, \\ 0, & \text{for } t > 1. \end{cases}$$

Solution: Denote $\mathcal{L}(y(t)) = Y(p)$. Then $\mathcal{L}(y'(t)) = pY(p) - y(0) = pY(p)$. For the right-hand side of the equation we get

$$\mathcal{L}(f(t)) = \frac{1}{p}(1-e^{-p}).$$

It is possible to get it by definition or by using the table of the Laplace transforms, the reader is asked to perform both as an exercise.

Now, substitute and construct operator equation:

$$pY(p) - Y(p) = \frac{1}{p}(1-e^{-p}),$$

$$Y(p) = \frac{1}{p(p-1)}(1-e^{-p}).$$

Now, after the decomposition into the partial fractions

$$\mathcal{L}^{-1}\left(\frac{1}{p(p-1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{p-1} - \frac{1}{p}\right) = e^t - 1.$$

The solution takes the from:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{p(p-1)}(1-e^{-p})\right) = \\ &= \mathcal{L}^{-1}\left(\frac{1}{p(p-1)}\right) - \mathcal{L}^{-1}\left(\frac{1}{p(p-1)}e^{-p})\right) = \\ &= (e^t-1)\eta(t) - (e^{t-1}-1)\eta(t-1), \end{aligned}$$

or equivalently

$$y(t) = \begin{cases} e^t - 1, \text{ for } 0 < t < 1, \\ e^t - e^{t-1}, \text{ for } t > 1. \end{cases}$$

The reader is asked to do a proof of the solution, which is optional at the final exam.

Example no. 8: the theory

8.1.
- Define *absolute value, argument and principal value of an argument* of a complex number.
- Formulate the residue theorem.

8.2.
- Define functions $\ln z$ and $\operatorname{Ln} z$ on a complex plane.
- Define Dirichlet's conditions.

8.2.
- Define *power series* and *adjoint complex number*.
- Write down formula for *a radius of convergency* of a power series at a centre in a point z_0.

Solution:
See textbooks for students.