An Introduction to Integral Trasforms

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Preface

Integral transforms as well as Fourier series are very interesting and powerful topics in the undergraduate mathematics course. Their importance to applications means that they can be studied both from a very pure perspective and a very applied perspective. This text book for students takes into account the varying needs and backgrounds of students in mathematics, science, and engineering. It covers two topics that feature in the course:

- Fourier series,
- Laplace transform.

Since the topics of Fourier series and integral transform are not elementary subjects, there are some reasonable assumptions about what the reader knows. The reader should be confident with the relevant topics taught as standard in the area of real analysis of real functions of one and multiple variables, complex analysis, sequences, and series.

This text is mostly a translation from the Czech original [8], and it is a natural continuation of the textbook *An Introduction to Complex Analysis* [2].

The authors are grateful to their colleagues for their comments that improved this text and to John Cawley who helped with the correction of many typos and English grammar.

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Chapter 1

Fourier series

In a numerous technical problems one can find functions whose course is repeating. These functions describe periodic events in many physical processes (vibrations of constructions, steady movement of the piston of internal combustion engines, steady rotation movements) or mechanical oscillations (acoustic waves) and alternating electric current.

Such functions are called *periodic*, that is if $f(t)$ is a function of a real variable t and there is a positive real number T such that for each t from the domain it holds that

$$f(t) = f(t+T). \tag{1.1}$$

The number T, for which equation (1.1) holds, is called *period*, hence the function f is *periodic with period* T. If T is a period of f then nT is also a period for each $n \in \mathbb{N}$. The smallest such T, for which equation (1.1) holds, is called the *prime period*. We point out that the prime period should not to exist. As an example, a constant function can be given that has as a period every positive real number, but does not have a prime period. Next, for simplicity we will use the notation of period, and from the context it will always be clear if the period is prime or not. For any $\alpha \in \mathbb{R}$ the interval $(\alpha, \alpha + T]$ is called the *interval of periodicity*; in particular the *basic interval of periodicity* is a special case where $\alpha = 0$ or $\alpha = -T/2$, that is the basic interval of periodicity takes the form $(0, T]$ or $(-T/2, T/2]$.

The next lemma shows that it is possible to restrict our attention to functions with the period 2π.

Lemma 1 *For every periodic function $f(t)$ with a period of T there is a transformation of an argument*[1] $t = \mathrm{tr}(x)$ *such that the transformed function $f(\mathrm{tr}(x))$ has a period of 2π.*

Proof: Let

$$t = \mathrm{tr}(x) = \frac{T}{2\pi}x.$$

[1]by a transformation of an argument we mean transformation of coordinates, such as in the case of transformation of Cartesian coordinates to polar, spherical of cylindrical ones.

Then

$$f(t) = f(\mathrm{tr}(x)) = f\left(\frac{T}{2\pi}x\right) = g(x).$$

The function $g(x)$ is defined for every x and is periodic with a period of 2π:

$$g(x+2\pi) = f\left(\frac{T}{2\pi}(x+2\pi)\right) = f\left(\frac{T}{2\pi}x + T\right) = f(t+T) = f(t) = g(x).$$

□

An elementary example to consider is *simple harmonic oscillation*, which is given by a general sine function

$$f(t) = A\sin(\omega t + \varphi). \tag{1.2}$$

Here, the variable t is interpreted as time, A is the *amplitude* indicating deviation from the equilibrium position, the argument $\omega t+\varphi$ is called the *phase of oscillation*, for $t = 0$ we get the *initial phase* and the constant ω, and indicating the number of oscillations from 2π seconds is called the *circular frequency (angular velocity)*. The time for one oscillation period is denoted by T, and in our case it is $T = 2\pi/\omega$.

In practice, however, we encounter more complex periodic functions (which we will show in the next sections) that can be written as the sum of an infinite series of simple harmonic oscillations, where the first term of this series has the same period as the given periodic function. The periods of the following oscillations are then a half, a third, etc., of the period of the first oscillation. This creates a periodic function expressing a compound harmonic oscillation, which is described by an infinite series with the terms

$$u_n = A_n\sin(n\omega t + \varphi_n). \tag{1.3}$$

These can be equivalently written in the form

$$u_n = a_n\cos(n\omega t) + b_n\sin(n\omega t), \tag{1.4}$$

where for simplicity we put

$$u_0 = \frac{a_0}{2}. \tag{1.5}$$

The series

$$\sum_{n=1}^{\infty} u_n = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\omega t) + b_n\sin(n\omega t)) \tag{1.6}$$

is called a *trigonometric series*. If the series converges, as will be shown later, then it converges to a function with a period[2] of $T = 2\pi/\omega$, that is with the period of the element with index 1. The coefficients a_n and b_n are called *Fourier coefficients of the function $f(t)$*.

[2]Observe that the period does not have to be prime. If $a_1 = b_1 = 0$ then $2\pi/\omega$ is a period, but is not the smallest one. In addition, if there at least one of the coefficients a_2 or b_2 is non-zero, then the prime period is π/ω.

1.1 Fourier series

In Section 1.5 will be shown that the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)) \tag{1.7}$$

is uniformly convergent in $\mathbb{R}$, and its sum is a continuous periodic function $f(t)$ with a period of the first term series[2], that is $T = 2\pi$ (here, Lemma 1 was applied, hence $T = 2\pi$ and $\omega = 1$). Consequently,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)). \tag{1.8}$$

Now, the task is to find coefficients' a_n and b_n uniformly convergent trigonometric series (1.8) using the function $f(t)$ which is its sum. To solve this problem the orthogonality of the systems function from Example 10 (page 31) on the interval $[-\pi, \pi]$, which is an interval of a length 2π.

The coefficient a_0 will be derived by integration of equation (1.8) from $-\pi$ to π. So,

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))\right)\mathrm{d}t = \pi a_0,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t, \tag{1.9}$$

here we used the fact that for each n, $\int_{-\pi}^{\pi}\cos(nt)\,\mathrm{d}t = 0$ and $\int_{-\pi}^{\pi}\sin(nt)\,\mathrm{d}t = 0$. The coefficients a_n will be derived from (1.8) multiplied by function $\cos(nt)$ and its integration on the same interval. Then using computations from Example 10 (on page 31) we get

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t = a_n\int_{-\pi}^{\pi}\cos^2(nt)\,\mathrm{d}t = a_n\pi,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t. \tag{1.10}$$

The coefficients b_n will be found analogously, as a_n under assumption (1.8) is multiplied by function $\sin(nt)$:

$$\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t = b_n\int_{-\pi}^{\pi}\sin^2(nt)\,\mathrm{d}t = b_n\pi,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t. \tag{1.11}$$

Formulae for the determination of coefficients are called *(Euler)-Fourier*. The given trigonometric series is called the *Fourier series of functions* $f(t)$ and coefficients a_n and b_n the *Fourier coefficients of the function* $f(t)$.

Naturally, this begs the question whether Fourier series (1.8) is convergent and if its sum equals $f(t)$ in the interval $[-\pi, \pi]$. The answer is given by the following theorem, which will be proved in Section 1.5:

Theorem 1 (Dirichlet's) *If the function $f(t)$ fulfills the so called Dirichlet conditions, then a Fourier series of the function $f(t)$ is convergent at every t to the value*

$$\frac{1}{2}(f(t+0)+f(t-0))$$

and it holds that

$$\frac{1}{2}(f(t+0)+f(t-0)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Moreover, in points t where $f(t)$ is continuous, it is

$$\frac{1}{2}(f(t+0)+f(t-0)) = f(t).$$

In the foregoing theorem we use standard notation[3]

$$f(t+0) = \lim_{t_1 \to t^+} f(t_1) \text{ a } f(t-0) = \lim_{t_1 \to t^-} f(t_1).$$

Dirichlet conditions are the following:

1. the function $f(t)$ is periodic,
2. the function $f(t)$ has on the interval of periodicity only finite numbers of discontinuities of the first type,
3. the function $f(t)$ has on the interval of periodicity piecewise continuous derivation.

Example 1 The following functions do not fulfill the Dirichlet conditions on the interval $[-\pi, \pi]$:

$$f_1(t) = \frac{2}{1-t}, \; f_2(t) = \sin\left(\frac{2}{2-t}\right).$$

Really, $f_1(t)$ has at the point $t_0 = 1$ discontinuity of the second kind and $f_2(t)$ has on the neighborhood of the point $t_0 = 2$ infinitely many extremes.

[3] sometimes a shortened version is used; $f(t+)$ resp. $f(t-)$

The relations (1.9) – (1.11) can be generalized for functions with the period $T = 2l$, hence also for functions with the interval of periodicity $[-l, l]$. Using Lemma 1, the transformation $t = \frac{\pi}{l}t$ can be done, and we get for $n \in \mathbb{N}$ the formulae:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,\mathrm{d}\,t, \tag{1.12}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos(n\frac{\pi}{l}t)\,\mathrm{d}\,t, \tag{1.13}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin(n\frac{\pi}{l}t)\,\mathrm{d}\,t \tag{1.14}$$

and the Fourier series takes the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{\pi}{l}nt) + b_n\sin(\frac{\pi}{l}nt)). \tag{1.15}$$

1.2 Fourier series in a complex field

In Section 1.1 the Fourier coefficients a_n and b_n were derived from a Fourier series of a periodic function with a period of 2π. These formulae take the form:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt)), \tag{1.16}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}\,t, \tag{1.17}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t, \tag{1.18}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t. \tag{1.19}$$

Now, let us write functions $\sin(nt)$ and $\cos(nt)$ in series (1.16) using the following exponential form:

$$\cos(nt) = \frac{1}{2}(e^{\mathrm{i}nt} + e^{-\mathrm{i}nt}), \tag{1.20}$$

$$\sin(nt) = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}nt} - e^{-\mathrm{i}nt}) = -\frac{\mathrm{i}}{2}(e^{\mathrm{i}nt} - e^{-\mathrm{i}nt}). \tag{1.21}$$

Substituting (1.20) and (1.21) into the series (1.16) we get

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n(\frac{e^{\mathrm{i}nt} + e^{-\mathrm{i}nt}}{2}) - \mathrm{i}b_n(\frac{e^{\mathrm{i}nt} - e^{-\mathrm{i}nt}}{2})\right) = \tag{1.22}$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{1}{2}(a_n - \mathrm{i}b_n)e^{\mathrm{i}nt} + \frac{1}{2}(a_n + \mathrm{i}b_n)e^{-\mathrm{i}nt}\right). \tag{1.23}$$

Now, put

$$c_0 = \frac{1}{2}a_0, \tag{1.24}$$

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n), \tag{1.25}$$

$$c_{-n} = \frac{1}{2}(a_n + \mathrm{i}b_n), \tag{1.26}$$

where c_n and c_{-n} are complex adjoint coefficients. Hence, Fourier complex coefficients c_n and c_{-n} are:

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\cos(nt) - \mathrm{i}\sin(nt))\,\mathrm{d}\,t = \tag{1.27}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\mathrm{i}nt}\,\mathrm{d}\,t,\ \ n = 1, 2, 3, \ldots, \tag{1.28}$$

$$c_{-n} = \frac{1}{2}(a_n + \mathrm{i}b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(\cos(nt) + \mathrm{i}\sin(nt))\,\mathrm{d}\,t = \tag{1.29}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{\mathrm{i}nt}\,\mathrm{d}\,t,\ \ n = 1, 2, 3, \ldots \tag{1.30}$$

For the coefficient c_0 we get

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}\,t.$$

So, we can observe that all coefficients c_i can be expressed by one formula

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\mathrm{i}nt}\,\mathrm{d}\,t,\ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{1.31}$$

Putting c_n into (1.23) we get the following form of the Fourier series

$$f(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{\mathrm{i}nt} + c_{-n}e^{-\mathrm{i}nt}) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}nt}, \tag{1.32}$$

where c_n is given by (1.31). The series (1.32) is called a *complex Fourier series of the function* $f(t)$. The coefficients c_n are called *complex Fourier coefficients*.

Note the benefit of the complex form of the Fourier series (1.31), that is to get its coefficients it suffices to compute only one integral (the integral of a complex function of a complex variable). Next, if $f(t)$ has a period of T, then (1.31) and (1.32) take the form

$$f(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{\mathrm{i}n\omega t} + c_{-n}e^{-\mathrm{i}n\omega t}), \tag{1.33}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-\mathrm{i}n\omega t}\,\mathrm{d}\,t,\ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{1.34}$$

where $\omega = 2\pi/T$.

Finally, if we would like to write Fourier series in a real form that is given in a complex form, then for the coefficients' expression it suffices to apply:

$$a_n = c_n + c_{-n}, \tag{1.35}$$

$$b_n = \mathrm{i}(c_n - c_{-n}). \tag{1.36}$$

Equations (1.35) and (1.36) were derived from (1.25) and (1.26).

Example 2 Find the complex and real form of the Fourier series of the function $f(t) = 1/2\, e^t$ with the basic interval of periodicity $(0, \pi]$ a $f(0) = f(\pi)$.

Let us follow the above mentioned remark, that is find out the complex form first, and then do the transformations to the real one. Hence by (1.34) it is (here $\omega = 2$)

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}e^t e^{-2\mathrm{i}nt}\,\mathrm{d}\,t = \frac{1}{2\pi}\int_0^{\pi} e^{(1-2\mathrm{i}n)t}\,\mathrm{d}\,t = \frac{1}{2\pi}\frac{1}{1-2\mathrm{i}n}\left[e^{(1-2\mathrm{i}n)t}\right]_0^{\pi} = \\ &= \frac{1}{2\pi}\frac{1}{1-2\mathrm{i}n}(e^{\pi}-1),\ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

The complex form of the Fourier series takes the form

$$\begin{aligned} f(t) &= \frac{1}{2\pi}(e^{\pi}-1) + \frac{e^{\pi}-1}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1}{1-2\mathrm{i}n}e^{2\mathrm{i}nt} + \frac{1}{1+2\mathrm{i}n}e^{-2\mathrm{i}nt}\right) = \\ &= \frac{e^{\pi}-1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1-2\mathrm{i}n}e^{2\mathrm{i}nt}. \end{aligned}$$

Let us transform this series into the real form. Firstly, by (1.35) and (1.36) find coefficients a_n and b_n:

$$\begin{aligned} a_n &= c_n + c_{-n} = \frac{1}{2\pi}(e^{\pi}-1)\left(\frac{1}{1-2\mathrm{i}n} + \frac{1}{1+2\mathrm{i}n}\right) = \\ &= \frac{e^{\pi}-1}{\pi}\frac{1}{1+4n^2},\ \ n = 0, 1, 2, 3, \ldots, \\ b_n &= \mathrm{i}(c_n - c_{-n}) = \frac{\mathrm{i}}{2\pi}(e^{\pi}-1)\left(\frac{1}{1-2\mathrm{i}n} - \frac{1}{1+2\mathrm{i}n}\right) = \\ &= -2\frac{e^{\pi}-1}{\pi}\frac{n}{1+4n^2},\ \ n = 1, 2, 3, \ldots \end{aligned}$$

Finally, the real form of the Fourier series is:

$$\begin{aligned} f(t) &= \frac{1}{2\pi}(e^{\pi}-1) + \frac{e^{\pi}-1}{\pi}\left(\frac{\cos(2t)}{1+4\cdot 1^2}+\frac{\cos(4t)}{1+4\cdot 2^2}+\ldots\right)- \\ &\quad - 2\frac{e^{\pi}-1}{\pi}\left(\frac{\sin(2t)}{1+4\cdot 1^2}+\frac{\sin(4t)}{1+4\cdot 2^2}+\ldots\right)= \\ &= \frac{1}{2\pi}(e^{\pi}-1) + \frac{e^{\pi}-1}{\pi}\sum_{n=0}^{\infty}\frac{1}{1+4n^2}\cos(2nt)- \\ &\quad - 2\frac{e^{\pi}-1}{\pi}\sum_{n=1}^{\infty}\frac{n}{1+4n^2}\sin(2nt). \end{aligned}$$

An integral part of harmonic analysis is *spectrum analysis*. Here we will address the question of the *phase* and the *amplitude spectrum*.

Firstly, by *one-sided spectrum* we mean an ordered pair of sequences

$$(\{A_n\}_{n=0}^{\infty}, \{\varphi_n\}_{n=1}^{\infty}),$$

where $\{A_n\}_{n=0}^{\infty}$ stands for *one-sided amplitude spectrum* and is defined by:

$$A_0 = \left|\frac{a_0}{2}\right| = |c_0|, \tag{1.37}$$

$$A_n = \sqrt{a_n^2+b_n^2} = 2|c_n|, \; n=1,2,\ldots \tag{1.38}$$

and $\{\varphi_n\}_{n=1}^{\infty}$ is a *one-sided phase spectrum* defined by

$$\varphi_n = -\arg c_n \in (-\pi,\pi], \; n=1,2,\ldots. \tag{1.39}$$

By *two-sided spectrum* we mean a pair of sequences

$$(\{|c_n|\}_{n=-\infty}^{\infty}, \{\varphi_{\pm n}\}_{n=1}^{\infty}),$$

where $\{|c_n|\}_{n=-\infty}^{\infty}$ stands for a *two-sided amplitude spectrum* and $\{\varphi_{\pm n}\}_{n=1}^{\infty}$ a *two-sided phase spectrum* defined by

$$\varphi_n = -\arg c_n \in (-\pi,\pi], \; n=\pm 1,\pm 2,\pm 3\ldots. \tag{1.40}$$

Note that the phase φ_0 is not defined. If it is analyzed as a complex function with a non-zero imaginary part, it holds that the coefficients c_n and c_{-n} are not a complex adjoint. So, the amplitude spectrum is not even, and the phase spectrum is not odd.

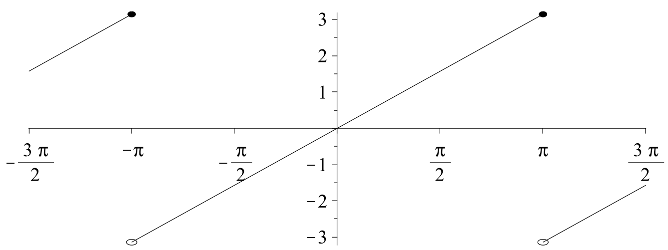

Figure 1.1: Graph of the function (1.41).

1.3 Development of periodic function

Let us first deal with a situation where we have to develop a Fourier series of periodic functions. If we have a periodic function, then we can easily develop it according to (1.12), (1.13) and (1.14). Let us now give two examples; based on Lemma 1, we can consider the basic interval of periodicity 2π without lost of generality.

Example 3 Let us develop the periodic function in the Fourier series $f(t)$ with a basic interval of periodicity $(-\pi, \pi]$ (see Figure 1.1) defined by

$$f(t) = \begin{cases} t & \text{for } t \in (-\pi, \pi], \\ \pi & \text{for } t = -\pi, \end{cases} \tag{1.41}$$

and do the spectral analysis.

Firstly, we need to verify the Dirichlet conditions that:
1. the function is obviously periodic,
2. the function is continuous in the interval of periodicity, and discontinuous in the border points $(2k+1)\pi$, $(k \in \mathbb{Z})$, which is just discontinuity of the first kind,
3. the function has in the interval of periodicity the derivative $(f'(t) = 1)$.

Hence, we can apply equations (1.9), (1.10) and (1.11) to determine Fourier coefficients:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} t \, \mathrm{d}t = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)\, \mathrm{d}t = \frac{1}{\pi}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_{-\pi}^{\pi} = 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\, \mathrm{d}t = \frac{1}{\pi}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_{-\pi}^{\pi} = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

Note that the developed function is odd, and all coefficients a_n are zero, hence Fourier series that will contain only sinusoidal elements, will be odd. This is not a

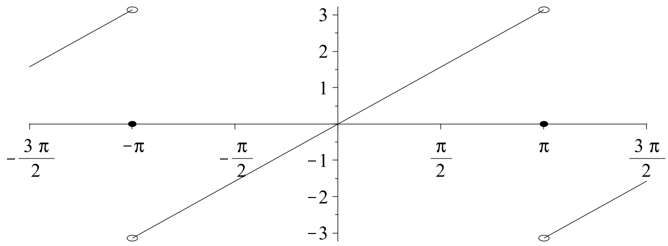

Figure 1.2: Graph of the sum of the development of the function (1.41).

coincidence, as we will show in the next section Therefore:

$$f(t) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nt)}{n}.$$

Through Dirichlet theorem 5 we see that the sum $f(t) = t$ for $t \in (-\pi, \pi)$. In points $\pm\pi$ of discontinuities of the first kind it holds that:

$$f(\pi_-) = \pi \text{ a } f(\pi_+) = -\pi,$$

$$f(-\pi_-) = \pi \text{ a } f(-\pi_+) = -\pi.$$

Hence

$$\frac{f(-\pi_+) + f(-\pi_-)}{2} = 0,$$

$$\frac{f(\pi_+) + f(\pi_-)}{2} = 0,$$

these values have the sum of the series in points $\pm\pi$, that is $f(\pi) = 0$ and $f(-\pi) = 0$; the graph of the sum is in Figure 1.2.

Partial sums of the first members

$$s_1(t) = 2\sin(t), \tag{1.42}$$

$$s_2(t) = 2\left(\sin(t) - \frac{\sin(2t)}{2}\right), \tag{1.43}$$

$$s_3(t) = 2\left(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3}\right), \tag{1.44}$$

$$s_4(t) = 2\left(\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4}\right), \tag{1.45}$$

are in the pictures 1.3, 1.4, 1.5 and 1.6, respectively.

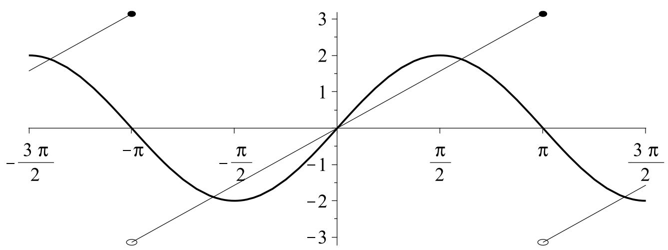

Figure 1.3: Graph of the sum $s_1(t)$ of the development of (1.41).

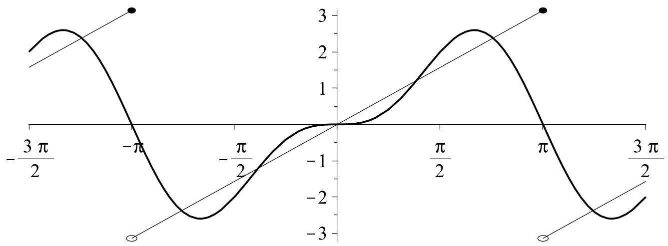

Figure 1.4: Graph of the sum $s_2(t)$ of the development of (1.41).

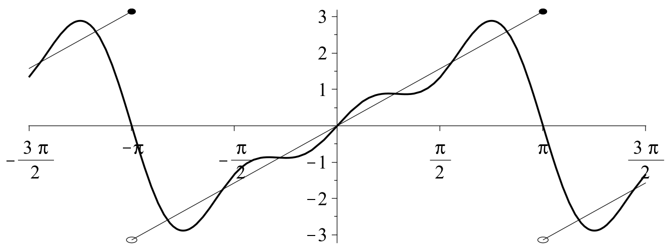

Figure 1.5: Graph of the sum $s_3(t)$ of the development of (1.41).

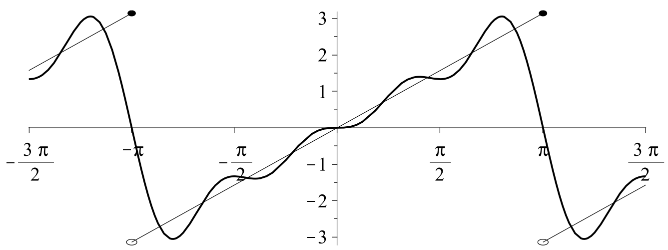

Figure 1.6: Graph of the sum $s_4(t)$ of the development of (1.41).

Let us now compose a one-sided and two-sided phase and amplitude spectrum, using formulas (1.37), (1.38), (1.39), (1.40) and (1.24), (1.25), (1.26):

$$\begin{aligned}
A_0 &= \left|\frac{a_0}{2}\right| = 0,\\
A_n &= \sqrt{a_n^2 + b_n^2} = \sqrt{0 + (-1)^{n+1}\frac{2}{n}} = \frac{2}{n},\\
c_n &= \frac{1}{2}(a_n - \mathrm{i}b_n) = \frac{1}{2}\left(0 - \mathrm{i}(-1)^{n+1}\frac{2}{n}\right) = \mathrm{i}(-1)^n\frac{1}{n},\\
\varphi_n &= -\arg c_n = \begin{cases} -\pi/2 & \text{for } n = \dots, -5, -3, -1, 2, 4, 6, \dots, \\ \pi/2 & \text{for } n = \dots, -6, -4, -2, 1, 3, 5, \dots. \end{cases}
\end{aligned}$$

The two-sided amplitude (resp. phase) spectrum is shown in Figure 1.7 (resp. 1.8). The values of the coefficients are given in table 1.1.

n	-3	-2	-1	0	1	2	3
a_n	—	—	—	0	0	0	0
b_n	—	—	—	—	2	-1	2/3
c_n	$\mathrm{i}/3$	$-\mathrm{i}/2$	i	0	$-\mathrm{i}$	$\mathrm{i}/2$	$-\mathrm{i}/3$
$\lvert c_n\rvert$	$1/3$	$1/2$	1	0	1	$1/2$	$1/3$
A_n	—	—	—	0	2	1	2/3
φ_n	$-\pi/2$	$\pi/2$	$-\pi/2$	—	$\pi/2$	$-\pi/2$	$\pi/2$

Table 1.1: Table of coefficients of harmonic analysis of a function (1.41).

Example 4 Let us develop the periodic function in the Fourier series $f(t)$ with a basic interval of periodicity $(-\pi, \pi]$ (see Figure 1.9) given by:

$$f(t) = \begin{cases} t & \text{for } t \in [0, \pi], \\ -t & \text{for } t \in (-\pi, 0), \end{cases} \tag{1.46}$$

and do a spectral analysis.

We will analyze the problem exactly as in the previous example.

Firstly, we verify Dirichlet's conditions:

1. the function is obviously periodic,

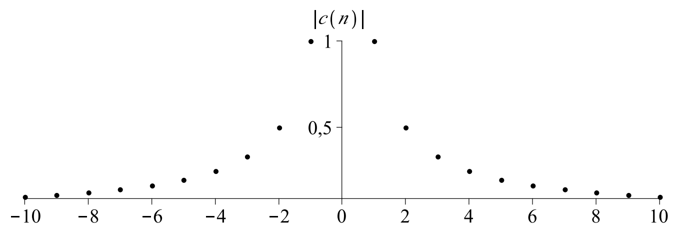

Figure 1.7: Two-sided amplitude spectrum of the function (1.41).

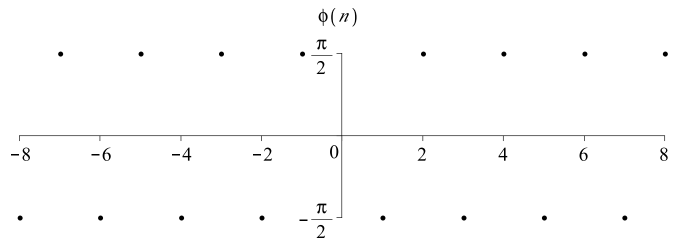

Figure 1.8: Two-sided phase spectrum of the function (1.41).

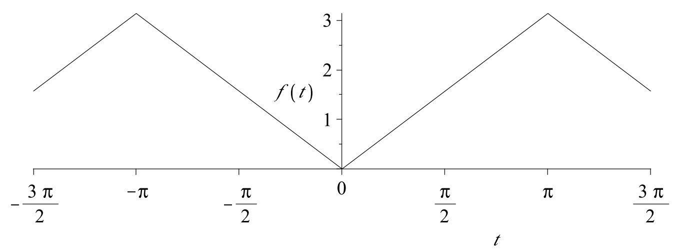

Figure 1.9: Graph of the function (1.46)

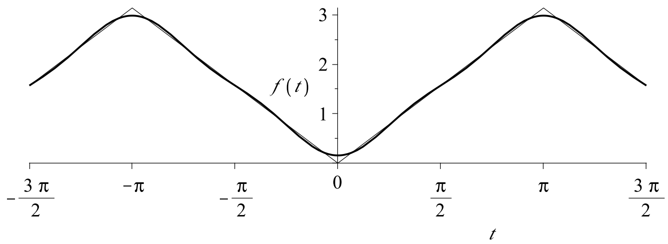

Figure 1.10: Graph of the sum $s_2(t)$ of the development of (1.46).

2. the function is continuous,
3. the function has on the interval $(k\pi, \pi + k\pi)$ a derivative for $k \in \mathbb{Z}$.

Hence, we can apply (1.9), (1.10) and (1.11) to get Fourier coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{1}{\pi}\left(\int_{-\pi}^{0} -t\,\mathrm{d}t + \int_0^{\pi} t\,\mathrm{d}t\right) = \pi, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t = \frac{1}{\pi}\left(\int_{-\pi}^{0} -t\cos(nt)\,\mathrm{d}t + \int_0^{\pi} t\cos(nt)\,\mathrm{d}t\right) = \\
&= \frac{2}{\pi n^2}((-1)^n - 1), \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t = \frac{1}{\pi}\left(\int_{-\pi}^{0} -t\sin(nt)\,\mathrm{d}t + \int_0^{\pi} t\sin(nt)\,\mathrm{d}t\right) = 0.
\end{aligned}$$

It is worth noting that if a developed function is even, and all coefficients b_n are zero, then the Fourier series will have only cosine elements; it will be even. This is not a coincidence, as we will show in the next section. The sought-after development of our function is therefore

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)t}{(2n-1)^2},$$

where we applied $a_n = 0$ for even n.

Due to Dirichlet's theorem 5, the sum of this series equal $f(t)$ for $t \in \mathbb{R}$; a graph of the sum is in Figure 1.9. Partial sums of the first members:

$$s_2(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9}\right), \tag{1.47}$$

$$s_3(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(t) + \frac{\cos(3t)}{9} + \frac{\cos(5t)}{25}\right), \tag{1.48}$$

are given in Figures 1.10 and 1.11 respectively.

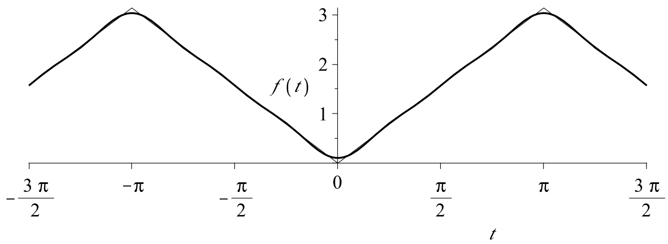

Figure 1.11: Graph of the sum $s_3(t)$ of the development of (1.46).

Let us now compose a one-sided and two-sided phase and amplitude spectrum, using formulas (1.37), (1.38), (1.39), (1.40) and (1.24), (1.25), (1.26):

$$\begin{aligned}
A_0 &= \left|\frac{a_0}{2}\right| = \pi/2, \\
A_n &= \sqrt{a_n^2 + b_n^2} = \sqrt{\frac{2}{\pi n^2}((-1)^n - 1) + 0} = \frac{2}{\pi n^2}|(-1)^n - 1|, \\
c_n &= \frac{1}{2}(a_n - \mathrm{i}b_n) = \frac{1}{2}\left(\frac{2}{\pi n^2}((-1)^n - 1) - \mathrm{i}0\right) = \frac{1}{\pi n^2}((-1)^n - 1), \\
\varphi_n &= -\arg c_n = \begin{cases} 0 & \text{for } n = \pm 2, \pm 4, \pm 6, \dots, \\ \pi & \text{for } n = \pm 1, \pm 3, \pm 5, \dots. \end{cases}
\end{aligned}$$

The two-sided amplitude (resp. phase) spectrum is shown in Figure 1.12 (resp. 1.13). The values of the coefficients are given in table 1.2.

n	-3	-2	-1	0	1	2	3
a_n	—	—	—	π	$-4/\pi$	0	$-4/(9\pi)$
b_n	—	—	—	—	0	0	0
c_n	$-2/(9\pi)$	0	$-2/\pi$	$\pi/2$	$-2/\pi$	0	$-2/(9\pi)$
$\|c_n\|$	$2/(9\pi)$	0	$2/\pi$	0	$2/\pi$	0	$2/(9\pi)$
A_n	—	—	—	$\pi/2$	$4/\pi$	0	$4/(9\pi)$
φ_n	π	0	π	—	π	0	π

Table 1.2: Table of coefficients of harmonic analysis of a function (1.46).

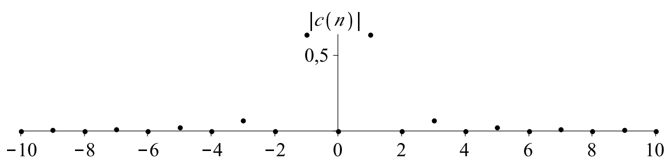

Figure 1.12: Two-sided amplitude spectrum of the function (1.46).

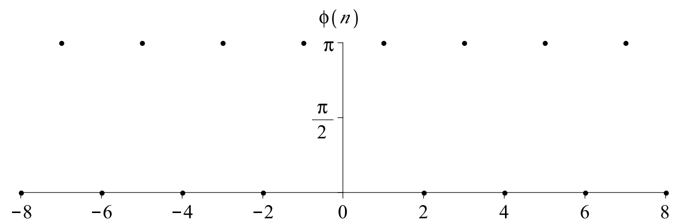

Figure 1.13: Two-sided phase spectrum of the function(1.46).

1.4 Sine and cosine series

In Exercises 3 and 4 we observed the interesting property that when the given function is odd (resp. even), then the Fourier series contains only sine (resp. cosine) elements. The following theorems describe this observation

Theorem 2 *Let $f(t)$ be an odd function with a period of 2π satisfying Dirichlet's conditions. Then its Fourier expansion contains only sine terms*

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt). \tag{1.49}$$

Proof: Firstly, by Dirichlet's theorem 5 the series (1.49) converges.

Now we will show that $a_n = 0$ for $n \in \mathbb{N} \cup \{0\}$ and $b_n \in \mathbb{R}$. Since the function $f(t)$ is even, that is $f(-t) = -f(t)$ for each t, and also $f(t)\cos(nt)$ is odd, we get for $n \in \mathbb{N}$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}\,t = \frac{1}{\pi}\left(\int_{-\pi}^{0} -f(-t)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\,\mathrm{d}\,t\right) = \\
&= \frac{1}{\pi}\left(-\int_{0}^{\pi} f(t)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\,\mathrm{d}\,t\right) = 0, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t = \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(t)\cos(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t\right) = \\
&= \frac{1}{\pi}\left(-\int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t\right) = 0,
\end{aligned}$$

where we used linear substitutions to calculate the first integral in the penultimate equality $t = -t$.

The function $f(t)\sin(nt)$ is obviously even, for $n \in \mathbb{N}$ we have

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t = \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(t)\sin(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t\right) = \\
&= \frac{1}{\pi}\left(\int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t\right) = \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t.
\end{aligned}$$

□

Theorem 3 *Let $f(t)$ be an even periodic function with a period of 2π satisfying Dirichlet's conditions. Then its Fourier evolution contains only cosine terms*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt). \tag{1.50}$$

Proof: We will lead the proof as before, now $f(-t) = f(t)$. Hence, the function $f(t)\sin(nt)$ is obviously odd and for $n \in \mathbb{N}$ we have

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t = \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(t)\sin(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t\right) = \\
&= \frac{1}{\pi}\left(-\int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\sin(nt)\,\mathrm{d}\,t\right) = 0.
\end{aligned}$$

Next, the function $f(t)\cos(nt)$ is even and for $n \in \mathbb{N} \cup \{0\}$ it is

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t = \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(t)\cos(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t\right) = \\ &= \frac{1}{\pi}\left(\int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t + \int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t\right) = \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos(nt)\,\mathrm{d}\,t. \end{aligned}$$

□

The previous proofs are also instructive for how to calculate the relevant coefficients.

If the function $f(t)$ is even with the period $T = 2l$ with the basic interval of periodicity $(-l, l]$, all coefficients will be $a_n = 0$ and

$$b_n = \frac{2}{l}\int_0^l f(t)\sin\left(n\frac{\pi}{l}t\right)\mathrm{d}\,t.$$

If the function $f(t)$ is odd with the period $T = 2l$ with the basic interval of periodicity $(-l, l]$, all coefficients will be $b_n = 0$ and

$$a_n = \frac{2}{l}\int_0^l f(t)\cos\left(n\frac{\pi}{l}t\right)\mathrm{d}\,t.$$

Let us assume that on the interval $(0, l]$ is given a function $f(t)$ satisfying Dirichlet's conditions and we would like develop it into a Fourier series. Such a task can be performed in various ways. The entered function can be extended to an interval $(-l, l]$, which we can do by defining the interval $(-l, 0)$ so that the extension is even or odd.

Definition 1 *Let $f(t)$ be a piecewise continuous function on the interval* $(0, l]$. Odd periodic extension of the function $f(t)$ *with the basic interval of periodicity* $(-l, l]$ *is a function* $g(t)$ *defined by*

$$g(t) = \begin{cases} f(t) & \textit{for } t \in [0, l], \\ -f(-t) & \textit{for } t \in (-l, 0). \end{cases} \tag{1.51}$$

Definition 2 *Let $f(t)$ be a piecewise continuous function on the interval* $(0, l]$. Even periodic extension of the function $f(t)$ *with the basic interval of periodicity* $(-l, l]$ *is a function* $g(t)$ *defined by*

$$g(t) = \begin{cases} f(t) & \textit{for } t \in (0, l], \\ f(-t) & \textit{for } t \in (-l, 0). \end{cases} \tag{1.52}$$

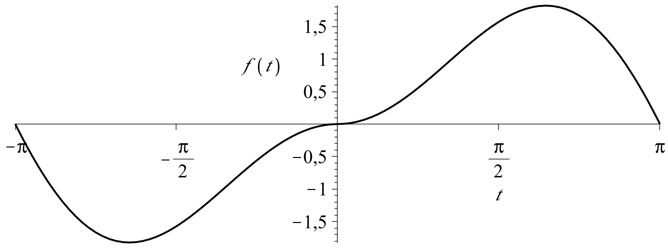

Figure 1.14: Graph of odd extension of the function (1.53).

The series (1.49) is called a *sine Fourier sequence* and the sequence (1.50) *cosine Fourier sequence*.

The attentive reader has noticed that we assume a basic interval $(0, l]$, which may seem restrictive when the specified function is defined on an interval $(a, b]$. Then just perform a coordinate transformation $\tau = t - a$ and transform the interval $(a, b]$ on $(0, l]$, where $l = b - a$.

Example 5 Let's develop the following function in the sine and cosine Fourier series

$$f(t) = t \sin(t) \text{ for } t \in (0, \pi]. \tag{1.53}$$

Sine Fourier sequence

Firstly, we make an odd extension (see Figure 1.14). The developing function has a period of 2π and the basic interval of periodicity $(-\pi, \pi]$. according to Theorem 2, $a_n = 0$ and $b_n = \frac{2}{l} \int_0^l f(t) \sin\left(n\frac{\pi}{l}t\right) \mathrm{d}\,t$. So, for $n = 2, 3, \dots$ it is

$$b_n = \frac{2}{\pi} \int_0^{\pi} t \sin(t) \sin(nt) \,\mathrm{d}\,t = \frac{4n}{\pi} \frac{(-1)^n - 1}{(n-1)^2(n+1)^2}.$$

For $n = 1$ we get

$$b_1 = \frac{2}{\pi} \int_0^{\pi} t \sin^2(t) \,\mathrm{d}\,t = \frac{\pi}{2}.$$

The series gets the form

$$f(t) = \frac{\pi}{2} \sin(t) + \sum_{n=2}^{\infty} \frac{4n}{\pi} \frac{(-1)^n - 1}{(n-1)^2(n+1)^2} \sin(nt).$$

Cosine Fourier sequence

Firstly, we make an even extension (see Figure 1.15). The developing function has a period of 2π and the basic interval of periodicity $(-\pi, \pi]$. according to Theorem 3, $b_n = 0$ and $a_n = \frac{2}{l} \int_0^l f(t) \cos\left(n\frac{\pi}{l}t\right) \mathrm{d}\,t$. So, for $n = 0, 2, 3, \dots$ it is

$$a_n = \frac{2}{\pi} \int_0^{\pi} t \sin(t) \cos(nt) \,\mathrm{d}\,t = 2\frac{(-1)^{n+1}}{(n-1)(n+1)}.$$

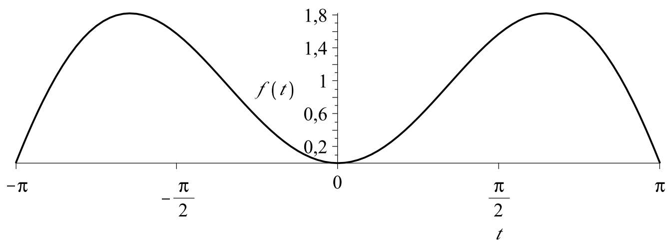

Figure 1.15: Graph of even extension of the function (1.53).

For $n = 1$ we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} t\sin(t)\cos(t)\,\mathrm{d}t = \frac{1}{2}.$$

The series gets the form

$$f(t) = 1 + \frac{1}{2}\cos(t) + \sum_{n=2}^{\infty} 2\frac{(-1)^{n+1}}{(n-1)(n+1)}\cos(nt).$$

1.5 Properties of Fourier series

Theorem 4 *For every piecewise continuous function $f(t)$ on the interval $[a,b]$ it holds that*

$$\lim_{n\to\infty}\int_a^b f(t)\sin(nt)\,\mathrm{d}t = 0, \tag{1.54}$$

$$\lim_{n\to\infty}\int_a^b f(t)\cos(nt)\,\mathrm{d}t = 0. \tag{1.55}$$

Proof: If the interval $[a,b]$ has the length 2π, then it is easy to see that formulae (1.54) and (1.55) hold. If the interval $[a,b]$ has a length greater than 2π, then split it into $k+1$ intervals

$$[a,b] = \left[\bigcup_{i=1}^{k}[a+2(i-1)\pi, a+2i\pi]\right] \cup [a+2k\pi, a+2k\pi+b],$$

where the length of the last one is less then 2π. Now, extend the function $f(t)$ on the right from point b such that it will equal zero on the interval $[b, a+2(k+1)\pi]$. Then

$$\int_a^b f(t)\sin(nt)\,\mathrm{d}t = \sum_{i=1}^{k+1}\int_{a+2(i-1)\pi}^{a+2i\pi} f(t)\sin(nt)\,\mathrm{d}t$$

and each of the integrals on the right is for $n\to\infty$ in a limit equal to zero. Analogously to (1.55), proving this theorem. □

The following theorem, which we mentioned in section 1.1 and used in section 1.3, gives the answer to the question of what conditions the functions must meet $f(t)$, so that the respective Fourier series converge.

Theorem 5 (Dirichlet's) *If the function $f(t)$ fulfills the so called Dirichlet conditions, then the Fourier series of the function $f(t)$ is convergent at every t to the value*

$$\frac{1}{2}(f(t+0)+f(t-0))$$

and it holds that

$$\frac{1}{2}(f(t+0)+f(t-0)) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Moreover in points t, where $f(t)$ is continuous, it is

$$\frac{1}{2}(f(t+0)+f(t-0)) = f(t).$$

Proof of Dirichlet's theorem: It is possible to show that (see [9]), for every n it holds that

$$1 = \frac{1}{\pi}\int_0^{\pi} \frac{\sin\left(\left(\frac{1}{2}+n\right)t\right)}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t. \tag{1.56}$$

The second relation will be multiplied by

$$\frac{f(t+0)+f(t-0)}{2} \tag{1.57}$$

and we get

$$\frac{f(t+0)+f(t-0)}{2} = \frac{1}{\pi}\int_0^{\pi} \left[\frac{f(t+0)+f(t-0)}{2}\right] \frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u. \tag{1.58}$$

Now, we introduce

$$R_n(t) = s_n(t) - \frac{f(t+0)+f(t-0)}{2}, \tag{1.59}$$

where

$$s_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kt) + b_k \sin kt). \tag{1.60}$$

To prove this theorem, it is now sufficient to show that

$$\lim_{n\to\infty} R_n(t) = 0.$$

For the partition sum it holds that (see [9])

$$s_n(t) = \frac{1}{\pi}\int_0^{\pi}\left[\frac{f(t+u)+f(t-u)}{2}\right]\frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u. \tag{1.61}$$

Then from (1.58) and (1.61) we get

$$\begin{aligned} R_n(t) &= \frac{1}{\pi}\int_0^{\pi}\left[\frac{f(t+u)+f(t-u)}{2}\right]\frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u - \\ &\quad -\frac{1}{\pi}\int_0^{\pi}\left[\frac{f(t+0)+f(t-0)}{2}\right]\frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u. \end{aligned}$$

We adjust the integral

$$\begin{aligned} R_n(t) &= \frac{1}{\pi}\int_0^{\pi}\left[\frac{f(t+u)-f(t+0)}{2}\right]\frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u + \\ &\quad +\frac{1}{\pi}\int_0^{\pi}\left[\frac{f(t-u)-f(t-0)}{2}\right]\frac{\sin\left(\left(\frac{1}{2}+n\right)u\right)}{\sin\left(\frac{u}{2}\right)}\,\mathrm{d}u = \\ &\quad = I_1 + I_2. \end{aligned}$$

On both integrals I_1 and I_2 it is possible to apply Theorem 4. Indeed, for I_1 we have

$$\begin{aligned}
\pi I_1 &= \int_0^{\pi} \left[\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)} \right] \sin\left(\left(\frac{1}{2}+n \right) u \right) \mathrm{d}\,u = \\
&= \int_0^{\pi} \left[\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)} \right] \sin\left(\frac{u}{2}\right) \cos(nu)\,\mathrm{d}\,u + \\
&\quad + \int_0^{\pi} \left[\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)} \right] \cos\left(\frac{u}{2}\right) \sin(nu)\,\mathrm{d}\,u = \\
&= \frac{1}{2}\int_0^{\pi} [f(t+u)-f(t+0)] \cos(nu)\,\mathrm{d}\,u + \\
&\quad + \int_0^{\pi} \left[\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)} \right] \cos\left(\frac{u}{2}\right) \sin(nu)\,\mathrm{d}\,u = \\
&= J_1 + J_2.
\end{aligned}$$

To the integral J_1 it is possible to apply Theorem 4 directly. For the integral J_2 it is necessary to investigate the behavior of the function

$$\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)}$$

in the point $u = 0$. We get it with a simple adjustment

$$\frac{f(t+u)-f(t+0)}{2\sin\left(\frac{u}{2}\right)} = \frac{f(t+u)-f(t+0)}{u} \frac{\frac{u}{2}}{\sin\left(\frac{u}{2}\right)}.$$

The first factor converges for $u \to 0$ to the limit

$$\lim_{u\to 0} \frac{f(t+u)-f(t+0)}{u} = \lim_{h\to 0} \frac{f(t+h)-f(t+0)}{h} = f'(t),$$

which is assumed to be final. The second factor has a limit for $u \to 0$ equal to 1 and this function is piecewise continuous. Hence, also on the integral J_2 it is possible to apply Theorem 4.

Consequently,

$$\lim_{n\to\infty} I_1 = 0.$$

It will be proved analogously

$$\lim_{n\to\infty} I_2 = 0.$$

Hence, this theorem is proved. □

1.6 The space $L_2(a, b)$

Let us denote by $L_2(a, b)$ the set of all appropriate (see Remark 1) complex functions $f : (a, b) \to \mathbb{C}$, for which the integral is

$$\int_a^b |f(t)|^2 \, \mathrm{d}\, t \tag{1.62}$$

finite. A function belonging to this space $L_2(a, b)$ is called *integrable with square*.

Remark 1 *The attentive reader may have noticed the highlighted word* appropriate *in the above definition. This expression is essential for the correctness of the text. It is not possible to build a space of integrable functions with a square without restrictions. Mathematicians can define this limitation precisely, in a single word* measurable*. Under the condition of* measurable *is all correct. In order to use this term, we must build a theory of measure that goes beyond this text and many engineering studies. Overall, therefore, the functions that are permissible for us,* suitable*, they are the ones we usually encounter in practice and in this text, unless otherwise stated. For further restrictions, pay attention to the following notes.*

The space $L_2(a, b)$ with the operation of adding $(h(t) = f(t) + g(t))$ and a multiplication of the scalar $(h(t) = c f(t))$ is linear. To see this, we must verify the following conditions:

1. the sum of each of the two functions from $L_2(a, b)$ is again a function from $L_2(a, b)$. We know that the inequality $|a+b|^2 \le 2(|a|^2+|b|^2)$ works. So, it is enough to put $a = f(t)$ and $b = g(t)$. Then $|f(t)+g(t)|^2 \le 2(|f(t)|^2+|g(t)|^2)$. By integration of a given function on the interval (a, b) we have $\int_a^b |f(t) + g(t)|^2 \,\mathrm{d}\, t \le 2(\int_a^b |f(t)|^2 \,\mathrm{d}\, t + \int_a^b |g(t)|^2 \,\mathrm{d}\, t)$. Both integrals on the right-hand side of the inequality are finite, so the integral on the left-hand side is finite, and a function $f(t) + g(t)$ is integrable with the square.

2. multiplication of a function from $L_2(a, b)$ by a complex number is again a function from $L_2(a, b)$. Indeed, $\int_a^b |c f(t)|^2 \,\mathrm{d}\, t = |c|^2 \int_a^b |f(t)|^2 \,\mathrm{d}\, t$, so the function $c f(t)$ is integrable with the square.

Example 6 By integration it is possible to verify
$f(t) = \dfrac{1+\mathrm{i}}{\sqrt{t}} \in L_2[1, 2]$, $f(t) = \dfrac{1+\mathrm{i}}{\sqrt{t}} \notin L_2(0, 1]$ a $f(t) = \dfrac{1+\mathrm{i}}{\sqrt[4]{t}} \in L_2(0, 1]$.

Remark 2 A special space is often used to solve many problems; $L_1(a, b)$, the *space of functions integrable on an interval* (a, b). This space is again linear. We say that the function f belongs to $L_1(a, b)$ if it is on the interval (a, b) *(absolute) integrable*, that is

$$\int_a^b |f(t)| \,\mathrm{d}\, t < \infty.$$

Apparently, if $f(t) \in L_2(a,b)$, then $f(t) \in L_1(a,b)$. The converse implication is not true in general, see Example 8.

Example 7 By integration it is possible to show
$f(t) = \dfrac{1+\mathrm{i}}{\sqrt{t}} \in L_1[1,2]$ a $f(t) = \dfrac{1+\mathrm{i}}{\sqrt[4]{t}} \in L_1(0,1]$.

Example 8 Let's decide if the function $f(t) = \dfrac{1}{\sqrt{t-1}}$ is integrable with a square, or at least absolute integrable on the interval $[1,2]$. Let's calculate the relevant integrals:

$$\int_1^2 \frac{1}{\sqrt{t-1}}\,\mathrm{d}t = \lim_{u\to 1}\int_u^2 \frac{1}{\sqrt{t-1}}\,\mathrm{d}t = \lim_{u\to 1}[2\sqrt{t-1}]_u^2 = 2,$$

$$\int_1^2 \left(\frac{1}{\sqrt{t-1}}\right)^2 \mathrm{d}t = \lim_{u\to 1}\int_u^2 \frac{1}{t-1}\,\mathrm{d}t = \lim_{u\to 1}[\ln|t-1|]_u^2 = \infty.$$

Hence, the function $\dfrac{1}{\sqrt{t-1}} \in L_1[1,2]$, but $\dfrac{1}{\sqrt{t-1}} \notin L_2[1,2]$.

Let us now state a sufficient condition for integrable functions with a square.

Theorem 6 *Each piecewise continuous function on a closed interval is integrable with a square on this interval.*

Remark 3 *The above given theorem characterizes the elements of space $L_2(a,b)$ and supplements Remark 1. For completeness, however, it is necessary to define a piecewise continuous function. It is a function that meets the following conditions:*

1. *there is a finite partition of the interval (a,b) such that partial intervals are pairwise disjoint and their union is exactly the interval (a,b),*

2. *the function restricted on each partial interval is continuous.*

The only thing to consider is the boundedness of the function on the respective partial interval, which the diligent reader can handle themselves.

Let f and g be functions from $L_2(a,b)$, then *scalar multiplication of the functions f and g on the interval* (a,b) is defined by

$$(f,g) = \int_a^b f(t)\overline{g}(t)\,\mathrm{d}t. \tag{1.63}$$

The *norm* of the function f from $L_2(a,b)$ is given by

$$\|f\| = \sqrt{(f,f)}. \tag{1.64}$$

We therefore understand the norm of a function as its distance from the zero function.

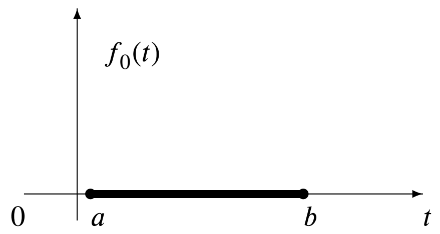

Figure 1.16: Graph of the function f_0

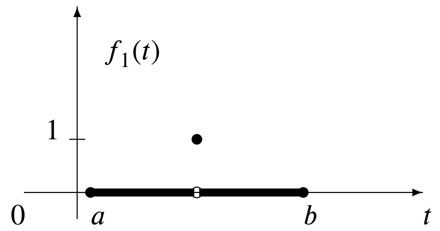

Figure 1.17: Graph of the function f_1

Remark 4 *The inquisitive reader may have noticed that if they take a function, for example $f_0(t)$ a $f_1(t)$ defined in Figures 1.16 and 1.17 then*

$$\|f_0\| = \|f_1\| = 0.$$

So it gets the same norm for two different functions, which is not right. Taking into account Remark 1, then according to the Lebesgue measure these functions are equal; they differ only on a set of zero measure.

Let us formulate the basic properties of the scalar product and norm; the proof is easy and that is why we leave it to the reader:

Lemma 2 *Let f, g and h be functions from $L_2(a, b)$ and $c \in \mathbb{C}$, then*

1. $(f, f) = \int_a^b f(t)\overline{f}(t)\,\mathrm{d}\,t = \int_a^b |f(t)|^2\,\mathrm{d}\,t = \|f\|^2$,
2. $(cf, g) = c(f, g)$,
3. $(f + h, g) = (f, g) + (h, g)$,
4. $(f, g) = \overline{(g, f)}$,
5. *Schwarz-Buňakovsky inequality:* $|(f, g)| \leq \|f\|\|g\|$,
6. $\|f\| = 0$ *if and only if* $f(t) = 0$ *for almost every* t,

7. $\|cf\| = |c|\|f\|$,

8. $\|f + g\| \leq \|f\| + \|g\|$.

The system of functions $\{f_n\}_{n=1}^{\infty}$ from $L_2(a, b)$ is *orthogonal*, if scalar multiplication of each of the two different functions equals zero, that is if for any form $m \neq n$ it holds that

$$(f_m, f_n) = 0. \tag{1.65}$$

In addition, if the norm of each sequence function is equal to one, we call such a system *orthonormal*, that is

$$(f_m, f_n) = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \tag{1.66}$$

Example 9 The system of functions $\{e^{\mathrm{i}nt}\}_{n=-\infty}^{\infty}$ is on the interval $[0, 2\pi]$ orthogonal, but not orthonormal. Firstly, $\int_0^{2\pi} |e^{\mathrm{i}nt}|^2 \,\mathrm{d}t = 2\pi$, hence $e^{\mathrm{i}nt} \in L_2[0, 2\pi]$ for each n. Next

$$(f_m, f_n) = \begin{cases} \int_0^{2\pi} e^{\mathrm{i}mt} e^{-\mathrm{i}nt} \,\mathrm{d}t = 2\pi & \text{for } m = n, \\ \int_0^{2\pi} e^{\mathrm{i}mt} e^{-\mathrm{i}nt} \,\mathrm{d}t = 0 & \text{for } m \neq n. \end{cases} \tag{1.67}$$

The given system of functions can be normalized in a simple way by multiplying each function by the inverse value of the norm, i.e.

$$\left\{ \frac{e^{\mathrm{i}nt}}{\|e^{\mathrm{i}nt}\|} \right\}_{n=-\infty}^{\infty} = \left\{ \frac{e^{\mathrm{i}nt}}{\sqrt{2\pi}} \right\}_{n=-\infty}^{\infty}.$$

Then obviously

$$\|\frac{e^{\mathrm{i}nt}}{\sqrt{2\pi}}\|^2 = \frac{1}{2\pi}(e^{\mathrm{i}nt}, e^{\mathrm{i}nt}) = 1.$$

Example 10 The sequence of trigonometric functions

$$1, \cos(t), \sin(t), \cos(2t), \sin(2t), \ldots$$

is on the interval $(-\pi, \pi)$ orthogonal, but not orthonormal. In really:

$$\text{for } m \neq n \text{ je } \begin{cases} \int_{-\pi}^{\pi} \cos(mt)\sin(nt) \,\mathrm{d}t = 0, \\ \int_{-\pi}^{\pi} \cos(mt)\cos(nt) \,\mathrm{d}t = 0, \\ \int_{-\pi}^{\pi} \sin(mt)\sin(nt) \,\mathrm{d}t = 0, \end{cases} \tag{1.68}$$

$$\text{for } m = n \text{ it is } \begin{cases} \int_{-\pi}^{\pi} \cos(mt)\sin(mt)\,\mathrm{d}t = 0, \\ \int_{-\pi}^{\pi} \cos(mt)\cos(mt)\,\mathrm{d}t = \pi, \\ \int_{-\pi}^{\pi} \sin(mt)\sin(mt)\,\mathrm{d}t = \pi. \end{cases} \tag{1.69}$$

By normalizing a given sequence, we obtain an orthonormal system of functions:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \frac{\cos(2t)}{\sqrt{\pi}}, \frac{\sin(2t)}{\sqrt{\pi}}, \dots$$

Example 11 The system of functions $\{e^{\mathrm{i}nt}\}_{n=-\infty}^{\infty}$ is not on the interval $[0,\pi]$ orthogonal. In really:

$$(f_m, f_n) = \int_0^{\pi} e^{\mathrm{i}mt}e^{-\mathrm{i}nt}\,\mathrm{d}t = \frac{(-1)^{m-n}-1}{m-n} \neq 0 \text{ for } m-n \text{ odd.}$$

Let a given sequence of functions be $\{f_n\}_{n=1}^{\infty}$ from $L_2(a,b)$. If there is a function $f \in L_2(a,b)$ such that

$$\lim_{n\to\infty} \|f_n - f\| = 0, \tag{1.70}$$

then the sequence is called $\{f_n\}_{n=1}^{\infty}$ *convergent to f in the norm $L_2(a,b)$*. Sometimes the designation *convergence in a diameter* or *convergence in terms of standard deviation* is also used.

The sequence of functions $\{f_n\}_{n=1}^{\infty}$ from $L_2(a,b)$ converges on M to a function f *uniformly*, if for any $\epsilon > 0$ there is n_0 such that for every $n > n_0$ and every $z \in M$ is $|f_n(z) - f(z)| < \epsilon$.

Note that if we know the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent, it is also convergent. The following example shows that this may not be the case.

Example 12 The geometric sequence $\{t^n\}_{n=1}^{\infty}$ is convergent but not uniformly on the interval $[0,1)$. It is easy to see that a given sequence converges to 0 on the interval $[0,1)$. Next, it holds that $\sup|t^n| = 1$ for $t \in [0,1)$, so 0 is not a limit, hence a given sequence is not convergent uniformly.

1.7 Generalized Fourier sequence

As we have already described in the previous paragraphs, the whole theory of Fourier series arose from the need to develop a given periodic function into a periodic function formed by trigonometric functions; see Example 10. Most of the theorems valid for Fourier series remain valid if we replace the trigonometric functions in the original considerations with a system of functions that are orthogonal or orthonormal (see Section 1.6).

Therefore, the question arises whether the given function $f(t)$, which is integrable with the square on the interval $[a, b]$, develops into an infinite series

$$\sum_{n=0}^{\infty} \alpha_n \varphi_n(t)$$

using an orthonormized system of functions $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi_n \in L^2(a, b)$ and determines the coefficients α_n.

1. Approximation

We will approximate the function $f(t)$ by a polynomial T_n based on the smallest standard deviation, provided that $\{\varphi_n(t)\}_{n=0}^{\infty}$ is an orthonormal system of functions. So the question is how to choose the coefficients α_n in the polynomial

$$T_k(t) = \alpha_0\varphi_0(t) + \alpha_1\varphi_1(t) + \cdots + \alpha_k\varphi_k(t),$$

to make the value of the integral

$$I_k = \frac{1}{b-a}\int_a^b [f(t) - T_k(t)]^2 \,\mathrm{d}\,t$$

minimal. Let's modify the given integral

$$I_k = \frac{1}{b-a}\left(\int_a^b [f(t)]^2 \,\mathrm{d}\,t - 2\int_a^b [f(t)T_k(t)] \,\mathrm{d}\,t + \int_a^b [T_k(t)]^2 \,\mathrm{d}\,t\right).$$

Then it is

$$\int_a^b f(t)T_k(t) \,\mathrm{d}\,t = \sum_{n=0}^{k} \alpha_n \int_a^b f(t)\varphi_n(t) \,\mathrm{d}\,t.$$

Denote $a_n = \int_a^b f(t)\varphi_n(t) \,\mathrm{d}\,t$ and let's call this number the *Fourier coefficient of the function $f(t)$ with respect to a given set of functions $\{\varphi_n(t)\}_{n=0}^{\infty}$*. Then we have

$$\int_a^b f(t)T_k(t) \,\mathrm{d}\,t = \sum_{n=0}^{\infty} \alpha_n a_n.$$

Now let's compute

$$\begin{aligned}
\int_a^b [T_k(t)]^2 \,\mathrm{d}\,t &= \int_a^b [\sum_{n=0}^{k} \alpha_n\varphi_n]^2 \,\mathrm{d}\,t = \\
&= \int_a^b \left(\sum_{n=0}^{k} \alpha_n^2\varphi_n^2 + 2\sum_{\substack{m,n=0\\ n<m}}^{k} \alpha_n\alpha_m\varphi_n\varphi_m\right) \mathrm{d}\,t = \\
&= \sum_{n=0}^{k} \alpha_n^2,
\end{aligned}$$

due to the orthonormality of the system$\{\varphi_n(t)\}_{n=0}^{\infty}$. The integral therefore has the form

$$I_k = \frac{1}{b-a}\left(\int_a^b f(t)^2\,\mathrm{d}\,t + \sum_{n=0}^{k}(\alpha_n^2 - 2\alpha_n a_n + a_n^2 - a_n^2)\right) = \tag{1.71}$$

$$= \frac{1}{b-a}\left(\int_a^b f(t)^2\,\mathrm{d}\,t + \sum_{n=0}^{k}(\alpha_n - a_n)^2 - \sum_{n=0}^{k} a_n^2\right). \tag{1.72}$$

This equation applies to any choice of coefficients α_n. Integral I_k therefore, has a minimum value when selected $\alpha_n = a_n$. Denote the polynomial T_k with coefficients $\alpha_n = a_n$ as P_k and relevant integrals J_k. Then

$$J_k = \frac{1}{b-a}\int_a^b [f(t) - P_k(t)]^2\,\mathrm{d}\,t = \frac{1}{b-a}\left(\int_a^b [f(t)]^2\,\mathrm{d}\,t - \sum_{n=0}^{k} a_n^2\right). \tag{1.73}$$

Because for every k

$$J_k \geq 0,$$

it is

$$\sum_{n=0}^{k} a_n^2 \leq \int_a^b [f(t)]^2\,\mathrm{d}\,t. \tag{1.74}$$

the inequality (1.74) is called *Bessel*. Because the formulae (1.73) and (1.74) hold for any k, infinite sequence

$$\sum_{n=0}^{\infty} a_n^2$$

is convergent, since all partial sums are smaller than the given fixed positive number according to (1.74) $\int_a^b [f(t)]^2\,\mathrm{d}\,t$.

2. Closedness

Let us now address the natural question of whether

$$\lim_{k\to\infty} J_k = 0.$$

Orthonormal systems that meet this property are called *closed*. Therefore, the following applies to them:

$$\lim_{k\to\infty}\int_a^b [f(t) - \sum_{n=0}^{k} a_n\varphi_n(t)]^2\,\mathrm{d}\,t = \int_a^b [f(t)]^2\,\mathrm{d}\,t - \sum_{n=0}^{\infty} a_n^2 = 0,$$

so

$$\int_a^b [f(t)]^2\,\mathrm{d}\,t = \sum_{k=0}^{\infty} a_k^2, \tag{1.75}$$

equality (1.75) is called *Parseval*. We have already proved that the sequence of trigonometric functions is in the interval $[0, 2\pi]$ orthonormal, see Example 10. Closedness can be proved by Fejér theorems.

3. Orthonormality

In the previous sections, we have already shown that the orthonormality of a system of functions significantly simplifies the development of a given function using these functions. Unfortunately, a power sequence of functions

$$\varphi_k(t) = t^k \text{ where } k = 0, 1, \dots \tag{1.76}$$

it is not orthonormal or normalized, despite the frequent task of developing a function $f(t)$ into a power series. The way to compile the coefficients of such a power series is known; it is enough to develop the given function $f(t)$ into a Taylor series. This method is theoretically good, but in practice we come across a problem that may not have a solution, namely determining the values of all orders of derivation of a given function. This problem can be eliminated by not developing the function with the system (1.76), but by a system composed of polynomials, which forms an orthonormed set of functions on a given interval.

Theorem 7 (Schmidt's) *Let*

$$\{\varphi_n(t)\}_{n=0}^{\infty} \tag{1.77}$$

be a sequence of continuous and nonzero functions on the interval $[a, b]$ *such that each finite block* $\varphi_0(t)$, $\varphi_1(t), \dots, \varphi_k(t)$ *denotes* $k+1$ *linearly independent functions. Then a sequence of functions can be created from this sequence*

$$\{\psi_n(t)\}_{n=0}^{\infty} \tag{1.78}$$

continuous on this interval $[a, b]$ *such that*

1. *each of its finite blocks* $\psi_0(t)$, $\psi_1(t), \dots, \psi_{k-1}(t)$ *denotes* k *linearly independent functions,*
2. *each function* $\psi_k(t)$ *is a linear combination of functions* $\varphi_0(t)$, $\varphi_1(t), \dots, \varphi_{k-1}(t)$,
3. *the sequence* (1.78) *forms an orthonormal sequence.*

The proof of this theorem has two parts. The first part is constructive; the appropriate orthonormal system is constructed. In the second, the validity of the properties described by the sentence is verified. Now let's do the construction, as the verification of the given properties is easy to do and has been left for the reader as an exercise.

Construction of the sequence(1.78):

1. Put

$$\psi_0(t) = \frac{\varphi_0(t)}{c_0}, \tag{1.79}$$

where $c_0^2 = \int_a^b \varphi_0^2(t)\,\mathrm{d}t$. The function $\psi_0(t)$ is obviously normalized. Hence,

$$\int_a^b \psi_0^2(t)\,\mathrm{d}t = 1/c_0^2 \int_a^b \varphi_0^2(t)\,\mathrm{d}t = 1.$$

2. Now, introduce

$$\chi_1(t) = \varphi_1(t) - a_{10}\psi_0(t), \tag{1.80}$$

where a_{10} is picked in such a way that the function $\chi_1(t)$ is orthogonal to the function $\psi_0(t)$, so it will be

$$\int_a^b \chi_1(t)\psi_0(t)\,\mathrm{d}t = \int_a^b [\varphi_1(t) - a_{10}\psi_0(t)]\psi_0(t)\,\mathrm{d}t = 0.$$

From here we have

$$a_{10} = \int_a^b a_{10}\psi_0^2(t)\,\mathrm{d}t = \int_a^b \varphi_1(t)\psi_0(t)\,\mathrm{d}t.$$

Denote

$$c_1^2 = \int_a^b \chi_1^2(t)\,\mathrm{d}t,$$

then the function

$$\psi_1(t) = \frac{\chi_1(t)}{c_1} \tag{1.81}$$

is on the interval $[a, b]$ orthonormal to the function $\psi_0(t)$.

Similarly, we introduce a function

$$\chi_2(t) = \varphi_2(t) - a_{20}\psi_0(t) - a_{21}\psi_1(t), \tag{1.82}$$

where we choose a_{20} and a_{21} such that $\chi_2(t)$ is orthogonal to the functions $\psi_0(t)$ and $\psi_1(t)$. We can easily deduce that

$$a_{20} = \int_a^b \varphi_2(t)\psi_0(t)\,\mathrm{d}t, \tag{1.83}$$

$$a_{21} = \int_a^b \varphi_2(t)\psi_1(t)\,\mathrm{d}t. \tag{1.84}$$

Then the function

$$\psi_2(t) = \frac{\chi_2(t)}{c_2} \text{ where } c_2^2 = \int_a^b \chi_2^2(t)\,\mathrm{d}t, \tag{1.85}$$

is normalized and orthogonal to $\psi_0(t)$ and $\psi_1(t)$.

3. To specify additional members of the sequence (1.78) it is enough to proceed analogously with the help of mathematical induction. Note the numbers c_k are generally different from zero with respect to the linear independence of each finite section of the sequence (1.77). From the construction (1.78), the linear independence of each finite block then follows.

Example 13 Apply Theorem 7 to the power sequence of functions (1.76) on the interval $[-1, 1]$. Using the process described above, we derive polynomials which, except for constant factors, are so-called *Legender polynomials*. Let's construct the first three members of the sought orthonormed system.

Firstly, $\varphi_0(t) = 1$ and

$$c_0^2 = \int_{-1}^{1} \varphi_0^2(t)\,\mathrm{d}\,t = 2,$$

so

$$c_0 = \sqrt{2}$$

and

$$\psi_0(t) = \frac{1}{\sqrt{2}}.$$

Next, put

$$\chi_1(t) = \varphi_1(t) - a_{10}\psi_0(t) = t - a_{10}\frac{1}{\sqrt{2}},$$

$$a_{10} = \int_{-1}^{1} \varphi_1(t)\psi_0(t)\,\mathrm{d}\,t = \int_{-1}^{1} \frac{t}{\sqrt{2}} = 0,$$

hence

$$\chi_1(t) = t$$

and

$$c_1^2 = \int_{-1}^{1} \chi_1^2(t)\,\mathrm{d}\,t = \int_{-1}^{1} t^2\,\mathrm{d}\,t = \frac{2}{3}.$$

So, we get

$$\psi_1(t) = \frac{\chi_1(t)}{c_1} = \frac{t}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}t.$$

Now, let

$$\chi_2(t) = \varphi_2(t) - a_{20}\psi_0(t) - a_{21}\psi_1(t) = t^2 - \frac{1}{3},$$

$$c_2^2 = \int_{-1}^{1} \chi_2^2\,\mathrm{d}\,t = \frac{8}{45}$$

and

$$\psi_2(t) = \frac{\chi_2(t)}{c_2} = \sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right).$$

The first three members of the search sequence are:

$$\begin{aligned}
\psi_0(t) &= \frac{1}{\sqrt{2}},\\
\psi_1(t) &= \sqrt{\frac{3}{2}}t,\\
\psi_2(t) &= \sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right).
\end{aligned}$$

Consequently, we can develop the function $f(t)$, which is continuous on the interval $[-1, 1]$, into a series made up of these polynomials.

1.8 Gibbs phenomenon

In the previous sections we dealt with the development of functions from $L_2(a, b)$ into the Fourier series. We already know that the Fourier series of functions $f \in L_2(0, 2\pi)$ converges in a norm of the space $L_2(0, 2\pi)$ to the function f. In addition, if other conditions are met (see Theorem 2.11 from [7]), then the Fourier series converges uniformly. A simple example of a function that does not meet these conditions is

$$f_0(t) = \frac{\pi - t}{2} \text{ pro } t \in (0, 2\pi),$$

which is periodically extended to the whole $\mathbb{R}$; see Figure 1.18. It can be verified (see Chapter 2 from [12]), that

$$f_0(t) = \sum_{k=1}^{\infty} \frac{\sin(kt)}{k}, \tag{1.86}$$

here equality applies in the sense of convergence in $L_2(l, l+2\pi)$, $l \in \mathbb{R}$, and also in terms of the uniform convergence on each interval with extreme points $2l\pi + \epsilon$ and $2(l+1)\pi - \epsilon$, where $\epsilon \in (0, \pi)$ (see Theorem 2.4 from [7]). The problem is therefore the points of discontinuity of the function f_0; for example, we do not know what is happening at the point $t = 0$. At this point the sum of the series (1.86) equals 0, thus, the average limit on the right and left is zero.

$$\frac{1}{2}\left[f_0(0+0) + f_0(0-0)\right] = 0.$$

Let us now focus on the error of the partial sum of the series (1.86)

$$R_n(t) = s_n(t) - f_0(t) = \sum_{k=1}^{n} \frac{\sin(kt)}{k} - \frac{\pi - t}{2} \text{ for } t \in (0, 2\pi).$$

We can easily verify that

$$R_n'(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2\sin\left(\frac{1}{2}t\right)},$$

$$R_n(0) = -\frac{\pi}{2}$$

and hence

$$R_n(t) = -\frac{\pi}{2} + \int_0^t \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{1}{2}x\right)} \,\mathrm{d}x.$$

The function $s_n(t)$ is increasing in a neighborhood of zero. Now let's look for such a point $t_n > 0$, in which the function $R_n(t)$ has a local extreme and is closest the zero. From the equation $R_n'(t) = 0$ we get

$$\sin\left(\left(n + \frac{1}{2}\right)t\right) = 0$$

and so

$$x_n = \pi\left(n + \frac{1}{2}\right)^{-1}.$$

If we denote $n + 1/2 = p$ and apply the substitution $px = s$ we get

$$R_n(t_n) = \int_0^{\pi/p} \frac{\sin(px)}{2\sin\left(\frac{1}{2}x\right)}\,\mathrm{d}\,x - \frac{\pi}{2} = \int_0^{\pi} \frac{\sin(s)}{2p\sin\left(\frac{s}{2p}\right)} - \frac{\pi}{2}.$$

For a large enough p (that is large enough for n) it is

$$2p\sin\left(\frac{s}{2p}\right) \geq 0,$$

for $s \in (0, \pi)$ and

$$\lim_{p\to\infty} 2p\sin\left(\frac{s}{2p}\right) = s.$$

Now we find the integral majorant and we get

$$\lim_{n\to\infty} R_n(t_n) = \int_0^{\pi} \frac{\sin(s)}{s}\,\mathrm{d}\,s - \frac{\pi}{2} \doteq 0,18\pi$$

and for a large n it is

$$s_n(t_n) \doteq 1,18\frac{\pi}{2}.$$

Each partial sum $s_n(t)$ has a maximum that exceeds by about 18% the maximum of the function f_0, see Figure 1.18. This phenomenon is called *Gibbs phenomenon*.

The maximum with increasing n is still significantly different to the maximum of the function f_0, just the point t_n is tending to zero, in which the maximum is reached. Therefore, we can never achieve a partial sum series (1.86) that approximates the function f_0 uniformly.

It is possible to demonstrate (see Chapter 6 in [7]) that each function f, which has a finite number of points of discontinuity of the first kind, can be written in the form

$$f(t) = g(t) + h(t),$$

where $g(t)$ is a function that meets our additional conditions imposed on the function f_0 and whose Fourier series converges uniformly, and where

$$h(t) = \sum_{i=1}^{m} c_i f_0(t - t_i)$$

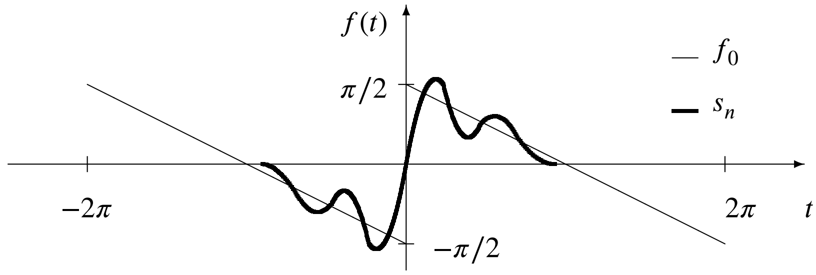

Figure 1.18: Gibbs phenomenon

is a function that captures the jumps of a function f.

From the properties of the function f_0 we know that the Fourier series of a function f converges at every point t to the value

$$\frac{1}{2}[f(t+0)+f(t-0)]$$

and that in every neighborhood of the function f, the Gibbs phenomenon manifests itself. Thus, a partial sum of the Fourier series of a function f will be in a neighborhood of each point of discontinuity t_i and acquire, except for a negligible deviation, a value

$$\frac{1}{2}[f(t_i+0)+f(t_i-0)] \pm \frac{1}{2}1,18[f(t_i+0)+f(t_i-0)],$$

therefore, the partial sums will not converge uniformly around the point of discontinuity.

1.9 Worked example

In this section we will solve one example in which we will look for the Fourier series of a periodic signal. As follows from the previous sections, the Fourier series of a periodic signal is a mathematical notation of the statement that a periodic signal $f(t)$ with a repetition frequency of $1/T$ can be composed of a constant signal and harmonic signals of frequencies $1/kT$ where $k = 1, 2, 3, \ldots$. Hence,

$$
\begin{aligned}
f(t) &= A_0 + \underbrace{A_1 \cos(\omega t + \varphi_1)}_{\textit{the first (basic) harmonic}} + \\
&\quad + A_2 \cos(2\omega t + \varphi_2) + A_3 \cos(3\omega t + \varphi_3) + \\
&\quad + \cdots + \underbrace{A_k \cos(k\omega t + \varphi_k)}_{\textit{k-th harmonic (higher)}} + \cdots \qquad (1.87) \\
&= \underbrace{A_0}_{\textit{direct component (mean value)}} + \underbrace{\sum_{k=1}^{\infty} A_k \cos(k\omega t + \varphi_k)}_{\textit{alternating component}},
\end{aligned}
$$

where A_k is the amplitude of k-th harmonic component, $k\omega$ is the circular repetition frequency k-th harmonic component and φ_k is the initial phase of k-th harmonic element.

From the above formula (1.87) it is obvious that each periodic signal has an *alternating* and *direct element*. The direct component is equal to the mean value of the signal over the repetition period. The alternating component consists of harmonic signals with zero mean values, so it is the original signal devoid of the direct component. The alternating component contains the so called *first harmonic* of a frequency, which is the same as the repetition frequency of the periodic signal, and from the *higher harmonic*, of which there is generally an infinite number and whose frequency is an integer multiple of the frequency of the first harmonic.

In decomposition of the above periodic signal (1.87) the sub-components are unambiguous and every two different periodic signals of the repetition frequency ω are unambiguously represented by different pairs of sets $\{A_0, A_1, \ldots A_k, \ldots\}$ and $\{\varphi_0, \varphi_1, \ldots \varphi_k, \ldots\}$; see the section Fourier series in the complex plane 1.2. The graphical representation of these sets in the form of spectral lines on the frequency axis is called the *spectrum of the signal*.

If a signal passes through an electrical circuit, we can understand it as the passage of a set of its harmonic components. Due to the different transmission capabilities of the circuit at different frequencies, the individual harmonic components at the output of the circuit will be differently attenuated and phase shifted, so that the output signal will also be periodic, but will be distorted compared to the input signal. The signal spectrum, resp. the distribution of its spectral lines on the frequency axis, together with the frequency characteristics of the circuit, provides a useful and illustrative tool for understanding the phenomena associated with signal-circuit interactions.

Because the harmonic signal can be written in other forms than those shown in the discussed formula (1.87) (specifically in the decomposition into sine and cosine

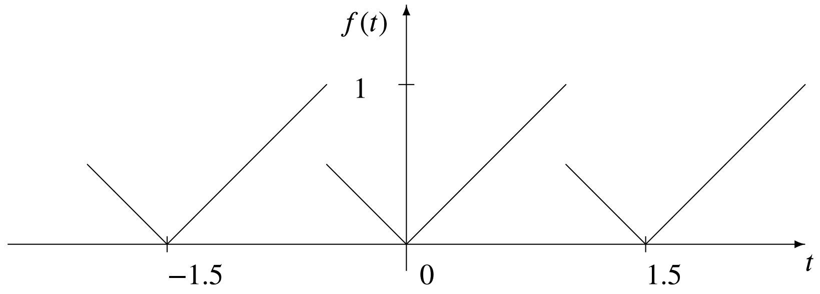

Figure 1.19: Graph of the periodic signal $f(t) = |t|$ with the basic periodicity interval $(-0.5, 1)$.

components and also in a complex form as the sum of two rotating phasors), there are corresponding shapes of the Fourier series, see sections 1.4 and 1.2. Further motivational considerations and applications can be found in the work [1], which are also motivated by the previous paragraphs.

Let us now construct a Fourier series using 10 harmonic signals

$$f(t) = |t|$$

on the basic periodicity interval $(-0.5, 1)$ see Figure 1.19. Next, let's perform a harmonic analysis, that is, construct an amplitude and phase spectrum.

The procedure and the necessary formulae for the calculations are given in the section Fourier series in the complex field 1.2. The result can therefore be found by direct calculation using a pencil and paper, or we can use a suitable algorithm. We will use the Matlab software; the relevant code of the m-file is given below with comments; compare the formulae from the Fourier series section in the complex field 1.2 with the kernel of the 1.1 algorithm written in lines 19 to 29.

After performing the calculation of the above algorithm in Matlab, we obtain the solution of the calculation, i.e. the Fourier series of the respective 10 harmonics (Figure 1.20), amplitude (Figure 1.21) and phase (Figure 1.22) spectrum.

Algorithm 1.1: Fourier series development algorithm

```
function Fourier_series(f,a,b,N)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Development of given function f(t) from L2(a,b)
% into Fourier series
% in complex plane using N harmonic components
% Calling sequence:   Fourier_series('abs(t)',-0.5,1,10)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

syms t real;     % Symbolic variable t
T=sym(b-a);    % Period
w=2*pi/T;         % Angular speed

% Computation of coefficients c_n of Fourier series f_N
% and its assembly
% Computation of amplitudes A_n
%and phases FIn for n=-N,...,N

fN=0; An=zeros(2*N+1,1); FIn=zeros(2*N+1,1);
for n=-N:N
  cn=1/T*int(f*exp(-i*w*n*t),t,a,b);
       % Coefficient of the n-th element of FR
  fN=fN+cn.*exp(i*w*n*t);
       % Assembly of Fourier series f_N
  An(n+N+1)=abs(double(cn));
       % Computation of amplitude (indexation vec. from 1)
  FIn(n+N+1)=-angle(double(cn));
       % Computation of phase (indexation vec. from 1)
end;

% Graphs of functions f and f_N
figure; hold on; grid on; box on;
set(gca,'FontSize',14);

% Original funkction f
hf=ezplot(f,[a,b]);
set(hf,'Color','Red','LineWidth',2);

% Approximation f_N by N harmonic elements
hfN=ezplot(fN,[a,b]);
set(hfN,'Color','Blue',
'LineWidth',2,'LineStyle','--');
xlabel('t');
title(['Fourierova rada funkce f(t)=',f]);
legend('f(t)', ['f_{',num2str(N),'}(t)'],
'Location','NorthEastOutside');
```

Algorithm 1.2: Algoritmus rozvoje ve Fourierovu řadu

```

% Amplitude spectrum
figure; hold on; grid on; box on;
set(gca,'FontSize',14);
bar(-N:N,An); xlabel('n');
title('Amplitudove spektrum')

% Phase spectrum
figure; hold on; grid on; box on;
set(gca,'FontSize',14);
bar(-N:N,FIn); xlabel('n');
title('Fazove spektrum')
```

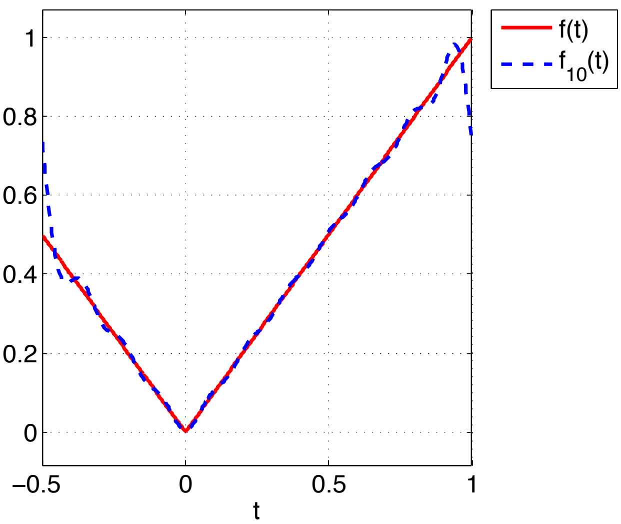

Figure 1.20: Graph of 10 harmonic.

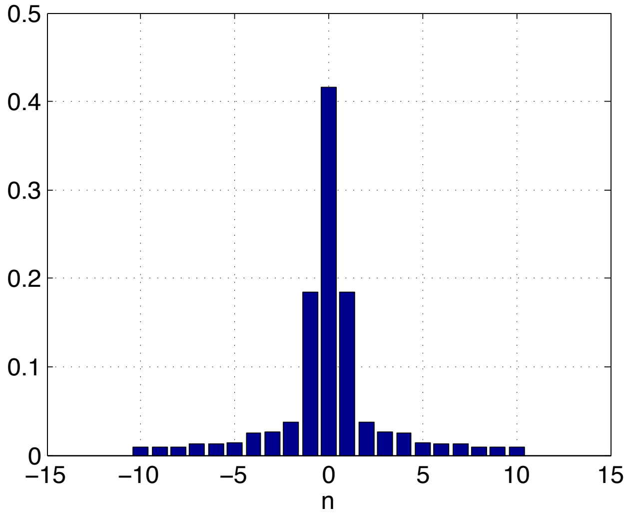

Figure 1.21: Amplitude spectrum of given signal.

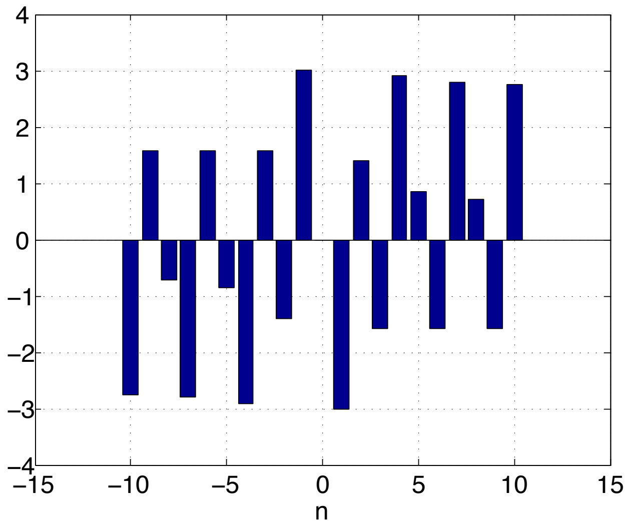

Figure 1.22: Phase spectrum of given signal

1.10 Appendix

Historically, Fourier series have emerged in the field of mathematical physics. The motivation was to solve the initial boundary value problem for the wave and diffusion equation at the finite interval. J. Fourier (1768 - 1830), a French mathematician and physicist, came up with a solution to the problem.

The method he proposed, which today bears his name, later led to a systematic study of trigonometric series, now called Fourier series. We will construct it for the wave equation; other constructions and examples can be found in [4].

Let us initially construct a boundary value problem that describes the oscillation of the string on a finite length interval l. The end points of the string are fixed in the zero position. Consider the initial deviation $\varphi(x)$ and the initial speed $\psi(x)$. So the problem has the shape

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && 0 < x < l, && t > 0, \\ &u(0,t) = u(l,t) = 0, && && t > 0, \\ &u(x,0) = \varphi(x), && u_t(x,0) = \psi(x), && 0 < x < l, \end{aligned} \tag{1.88}$$

where $u = u(x,t)$ describes the deflection of the string at a point x and time t.

Let's assume that the solution exists and has the shape

$$u(x,t) = X(x)T(t), \tag{1.89}$$

where $X = X(x)$ and $T = T(t)$ are real functions of one real variable having a continuous second derivative. The variables x and t in this case are separated from each other. Letting (1.89) into the equation of the problem (1.88) we get

$$XT'' = c^2 X'' T.$$

We divide by an element $-c^2XT$; here we assume that $XT \neq 0$, so

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)}.$$

This relationship tells us that the expression on the left depends only on the time variables t and the expression on the right depends only on the spatial variables x. Moreover, this equality must apply to all $t > 0$ and $x \in (0,l)$ and hence

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

where λ is a constant.

So we converted the original partial differential equation to an ordinary differential equation with separated variables with unknown functions $X(x)$ and $T(t)$

$$X''(x) + \lambda X(x) = 0, \tag{1.90}$$

$$T''(t) + c^2 \lambda T(t) = 0. \tag{1.91}$$

Next, we have the boundary conditions of the problem (1.88), so we have boundary conditions

$$X(0) = X(l) = 0. \tag{1.92}$$

We will first solve the marginal problem (1.90), (1.92). The obvious trivial solution$X(x) = 0$ is not interesting for us, therefore, if we rule out the case $\lambda \leq 0$. If $\lambda > 0$, then the equation (1.90) has the solution

$$X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and from the boundary conditions (1.92) it follows that

$$X(0) = C = 0,$$

$$X(l) = D\sin(\sqrt{\lambda}l) = 0.$$

So we get a non-trivial solution where

$$\sin(\sqrt{\lambda}l) = 0,$$

or

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, n \in \mathbb{N}.$$

Every λ_n corresponds to the solution

$$X_n(x) = C_n \sin\left(\frac{n\pi x}{l}\right), n \in \mathbb{N}, \tag{1.93}$$

where C_n are arbitrary constants.

Now let's solve the equation (1.91); the solution for $\lambda = \lambda_n$ takes the form

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right), n \in \mathbb{N}, \tag{1.94}$$

where A_n and B_n are again arbitrary constants. The original partial differential equation (1.88) solves the sequence of functions

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right), n \in \mathbb{N},$$

which meets the prescribed boundary conditions. The attentive reader may have noticed that instead of A_nC_n (resp. B_nC_n) we write A_n (resp. B_n), since they are arbitrary real constants. The problem is linear, and therefore any finite sum is again the solution

$$u(x,t) = \sum_{n=1}^{N}\left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right), n \in \mathbb{N}. \tag{1.95}$$

We still have to take into account the initial conditions. So function (1.95) will fulfill the initial conditions, if applicable

$$\varphi(x) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{l}\right), \tag{1.96}$$

$$\psi(x) = \sum_{n=1}^{N} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right). \tag{1.97}$$

The task for any initial data (1.88) is uniquely solvable and the appropriate solution is given by formula (1.95). The conditions (1.96) and (1.97) are very restrictive and difficult to guarantee. For this reason, we are looking for a solution to problem (1.88) in the form of an infinite sum, and we express it in the form of a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right), \; n \in \mathbb{N}. \qquad (1.98)$$

The constants A_n and B_n are then given as Fourier coefficients of sine developments of functions $\varphi(x)$ and $\psi(x)$, hence

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right),$$

$$\psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi x}{l}\right).$$

In other words, solving problem (1.88) for the wave equation, it is at all times t expressed in the form of a Fourier sine series in the variable x, if the initial conditions of $\varphi(x)$ and $\psi(x)$ can be expressed. It turns out that for a sufficiently wide class of functions, such a decomposition is possible, and the respective series converge. In this case, we calculate the coefficients using the formulae

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}\, x,$$

$$B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) \mathrm{d}\, x.$$

Finally, it should be noted that convergence needs to be discussed for correctness. Convergence considerations need to be applied to functions φ and ψ;, this have been done in the previous sections.

Chapter 2

Laplace transform

The Laplace transform is an effective method for solving various practical tasks in the fields of mathematical physics, electrical engineering, and control systems.

This chapter is motivated by the works [5] - [13] and is organized as follows. We will first introduce the necessary terms, then in section 2.1 we will pay attention to the properties of the Laplace transform. In section 2.2 we will learn to perform the inverse Laplace transformation, and finally in section 2.3 we will look at the following application examples: solution of differential equations (and systems), examples from areas of electrical engineering and control systems.

We will consider complex functions f of the real variable $t \in (-\infty, \infty)$, that is $f : \mathbb{R} \to \mathbb{C}$, and complex variable $p = x + \mathrm{i}y \in \mathbb{C}$. Let us assume that the improper integral

$$\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t \tag{2.1}$$

exists and has a finite value for at least one p. Then the integral (2.1) is called the *Laplace integral of the function f*.

Example 14 Let's calculate the Laplace integral of a function $f(t) = 1$. By (2.1) we have

$$\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t = \int_0^\infty e^{-pt}\,\mathrm{d}\,t = \lim_{\alpha\to\infty}\int_0^\alpha e^{-pt}\,\mathrm{d}\,t = \lim_{\alpha\to\infty}\left(\frac{1}{p} - \frac{1}{p}e^{-p\alpha}\right).$$

Since $\alpha \in \mathbb{R}$, for $p = x + \mathrm{i}y$ it holds that $|e^{-p\alpha}| = e^{-x\alpha}$. So, for $\operatorname{Re} p > 0$ it holds that $\lim_{\alpha\to\infty} e^{-p\alpha} = 0$, and the Laplace integral of the function $f(t) = 1$ for $\operatorname{Re} p > 0$ converges and equals the function $1/p$. For $\operatorname{Re} p \leq 0$ the Laplace integral does not exist.

Example 15 Let's calculate the Laplace integral of a function $f(t) = e^{at}$ where

$a \in \mathbb{C}$. By (2.1) we have

$$\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t = \int_0^\infty e^{at}e^{-pt}\,\mathrm{d}t = \lim_{\alpha\to\infty}\int_0^\alpha e^{(a-p)t}\,\mathrm{d}t =$$
$$= \lim_{\alpha\to\infty}\left(\frac{1}{a-p}e^{(a-p)\alpha} - \frac{1}{a-p}\right) = \frac{1}{p-a}$$

for $\operatorname{Re}(p-a) > 0$. So, the Laplace integral $f(t) = e^{at}$ converges for $\operatorname{Re} p > \operatorname{Re} a$ to the function $1/(p-a)$ and otherwise diverges.

Definition 3 *Let f be a complex function of a real variable $t \in (-\infty, \infty)$. Let $M \subset \mathbb{C}$ be a set of all p, for which the Laplace integral* (2.1) *converges. Then the complex function F defined by*

$$F(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t \quad (p \in M) \tag{2.2}$$

is called the Laplace image of the function f. *A given map that assigns a function f to its Laplace image F, is called a* Laplace transform*, and is denoted by*

$$\mathcal{L}(f(t)) = F(p).$$

Definition 4 *The function f is called a* subject *(sometimes also* preimage or original*), if the following conditions are fulfilled:*

1. *f is on the interval* $[0, \infty)$ *piecewise continuous,*
2. *$f(t) = 0$ for each $t < 0$,*
3. *there is a real number $M > 0$ and α such that for each $t \in [0, \infty)$ it holds that*
$$|f(t)| \le Me^{\alpha t}. \tag{2.3}$$

Definition 5 *Let $\alpha_0 = \inf\{\alpha \in \mathbb{R} : \ \alpha \text{ satisfies } (2.3)\}$. The number α_0 is called* the growth index of the function f.

An important example of the subject is the Heaviside function shown in Figure 2.1, defined by

$$\eta(t) = \begin{cases} 0, & \text{for } \ t < 0, \\ 1, & \text{for } \ t \ge 0. \end{cases} \tag{2.4}$$

Theorem 8 (on the existence of the Laplace image) *Let f be a preimage with the growth index α_0. Then the Laplace integral*

$$F(p) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t$$

converges in the half-plane $\operatorname{Re} p > \alpha_0$ *(see Figure 2.1) absolutely and defines Laplace image $\mathcal{L}(f(t)) = F(p)$, which is in this half-plane analytical function.*

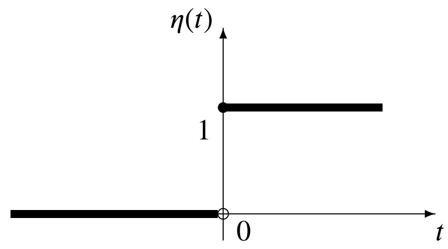

Figure 2.1: Heaviside function.

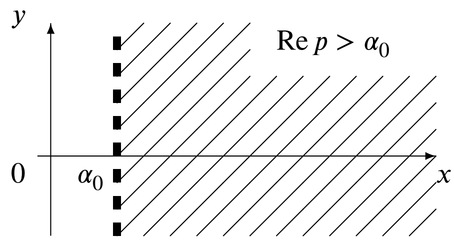

Figure 2.2: Half-plane $\operatorname{Re} p > \alpha_0$.

Proof: First, let's prove the absolute convergence of the integral in the half-plane $\operatorname{Re} p > \alpha_0$. The existence of the integral $\int_0^\tau f(t)e^{-pt}\,\mathrm{d}t$, for every $\tau > 0$, follows from the fact that $f(t)$ is on $[0,\infty)$ piecewise continuous. If $p = x + \mathrm{i}y$ and $\operatorname{Re} p > \alpha_0$, then $|e^{-pt}| = e^{-xt}$. From the third condition to the subject, then for each α such that $x > \alpha > \alpha_0$ it follows that

$$|f(t)e^{-pt}| \le Me^{\alpha t}e^{-xt} = Me^{(\alpha-x)t}.$$

So,

$$\left|\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t\right| \le \int_0^\infty |f(t)e^{-pt}|\,\mathrm{d}t \le M\int_0^\infty e^{(\alpha-x)t}\,\mathrm{d}t = \left[M\frac{e^{(\alpha-x)t}}{\alpha-x}\right]_0^\infty.$$

Since $\alpha - x < 0$, it is $\lim_{t\to\infty} e^{(\alpha-x)t} = 0$ and

$$\left|\int_0^\infty f(t)e^{-pt}\,\mathrm{d}t\right| \le \frac{M}{x-\alpha}. \tag{2.5}$$

Thus we have proved that the Laplace integral converges absolutely in the half-plane $\operatorname{Re} p > \alpha_0$.

Now, we will show that F is analytic in the half-plane $\operatorname{Re} p > \alpha_0$. Let α_1 be such that $x \ge \alpha_1 > \alpha > \alpha_0$. Then from the third condition we have the subject

$$|f(t)e^{-pt}| \le Me^{(\alpha-\alpha_1)t}.$$

The expression of the right side of inequality does not depend on p and since $\alpha - \alpha_1 < 0$, integral

$$\int_0^\infty Me^{(\alpha-\alpha_1)t}\,\mathrm{d}t$$

converges. Thus the Laplace integral converges in the half-plane $\operatorname{Re} p \geq \alpha_1$ absolutely. Moreover, it holds that

$$\frac{\partial}{\partial p}[f(t)e^{-pt}] = -tf(t)e^{-pt}$$

and

$$|-tf(t)e^{-pt}| \leq Mte^{(\alpha-\alpha_1)t},$$

where

$$\int_0^\infty Mte^{(\alpha-\alpha_1)t}\,\mathrm{d}t = \frac{M}{(\alpha-\alpha_1)^2}.$$

That means, the integral

$$\int_0^\infty -tf(t)e^{-pt}\,\mathrm{d}t$$

converges in the half-plane $\operatorname{Re} p \geq \alpha_1$.

Overall, therefore, the Laplace integral of a function f can be integrable according to the parameter p. That is, the function F is analytical in the half-plane $\operatorname{Re} p \geq \alpha_1 > \alpha_0$. Since α_1 was arbitrarily taken, it follows that F is analytical in the half-plane $\operatorname{Re} p > \alpha_0$. □

Corollary 1 *Let $\mathcal{L}(f(t)) = F(p)$ and $x = \operatorname{Re} p$. Then $\lim_{x\to\infty} F(p) = 0$.*

Proof: The statement follows from (2.5) and the proof of Theorem 8. □

Example 16 Let's find a function $f(t)$ such that its Laplace image equals the function $\sqrt{p}$.

Firstly, $\lim_{x\to\infty} \sqrt{x + \mathrm{i}y} \neq 0$. That is, according to the previous Corollary 1 the function $\sqrt{p}$ cannot be a Laplace image of any function $f(t)$.

Proof of the following statement, which gives us complete information about the behavior of the function F in the neighborhood of ∞, exceeds the scope of this text and can be found, for example, in [7].

Theorem 9 (the first limit) *Let f be a preimage with a growth rate α_0 a $\alpha > \alpha_0$. Then for the Laplace image F of the function f it holds that*

$$\lim_{\substack{p\to\infty \\ \operatorname{Re} p \geq \alpha}} F(p) = 0.$$

2.1 Properties of the Laplace transform

In this section we will deal with the basic properties of Laplace transformation. We will formulate a theorem, which will be the corner stone for the operator calculus, and give a number of examples to illustrate it. These examples will give us a large number of images of the functions that are important and used in practice. We close the section with the second and third limit theorem and Duhamel's formula.

Theorem 10 (rules of the operator calculus) *Let f_k be preimages, $\mathcal{L}(f_k(t)) = F_k(p)$ and $c_k \in \mathbb{C}$ for $k = 1, 2, \dots n$. Then*
I. linearity

$$\mathcal{L}(\sum_{k=1}^{n} c_k f_k(t)) = \sum_{k=1}^{n} c_k F_k(p),$$

II. time scaling

$$\mathcal{L}(f(\lambda t)) = \frac{1}{\lambda} F\left(\frac{p}{\lambda}\right), \quad \lambda > 0,$$

III. Laplace domain shifting

$$\mathcal{L}(e^{at} f(t)) = F(p - a),$$

IV. derivative by a parameter

$$\frac{\partial f(t,\lambda)}{\partial \lambda} = \frac{\partial F(p,\lambda)}{\partial \lambda}, \text{ kde } \mathcal{L}(f(t,\lambda)) = F(p,\lambda),$$

V. time shifting *for every $\tau > 0$ it holds that*

$$\mathcal{L}(f(t-\tau)\eta(t-\tau)) = e^{-\tau p} F(p),$$

VI. time domain derivative

$$\mathcal{L}(f^{(n)}(t)) = p^n F(p) - p^{n-1} f(0_+) - \cdots - f^{(n-1)}(0_+),$$

where f and its derivatives up to the order $n-1$ are continuous and $f^{(i)}(0_+) = \lim_{t\to 0_+} f^{(i)}(t)$,
VII. Laplace domain derivative

$$\mathcal{L}(-t f(t)) = F'(p),$$

VIII. time domain integration

$$\mathcal{L}\left(\int_0^t f(\tau)\,\mathrm{d}\,\tau\right) = \frac{F(p)}{p},$$

IX. Laplace domain integration

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_p^\infty F(z)\,\mathrm{d}\,z = \lim_{\mathrm{Re}\,q\to\infty}\int_p^q F(z)\,\mathrm{d}\,z,$$

where $f(t)/t$ is a preimage with a growth rate α_0, $\int_p^\infty F(z)\,\mathrm{d}\,z$ exists, and graph of the integrating curve$\int_p^\infty F(z)\,\mathrm{d}\,z$ is a subset of $\mathrm{Re}\,p > \alpha_0$.

Proof:

I. linearity

From the linearity of the integral we have

$$\begin{aligned}\mathcal{L}(\sum_{k=1}^n c_k f_k(t)) &= \int_0^\infty \left(\sum_{k=1}^n c_k f_k(t)\right) e^{-pt}\,\mathrm{d}\,t = \\ &= \sum_{k=1}^n c_k \int_0^\infty f_k(t)e^{-pt}\,\mathrm{d}\,t = \sum_{k=1}^n c_k F_k(p),\end{aligned}$$

where $\int_0^\infty f_k(t)e^{-pt}\,\mathrm{d}\,t = F_k(p)$. Note that the integral

$$\int_0^\infty \left(\sum_{k=1}^n c_k f_k(t)\right) e^{-pt}\,\mathrm{d}\,t$$

converges in half-plane $\mathrm{Re}\,p > \alpha_0$ where $\alpha_0 = \max_{i=1,2,\ldots n}\{\alpha_0^i\}$ and α_0^i is the growth index f_i for every i.

II. time scaling

By (2.2) we have

$$\mathcal{L}(f(\lambda t)) = \int_0^\infty f(\lambda t)e^{-pt}\,\mathrm{d}\,t.$$

Let's substitute on the right side $u = \lambda t$, (that is $\mathrm{d}\,t = 1/\lambda\ \mathrm{d}\,u$), then

$$\mathcal{L}(f(\lambda t)) = \frac{1}{\lambda}\int_0^\infty f(u)e^{-pu/\lambda}\,\mathrm{d}\,u = \frac{1}{\lambda}F\left(\frac{p}{\lambda}\right).$$

Obviously, if α_0 is a growth index of f depending on t, then $\lambda\alpha_0$ is a growth index of f depending on u.

III. Laplace domain shifting

By (2.2) we have

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{at}f(t)e^{-pt}\,\mathrm{d}\,t = \int_0^\infty f(t)e^{-(p-a)t}\,\mathrm{d}\,t = F(p-a),$$

where $\int_0^\infty f(t)e^{-(p-a)t}\,\mathrm{d}\,t$ converges to $\operatorname{Re}(p-a) > \alpha_0$, α_0 is a growth index of f.

IV. derivative by a parameter

The proof of this part goes beyond the complexity of this text, and can be found in [7].

V. time shifting

By (2.2) we have

$$\mathcal{L}(f(t-\tau)\eta(t-\tau)) = \int_0^\infty f(t-\tau)\eta(t-\tau)e^{-pt}\,\mathrm{d}\,t.$$

Moreover,$f(t-\tau)\eta(t-\tau) = 0$ for every $t \in (0,\tau)$, so

$$\mathcal{L}(f(t-\tau)\eta(t-\tau)) = \int_\tau^\infty f(t-\tau)\eta(t-\tau)e^{-pt}\,\mathrm{d}\,t.$$

If we introduce substitution$t-\tau = u$, then we get

$$\mathcal{L}(f(t-\tau)\eta(t-\tau)) = \int_0^\infty f(u)\eta(u)e^{-p(u+\tau)}\,\mathrm{d}\,u = e^{-p\tau}\int_0^\infty f(u)e^{-pu}\,\mathrm{d}\,u,$$

consequently,

$$\mathcal{L}(f(t-\tau)\eta(t-\tau)) = e^{-p\tau}\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t = e^{-p\tau}F(p).$$

VI. time domain derivative

Firstly, let's prove the case for $i = 1$, i.e.

$$\mathcal{L}(f'(t)) = pF(p) - f(0_+).$$

By (2.2) we have

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-pt}\,\mathrm{d}\,t,$$

for $\operatorname{Re} p > \alpha_0$. We note that if α_0 is the growth index of the function f', then it is also the growth index of f. We now calculate the integral on the right hand using the method of integration by parts, i.e. for $t \in (0,\infty)$ putting

$$\begin{aligned} u(t) &= f(t), & v(t) &= e^{-pt}, \\ u'(t) &= f'(t), & v'(t) &= -pe^{-pt}. \end{aligned}$$

Then

$$\mathcal{L}(f'(t)) = [f(t)e^{-pt}]_0^\infty + p\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t. \tag{2.6}$$

Since $|f(t)| \le Me^{\alpha t}$ $(\alpha > \alpha_0)$ for every p such that $\operatorname{Re} p > \alpha > \alpha_0$ then it holds that

$$|f(t)e^{-pt}| \le Me^{(\alpha - \operatorname{Re} p)t}.$$

Moreover $\lim_{t\to\infty} M e^{(\alpha-\operatorname{Re} p)t} = 0$ and

$$\lim_{t\to\infty} f(t)e^{-pt} = 0. \tag{2.7}$$

If we substitute (2.7) into (2.6) we get

$$\mathcal{L}(f'(t)) = pF(p) - f(0_+).$$

For arbitrary n the proof can be done by mathematical induction.

VII. Laplace domain derivative

From the proof of Theorem 8 it follows that

$$\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t$$

and

$$\int_0^\infty -t f(t)e^{-pt}\,\mathrm{d}\,t$$

and they are uniformly convergent in the half-plane $\operatorname{Re} p \geq \alpha > \alpha_0$, where α_0 is growth index f. The Laplace integral of the function f can be differentiated according to the parameter p, so, if

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t = F(p),$$

then

$$F'(p) = \int_0^\infty \frac{\partial}{\partial p}(f(t)e^{-pt})\,\mathrm{d}\,t = \int_0^\infty -t f(t)e^{-pt}\,\mathrm{d}\,t.$$

Finally, by (2.2) we have

$$\mathcal{L}(-t f(t)) = F'(p).$$

VIII. time domain integration

Denote

$$\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(\tau)\,\mathrm{d}\,\tau\right) = G(p).$$

Apparently it is true that

$$g'(t) = f(t),\ \ g(0) = 0.$$

It follows from the property of deriving an object that

$$\mathcal{L}(g'(t)) = pG(p) - g(0_+).$$

Hence,

$$\mathcal{L}(f(t)) = pG(p).$$

Consequently,

$$G(p) = \frac{F(p)}{p}.$$

IX. Laplace domain integration

Let $\mathcal{L}(f(t)/t) = G(p)$. Then by Theorem 8 the function G is analytical in the half-plane $\operatorname{Re} p > \alpha_0$. Since

$$|G(p)| \leq \int_0^\infty \left|\frac{f(t)}{t}e^{-pt}\right| \mathrm{d}\, t \leq M \int_0^\infty e^{-(\sigma-\alpha_0)t} \,\mathrm{d}\, t = \frac{M}{\sigma - \alpha_0},$$

where M is a positive constant and $\operatorname{Re} p = \sigma > \alpha_0$, we have

$$\lim_{\sigma\to\infty} |G(p)| = 0. \tag{2.8}$$

From the property of image derivation we have

$$\mathcal{L}(-f(t)) = G'(p)$$

and

$$F(p) = -G'(p).$$

Then for integrals of functions F a $-G'$ we have

$$G(p) - G(q) = \int_p^q F(z) \,\mathrm{d}\, z,$$

here we integrate along the curve with endpoints p and q fulfilling the condition $\operatorname{Re} q > \operatorname{Re} p > \alpha_0$.

Now just go to the limit in the previous formula $\operatorname{Re} q = \sigma \to \infty$; we apply (2.8) and get

$$G(p) = \int_p^\infty F(z) \,\mathrm{d}\, z.$$

□

Examples 17 to 24 demonstrate how to use properties of Theorem 10.

Example 17 Find the Laplace image of a function $f(t) = \sin(\omega t)$.

According to Euler's formulae, it holds that

$$\sin(\omega t) = \frac{e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t}}{2\mathrm{i}}.$$

If we put in Exercise 15 parameter $a = \pm\mathrm{i}\omega$ and apply I. from Theorem 10, then for $\operatorname{Re} p > \operatorname{Re}(\pm\mathrm{i}\omega) = |\operatorname{Im}\omega|$ we get

$$\mathcal{L}(\sin(\omega t)) = \frac{1}{2\mathrm{i}}\left(\frac{1}{p - \mathrm{i}\omega} - \frac{1}{p + \mathrm{i}\omega}\right) = \frac{\omega}{p^2 + \omega^2}.$$

Example 18 Find the Laplace image of a function $f(t) = e^{at}\sin(\omega t)$. We will use the result of Example 17 for the calculation and property III. from Theorem 10. So, for $\operatorname{Re} p > \operatorname{Re} a + |\operatorname{Im}\omega|$ we have

$$\mathcal{L}(e^{at}\sin(\omega t)) = \frac{\omega}{(p-a)^2 + \omega^2}.$$

Example 19 Find the Laplace image of a function $f(t) = t^n e^{at}$. From Exercise 15 we know that

$$\mathcal{L}(e^{at}) = \frac{1}{p-a}.$$

Let us now use property IV. from Theorem 10 and derive the left and right sides according to the parameter a

$$\mathcal{L}(te^{at}) = \frac{1}{(p-a)^2}.$$

The next derivatives take the form

$$\mathcal{L}(t^2e^{at}) = \frac{2}{(p-a)^3},$$

$$\mathcal{L}(t^3e^{at}) = \frac{3!}{(p-a)^4},$$

$$\vdots$$

$$\mathcal{L}(t^ne^{at}) = \frac{n!}{(p-a)^{n+1}}.$$

Example 20 Find the Laplace image of a function $f(t) = \eta(t-\tau)$. Note that the function f is a Heaviside function shifted by τ, hence

$$\eta(t-\tau) = \begin{cases} 0, & \text{for } \ t < \tau, \\ 1, & \text{for } \ t \geq \tau. \end{cases}$$

From Example 14 we know that $\mathcal{L}(\eta(t)) = 1/p$. Now, from property V. of Theorem 10 it follows that

$$\mathcal{L}(\eta(t-\tau)) = \frac{e^{-\tau p}}{p}.$$

Example 21 Find the Laplace image of a function $f(t) = \sin(\omega t - \varphi)\eta(\omega t - \varphi)$ where $\varphi > 0$ and $\omega > 0$. From property II. from Theorem 10 it follows that

$$\mathcal{L}(\sin(\omega t - \varphi)\eta(\omega t - \varphi)) = \frac{1}{\omega} F\left(\frac{p}{\omega}\right),$$

where $\mathcal{L}(\sin(t-\varphi)\eta(t-\varphi)) = F(p)$. From Example 17 we know that

$$\mathcal{L}(\sin(t)) = \frac{1}{p^2+1}$$

and from property V. of Theorem 10 we get

$$F(p) = \mathcal{L}(\sin(t-\varphi)\eta(t-\varphi)) = \frac{1}{p^2+1}e^{-\varphi p}.$$

Consequently,

$$\mathcal{L}(\sin(\omega t - \varphi)\eta(\omega t - \varphi)) = \frac{1}{\omega}\frac{1}{(p/\omega)^2+1}e^{-\varphi p/\omega} = \frac{\omega e^{-\varphi p/\omega}}{p^2+\omega^2}.$$

Example 22 Find the Laplace image of a function $f(t) = \sin^3(t)$. Firstly, denote $\mathcal{L}(\sin^3(t)) = F(p)$. Next

$$(\sin^3(t))' = 3\sin^2(t)\cos(t)$$

and

$$(\sin^3(t))'' = 6\sin(t)\cos^2(t) - 3\sin^3(t) = 6\sin(t) - 9\sin^3(t),$$

then by Example 17 we have

$$\mathcal{L}((\sin^3(t))'') = \frac{6}{p^2+1} - 9\,\mathcal{L}(\sin^3(t)) = \frac{6}{p^2+1} - 9F(p). \tag{2.9}$$

Moreover,

$$(\sin^3(t))|_{t=0_+} = (\sin^3(t))'|_{t=0_+} = 0$$

and by property VI. from Theorem 10 we have

$$\mathcal{L}(\sin^3(t))'') = p^2F(p) - p\cdot 0 - 0 = p^2F(p). \tag{2.10}$$

Let's now compare (2.9) and (2.10); we get

$$\frac{6}{p^2+1} - 9F(p) = p^2F(p),$$

so

$$F(p) = \frac{6}{(p^2+1)(p^2+9)}.$$

Example 23 Find the Laplace image of a function $f(t) = t^n$.

From Example 14 we know that

$$\mathcal{L}(\eta(t)) = \frac{1}{p}.$$

Then by property VIII. from Theorem 10 we get

$$\mathcal{L}(t) = \mathcal{L}\left(\int_0^t 1\,\mathrm{d}\,\tau\right) = \frac{1}{p^2},$$

$$\mathcal{L}\left(\frac{t^2}{2}\right) = \mathcal{L}\left(\int_0^t \tau\,\mathrm{d}\,\tau\right) = \frac{1}{p^3},$$

$$\vdots$$

$$\mathcal{L}\left(\frac{t^n}{n!}\right) = \mathcal{L}\left(\int_0^t \frac{\tau^{n-1}}{(n-1)!}\,\mathrm{d}\,\tau\right) = \frac{1}{p^{n+1}}.$$

Hence,

$$\mathcal{L}\left(t^n\right) = \frac{n!}{p^{n+1}}.$$

Example 24 Find the Laplace image of a function $f(t) = \sin(t)/t$.

From Example 15 we know that

$$\mathcal{L}(\sin(t)) = \frac{1}{p^2+1}.$$

Then by property IX. from Theorem 10 we get

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \int_p^\infty \frac{1}{z^2+1}\,\mathrm{d}\,z = [\arctan(z)]_p^\infty = \frac{\pi}{2} - \arctan(p) = \operatorname{arccotan}(p).$$

Theorem 11 (the second limit) *Let f and f' be preimages and f be on the interval $(0,\infty)$ continuous. If $\mathcal{L}(f(t)) = F(p)$ and α_0 is the growth index of the function f', then*

$$\lim_{p\to\infty} pF(p) = f(0_+).$$

Proof: From property VI. of Theorem 10 it follows that

$$\mathcal{L}(f'(t)) = pF(p) - f(0_+).$$

Since $f'(t)$ is a preimage with the growth rate α_0, then from Theorem 9 it follows that

$$\lim_{p\to\infty}(pF(p) - f(0_+)) = 0$$

in the half-plane $\operatorname{Re} p \geq \alpha > \alpha_0$. So,

$$\lim_{p\to\infty} pF(p) = f(0_+).$$

□

Theorem 12 (the third limit) *Let f and f' bee preimages and f be on the interval $(0,\infty)$ continuous. If $\mathcal{L}(f(t)) = F(p)$ and $\lim_{t\to\infty} f(t) \neq \infty$ exists, then*

$$\lim_{p\to 0} pF(p) = \lim_{t\to\infty} f(t).$$

Proof: We know that

$$\mathcal{L}(f'(t)) = pF(p) - f(0_+).$$

Then by (2.2) we have

$$\mathcal{L}(f'(t)) = \int_0^\infty f'(t)e^{-pt}\,\mathrm{d}\,t = pF(p) - f(0_+). \tag{2.11}$$

Moreover it holds that

$$\lim_{p\to 0}\int_0^\infty f'(t)e^{-pt}\,\mathrm{d}\,t = \int_0^\infty \lim_{p\to 0} f'(t)e^{-pt}\,\mathrm{d}\,t. \tag{2.12}$$

Then from (2.11) and (2.12) it follows that

$$\lim_{p\to 0}\left(pF(p) - f(0_+)\right) = \int_0^\infty f'(t)\,\mathrm{d}\,t = \lim_{t\to\infty} f(t) - f(0_+).$$

However, we assume that it exists that $\lim_{t\to\infty} f(t) \neq \infty$, so

$$\lim_{p\to 0} pF(p) = \lim_{t\to\infty} f(t).$$

□

Definition 6 A convolution of functions f and g *is called a function h defined by*

$$h(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\,\tau,\ \ t \in \mathbb{R}$$

and denoted by

$$h = f * g.$$

If functions f and g are preimages, then

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\,\tau,$$

which follows from the properties of preimages, i.e. $f(\tau) = 0$ for $\tau < 0$ and $g(t-\tau) = 0$ for $t < \tau$. The proof of the following properties of convolution is simple and we leave it to the reader.

Theorem 13 *The convolution has the following properties:*

1. comutativity: $f * g = g * f$,
2. associativity: $(f * g) * h = f * (g * h)$,
3. distributivity on adding $f * (h + g) = (f * h) + (f * g)$,
4. $(cf) * g = f * (cg) = c(f * g)$, *where c is a constant.*

Example 25 Find the convolution of preimages $f(t) = t$ and $g(t) = \sin(t)$.

Let us follow the definiton:

$$(f * g)(t) = \sin(t) * t = \int_0^t \sin(\tau)(t-\tau)\,\mathrm{d}\,\tau = t - \sin(t).$$

Note that the second equality holds due to the assumption that both functions are preimages (that is $f(\tau) = 0$ for $\tau < 0$ and $g(t-\tau) = 0$ for $t < \tau$).

Theorem 14 (multiplication of images) *Let f and g be preimages with growth rates α_0^f and α_0^g, $\mathcal{L}(f(t)) = F(p)$ and $\mathcal{L}(g(t)) = G(p)$. Then $\alpha_0 = \max\{\alpha_0^f, \alpha_0^g\}$ is a growth index of the function $h = f * g$. Moreover, it holds that*

$$\mathcal{L}((f * g)(t)) = F(p)G(p).$$

Corollary 2 (Duhamel's formula) *Let f and g be preimages, $\mathcal{L}(f(t)) = F(p)$ and $\mathcal{L}(g(t)) = G(p)$. Let f' be a preimage and f be continuous on the interval $[0, \infty)$. Then*

$$pF(p)G(p) = \mathcal{L}(f(0_+)g(t) + (f' * g)(t)).$$

Example 26 Find the preimage of the Laplace image

$$\frac{1}{(p^2+1)^2}.$$

From Example 17 we know that

$$\mathcal{L}(\sin(t)) = \frac{1}{p^2+1}.$$

Put

$$F(p) = G(p) = \frac{1}{p^2+1}.$$

Now, use Theorem 14

$$\begin{aligned} \frac{1}{(p^2+1)^2} &= F(p)G(p) = \mathcal{L}(\sin(t) * \sin(t)) = \\ &= \int_0^t \sin(\tau)\sin(t-\tau)\,\mathrm{d}\,\tau = 1/2\ \sin(t) - 1/2\,t\cos(t). \end{aligned}$$

Theorem 15 *Let f and g be preimages with growth rates α_0^f and α_0^g, $\alpha_0 = \max\{\alpha_0^f, \alpha_0^g\}$, $\mathcal{L}(f(t)) = F(p)$ and $\mathcal{L}(g(t)) = G(p)$. Then*

$$\mathcal{L}(f(t)g(t)) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} F(z)G(p-z)\,\mathrm{d}\,z,$$

where $p \in \mathbb{C}$, $\operatorname{Re} p > a + \alpha_0$ and $\alpha_0 < a$.

By a symbol $\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty}$ we mean a limit $\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} = \lim_{b\to\infty} \int_{a-\mathrm{i}b}^{a+\mathrm{i}b}$ and in the integral $\int_{a-\mathrm{i}b}^{a+\mathrm{i}b}$ we are integrating on the integration curve $z = a + \mathrm{i}t$, $t \in [-b, b]$, where we pick $0 < b \in \mathbb{R}$ and $a \in \mathbb{R}$ such that $\alpha_0 < a$. The proofs of Theorems 14 and 15 we leave to the reader, they can be found, e.g., in [6].

Example 27 By Theorem 15 find the Laplace image of the function $e^t \sin(t)$.

From Examples 17 and 15 we know that

$$\mathcal{L}(f(t)) = \mathcal{L}(\sin(t)) = \frac{1}{p^2+1}$$

and

$$\mathcal{L}(g(t)) = \mathcal{L}(e^t) = \frac{1}{p-1}.$$

Moreover, $\alpha_0^f = 0$, $\alpha_0^g = 1$ and $\alpha_0 = \max\{0, 1\} = 1$. Then by Theorem 15

$$\mathcal{L}(\sin(t)e^t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{z^2+1} \frac{1}{(p-z)-1} \,\mathrm{d}\, z,$$

where $a > 1$ and $\operatorname{Re} p > a + 1$. Then by a basic theorem on residua the integral can be computed

$$\mathcal{L}(\sin(t)e^t) = -\operatorname{Res}\left[\frac{1}{z^2+1} \frac{1}{(p-z)-1}\right]_{z=p-1} = \frac{1}{(p-1)^2+1}.$$

2.2 Inverse Laplace transform

In the previous section, we dealt with the calculation of the Laplace image F to the given preimage f. We defined the Laplace transform as a representation

$$\mathcal{L} : P \to O,$$

which assigns an preimage of the set P to its image of the set O. Let us now examine the inverse operation, the inverse representation *inverse Laplace transform*

$$\mathcal{L}^{-1} : O \to P,$$

which assigns a given complex function to the complex variable F preimage f, for which it holds that $\mathcal{L}(f(t)) = F(p)$.

To do this, you need to answer questions about:

Q1: the existence of inverse Laplace transform,

Q2: how to define the domain of $\mathcal{L}^{-1}$.

The following theorem gives a partial answer to the question Q1:

Theorem 16 (Lerch's) *Let $\mathcal{L}(f(t)) = F(p)$ and $\mathcal{L}(g(t)) = F(p)$. Then $f = g$ up to isolated points, in which at least one of the functions is not continuous.*

Note that the condition on isolated points from Lerch's theorem is not limiting, in practice we do not care about the values in isolated points.

To solve the question Q2, it is obvious that F must satisfy the necessary conditions of Laplace's image F of the object f. That is

1. there is $\alpha_0 \in \mathbb{R}$ such that F is in the half-plane $\operatorname{Re} p > \alpha_0$ analytic,

2. in the arbitrary half-plane $\operatorname{Re} p \geq \alpha > \alpha_0$ it holds that $\lim_{p\to\infty} F(p) = 0$.

Theorem 17 *Let F be analytic in $\mathbb{C}$ up to finitely many singular points $a_i \in \mathbb{C}$ ($i = 1, 2, \dots n$). Let for every $a \in \mathbb{R}$ such that for $a > \max_{i=1,2,\dots,n}\{|a_i|\}$ it holds that:*

1. *there is a sequence of circle lines k_i with a center at 0 and radii R_i, for which it holds that $|a| < R_1 < \cdots < R_n < \dots$ a $\lim_{n\to\infty} R_n = \infty$ such that*

$$\lim_{n\to\infty} \max_{p\in k_n}\{|F(p)|\} = 0,$$

2. *the integral $\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} |F(p)|\,\mathrm{d}\,p$ has a finite value.*

Then on $\mathbb{R}$ there is the continuous preimage f that is given by

$$f(t) = \begin{cases} \sum_{i=1}^{n} \mathrm{Res}[F(p)e^{pt}]_{p=a_i} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases} \tag{2.13}$$

Remark 5 Note that the residues in formula (2.13) are calculated in the singularities of the function $F(p)e^{pt}$. The functions e^{pt} are analytic in $\mathbb{C}$, therefore, they are crucial for calculating the residua function $F(p)$.

If, for example, a_i is a simple pole, then according to the rules for counting residues

$$\mathrm{Res}[F(p)e^{pt}]_{p=a_i} = e^{a_i t}\,\mathrm{Res}[F(p)]_{p=a_i}.$$

If a_i is a pole of the second order we get

$$\begin{aligned} \mathrm{Res}[F(p)e^{pt}]_{p=a_i} &= \lim_{p\to a_i} \frac{\mathrm{d}}{\mathrm{d}\,p}[(p-a_i)^2 F(p)e^{pt}] = \\ &= \mathrm{Res}[F(p)]_{p=a_i} e^{a_i t} + \lim_{p\to a_i}[(p-a_i)^2 F(p)]te^{a_i t}. \end{aligned}$$

We notice that images play a significant role, F, which has a rational form, see Examples 17 - 24. Let us now consider the inverse Laplace transform of the functions

$$F(p) = \frac{P(p)}{Q(p)}, \tag{2.14}$$

where $P(p)$ and $Q(p)$ are polynomials over a complex field.

Theorem 18 *The function* (2.14) *is a Laplace image of some preimage if and only if* $\deg(P(p)) < \deg(Q(p))$.

Proof: Firstly, in the half-plane $\mathrm{Re}\,p \leq \alpha$ it holds that

$$\lim_{p\to\infty} F(p) = \lim_{p\to\infty} \frac{P(p)}{Q(p)} = 0.$$

This only happens in cases where $\deg(P(p)) < \deg(Q(p))$ (deg denotes degree of the polynomial).

On the other hand, let $\deg(P(p)) < \deg(Q(p))$. This means that there is decomposition of the right hand (2.14) on partial fractions over the field of complex numbers

$$F(p) = \frac{P(p)}{Q(p)} = \sum_{k=1}^{m} \sum_{l=1}^{r_k} \frac{P_{kl}}{(p - a_k)^l}, \tag{2.15}$$

where $P_{kl} \in \mathbb{C}$, r_k is a multiple of the root a_k of polynomial $Q(p)$ and m is the number of different zero points of the polynomial $Q(p)$.

From Example 19 we know that

$$\mathcal{L}\left(\frac{t^{l-1}}{(l-1)!} e^{a_{kt}}\right) = \frac{1}{(p - a_k)^l}.$$

Due to the linearity of the Laplace transform and (2.15) we have for $t > 0$

$$F(p) = \frac{P(p)}{Q(p)} = \mathcal{L}\left(\sum_{k=1}^{m} \sum_{l=1}^{r_k} \frac{P_{kl} t^{l-1}}{(l-1)!} e^{a_k t}\right).$$

Then from Theorem 16 it follows that

$$f(t) = \begin{cases} \sum_{k=1}^{m} \sum_{l=1}^{r_k} \frac{P_{kl} t^{l-1}}{(l-1)!} e^{a_k t} & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

□

From the proof of the foregoing theorem it follows that

Theorem 19 (the second theorem on decomposition) *The Laplace image F of the function f is a rational function if and only if if for $t > 0$ we can describe the formula f as a linear combination of functions in the form $t^n e^{at}$, where $n \in \mathbb{N}_0$ and $a \in \mathbb{C}$.*

Example 28 Find out the preimage of the function $F(p) = \dfrac{p+1}{p^2 - p}$.

To do this we use the notes in Remark 5. The function F has two single poles, 0 and 1. So, we have

$$\operatorname{Res}[F(p)e^{pt}]_{p=0} = -1, \ \operatorname{Res}[F(p)e^{pt}]_{p=1} = 2e^t.$$

Then by Equation (2.13) of Theorem 17 we have for $t > 0$

$$f(t) = -1 + 2e^t.$$

Example 29 Find out the preimage of the function $F(p) = \dfrac{1}{(p+1)(p-1)^3(p^2+1)}$.

The computations will be done analogously to the previous example, hence using Remark 5. The function F has three single poles -1, $\pm\mathrm{i}$ and one pole of the third order. Hence,

$$\mathrm{Res}[F(p)e^{pt}]_{p=-1} = -1/16\, e^{-t},\ \mathrm{Res}[F(p)e^{pt}]_{p=\mathrm{i}} = 1/8\, e^{\mathrm{i}t}$$

$$\mathrm{Res}[F(p)e^{pt}]_{p=-\mathrm{i}} = -1/8\, e^{-\mathrm{i}t},\ \mathrm{Res}[F(p)e^{pt}]_{p=1} = \frac{2t^2-6t+5}{2}\, e^t.$$

Then by Equation (2.13) Theorem 17 we have for $t > 0$

$$f(t) = -1/16\, e^{-t} + 1/8\, e^{\mathrm{i}t} - 1/8\, e^{-\mathrm{i}t} + \frac{2t^2-6t+5}{2}\, e^t.$$

Table 2.1: Table of Laplace transforms of common functions

Time domain	Laplace domain	Region of convergence
1	$\frac{1}{p}$	$\mathrm{Re}(p) > 0$
e^{at}	$\frac{1}{p-a}$	$\mathrm{Re}(p) > \mathrm{Re}(a)$
$\sin(\omega t)$	$\frac{\omega}{p^2+\omega^2}$	$\mathrm{Re}(p) > 0$
$\cos(\omega t)$	$\frac{p}{p^2+\omega^2}$	$\mathrm{Re}(p) > 0$
$\sinh(\omega t)$	$\frac{\omega}{p^2-\omega^2}$	$\mathrm{Re}(p) > \lvert\omega\rvert$
$\cosh(\omega t)$	$\frac{p}{p^2-\omega^2}$	$\mathrm{Re}(p) > \lvert\omega\rvert$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(p-a)^2+\omega^2}$	$\mathrm{Re}(p) > a$

Table 2.1: Table of Laplace transforms of common functions

Time domain	Laplace domain	Region of convergence
$e^{at}\cos(\omega t)$	$\dfrac{p-a}{(p-a)^2+\omega^2}$	$\mathrm{Re}(p) > a$
$t^n,\ n \in \mathbb{N}$	$\dfrac{n!}{p^{n+1}}$	$\mathrm{Re}(p) > 0$
$t^n e^{at},\ n \in \mathbb{N}$	$\dfrac{n!}{(p-a)^{n+1}}$	$\mathrm{Re}(p) > \mathrm{Re}(a)$
$t\sin(\omega t)$	$\dfrac{2p\omega}{(p^2+\omega^2)^2}$	$\mathrm{Re}(p) > 0$
$t\cos(\omega t)$	$\dfrac{p^2-\omega^2}{(p^2+\omega^2)^2}$	$\mathrm{Re}(p) > 0$

2.3 Applications of the Laplace transform

Solution of ordinary differential equations

Consider the Cauchy problem for a linear differential equation with constant coefficients a_i $(i = 1, 2, \dots, n)$ and the initial conditions

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{(n-1)} x' + a_n x = f, \tag{2.16}$$

$$x(t_0) = x_0,\ x'(t_0) = x'_0, \dots,\ x^{(n-1)}(t_0) = x_0^{(n-1)}. \tag{2.17}$$

Next, suppose the right side of the equation f and a solution x including their derivatives up to the order of n are preimages. Under these conditions, we can solve the given problem by the Laplace transform.

Without loss to generality, we can assume that the initial conditions are given in point $t_0 = 0$, hence

$$x(0_+) = x_0,\ x'(0_+) = x'_0, \dots,\ x^{(n-1)}(0_+) = x_0^{(n-1)}. \tag{2.18}$$

Denote $\mathcal{L}(x(t)) = X(p)$ and $\mathcal{L}(f(t)) = F(p)$. Then the equation (2.16) can be written as (Theorem 10)

$$\begin{aligned}
[p^n X(p) - p^{n-1}x_0 - p^{n-2}x_0' \cdots - x_0^{(n-1)}] & \quad + \\
a_1[p^{n-1}X(p) - p^{n-2}x_0 - p^{n-3}x_0' \cdots - x_0^{(n-2)}] & \quad + \\
& \quad \vdots \\
a_{(n-1)}[pX(p) - x_0] & \quad + \\
a_n X(p) & \quad = \quad F(p).
\end{aligned} \tag{2.19}$$

After the adjustments we get

$$X(p) = \frac{F(p) - P(p)}{Q(p)}, \tag{2.20}$$

where $Q(p) = p^n + a_1 p^{n-1} + \cdots + a_{n-1}p + a_n$ is a characteristic polynomial of Equation (2.16) and the degree of the polynomial P is at most $(n-1)$.

Now simply find for the function X its preimage x. According to the uniqueness of such an inverse Laplace transform, such an object is then (Theorem 16) a solution of differential equation (2.16) on the interval $(0, \infty)$.

Remark 6

1. The procedure described above is called an *operator method*.
2. The Equation (2.19) is called an *operator*.
3. The advantage of the operator method is the simplicity of the solution operations.
4. With the solution we get a straight particular solution (if the initial conditions are not known, we get a general solution).

Example 30 Let us solve the differential equation

$$\begin{cases} x'' - 2x' + x = 4, \\ x(0_+) = 0, \; x'(0_+) = 1. \end{cases} \tag{2.21}$$

We will proceed with the operator method described above. So, put $\mathcal{L}(x(t)) = X(p)$, then

$$\mathcal{L}(x'(t)) = pX(p),$$
$$\mathcal{L}(x''(t)) = p^2 X(p) - 1.$$

Next $\mathcal{L}(4) = 4/p$, $\operatorname{Re} p > 0$. The corresponding operator equation has the form

$$p^2 X(p) - 1 - 2pX(p) + X(p) = 4/p.$$

We express $X(p)$

$$X(p) = \frac{p+4}{p(p-1)^2}, \ \operatorname{Re} p > 1.$$

After decomposition into partial fractions we get

$$X(p) = \frac{4}{p} - \frac{4}{p-1} + \frac{5}{(p-1)^2}.$$

The inverse Laplace transform gives for $t \geq 0$ the solution

$$x(t) = 4 - 4e^t + 5te^t.$$

In the procedure described above, decomposition into partial fractions can be avoided. Notice that the function

$$X(p) = \frac{p+4}{p(p-1)^2}$$

has at the point 0 a single pole and at the point 1 a pole of the second order, and uses the algorithm explained in Chapter 2.2 (see Theorem 17 and Example 29). Then according to the known formulas for calculating residues, we get

$$\operatorname{Res}[X(p)e^{pt}]_{p=0} = 4,$$

$$\operatorname{Res}[X(p)e^{pt}]_{p=1} = 5te^t - 4e^t.$$

Based on the inverse Laplace transform, we get the solution

$$x(t) = 4 - 4e^t + 5te^t.$$

Example 31 Let us solve the differential equation

$$\begin{cases} x'' + 4x = 2\cos(2t), \\ x(0_+) = 0, \ x'(0_+) = 4. \end{cases} \tag{2.22}$$

We will proceed with the operator method described above. So, put $\mathcal{L}(x(t)) = X(p)$, then

$$\mathcal{L}(x'(t)) = pX(p),$$

$$\mathcal{L}(x''(t)) = p^2 X(p) - 4.$$

Next $\mathcal{L}(2\cos(2t)) = 2p/(p^2+4)$. The corresponding operator equation has the form

$$(p^2+4)X(p) - 4 = \frac{2p}{p^2+4}.$$

We express $X(p)$:

$$X(p) = \frac{4}{p^2+4} + \frac{2p}{(p^2+4)^2}.$$

The inverse Laplace transform gives the solution

$$x(t) = 1/2\,(4+t)\sin(2t).$$

Example 32 (discontinuous right-hand side I) Let us solve the differential equation

$$\begin{cases} x'' + x = f(t), \\ x(0_+) = 1,\ x'(0_+) = -1, \end{cases} \tag{2.23}$$

where

$$f(t) = \begin{cases} 1, \text{ for } 0 \le t \le 1 \\ 0, \text{ for } t > 1. \end{cases} \tag{2.24}$$

Let's proceed similarly to the previous examples. Put $\mathcal{L}(x(t)) = X(p)$, then

$$\mathcal{L}(x'(t)) = pX(p) - 1,$$

$$\mathcal{L}(x''(t)) = p^2X(p) - p + 1.$$

Next $\mathcal{L}(f(t))$ can be calculated directly from definition (2.2):

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-pt}\,\mathrm{d}t = \int_0^1 e^{-pt}\,\mathrm{d}t = \frac{1}{p}(1-e^{-p}). \tag{2.25}$$

Or, notice that $f(t) = \eta(t) - \eta(t-1)$ and by property V. of Theorem 10 again we get (2.25). The corresponding operator equation has the form

$$(p^2+1)X(p) - p + 1 = \frac{1}{p}(1-e^{-p}).$$

We express $X(p)$ after decomposition to partial fractions

$$X(p) = \frac{1}{p} - \frac{1}{(p^2+1)} - \left(\frac{1}{p} - \frac{p}{p^2+1}\right)e^{-p}.$$

The inverse Laplace transform gives the solution

$$x(t) = (1-\sin(t))\eta(t) - (1-\cos(t))\eta(t-1)$$

or without using $\eta(t)$

$$x(t) = \begin{cases} 1-\sin(t), & t \in [0,1), \\ \cos(t) - \sin(t), & t \ge 1. \end{cases}$$

Example 33 (discontinuous right-hand side II) Let us solve the differential equation

$$\begin{cases} x'' + x = f(t), \\ x(0_+) = 1,\ x'(0_+) = 0, \end{cases} \tag{2.26}$$

where

$$f(t) = \begin{cases} 1, \text{ for } 0 \le t \le 3 \\ 2, \text{ for } t > 3. \end{cases} \tag{2.27}$$

Let's proceed analogously to the previous example. Put $\mathcal{L}(x(t)) = X(p)$, then

$$\mathcal{L}(x''(t)) = p^2 X(p) - p.$$

Next, it is enough to observe that $f(t) = \eta(t) + \eta(t-3)$, and by property V. of Theorem 10 we get

$$\mathcal{L}(f(t)) = \frac{1}{p}(1 + e^{-3p}). \tag{2.28}$$

The corresponding operator equation has the form

$$(p^2 X(p) - p) + X(p) = \frac{1}{p}(1 + e^{-3p}).$$

We express $X(p)$:

$$X(p) = \frac{p}{(p^2+1)} + \frac{1}{p(p^2+1)}(1 + e^{-3p}).$$

The inverse Laplace transform gives the solution

$$x(t) = 1 + (1 - \cos(t-3))\eta(t-3)$$

or without using $\eta(t)$

$$x(t) = \begin{cases} 1, & t \in [0,3), \\ 2 - \cos(t-3), & t \ge 3. \end{cases}$$

Example 34 (shifted initial conditions) Let us solve the differential equation

$$\begin{cases} x'' + 3x' + 2x = e^t, \\ x(1_+) = 1,\ x'(1_+) = 1. \end{cases} \tag{2.29}$$

Because the initial conditions are not given in point $t_0 = 0$, we have to do the substitution $t = \tau + 1$ a $x(t) = x(\tau + 1) = y(\tau)$. Then the equation takes the form

$$\begin{cases} y'' + 3y' + 2y = e^{\tau+1}, \\ y(0_+) = 1,\ y'(0_+) = 1. \end{cases} \tag{2.30}$$

Hence, put $\mathcal{L}(y(t)) = Y(p)$, then

$$\mathcal{L}(y'(\tau)) = pY(p) - 1,$$

$$\mathcal{L}(y''(t)) = p^2 Y(p) - p - 1.$$

Next

$$\mathcal{L}(e^{\tau+1}) = e\,\mathcal{L}(e^{\tau}) = e\frac{1}{p-1}.$$

We express $Y(p)$ after the decomposition to partial fractions

$$Y(p) = \frac{e/6}{p-1} + \frac{3 - e/2}{p+1} + \frac{e/3 - 2}{p+2}.$$

We get from the inverse Laplace transform for $\tau \geq 0$, the solution

$$y(\tau) = e/6\, e^{\tau} + (3 - e/2)\, e^{-\tau} + (e/3 - 2)\, e^{-2\tau}.$$

By the inverse substitution $\tau = t - 1$ and $y(\tau) = x(t)$ we have for $t \geq 1$ the solution

$$x(t) = e/6\, e^{t-1} + (3 - e/2)\, e^{1-t} + (e/3 - 2)\, e^{2-2t}.$$

Example 35 Let us solve the system of differential equations

$$\begin{cases} x' - x + y = 2, \\ x - y' - y = e^t, \\ x(0_+) = 1,\ y(0_+) = 1. \end{cases} \tag{2.31}$$

Let $\mathcal{L}(x(t)) = X(p)$ and $\mathcal{L}(y(t)) = Y(p)$, then $\mathcal{L}(x'(t)) = pX(p) - 1$ and $\mathcal{L}(y'(t)) = pY(p) - 1$. The corresponding operator system has the form

$$\begin{cases} (p-1)X(p) + Y(p) = \dfrac{p+2}{p}, \\ X(p) - (p+1)Y(p) = -\dfrac{p}{p-1}. \end{cases} \tag{2.32}$$

After treatment and decomposition into partial fractions

$$\begin{cases} X(p) = \dfrac{2}{p^3} + \dfrac{1}{p^2} + \dfrac{1}{p-1}, \\ Y(p) = \dfrac{2}{p^3} - \dfrac{1}{p^2} + \dfrac{1}{p}. \end{cases} \tag{2.33}$$

The inverse Laplace transform gives the solution

$$\begin{cases} x(t) = t^2 + t + e^t, \\ y(t) = t^2 - t + 1. \end{cases} \tag{2.34}$$

Tasks in electrical engineering

Consider first the simple oscillation circuit shown in Figure 2.3 described by an integral-differential equation

$$L\frac{\mathrm{d}\, i(t)}{\mathrm{d}\, t} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\,\tau = u(t), \tag{2.35}$$

where L, R, and C are the induction, resistance and capacitance constants. Next, u is the electromotive voltage and i is the current.

Without prejudice to generality, we can assume that no current flows through the entire circuit at the beginning. This initial condition corresponds to a switching situation. Thus, no current a passes through the circuit $i(0_+) = 0$. Then the last member left parties (2.35) represents the voltage on the capacitor plates, which at the beginning is zero.

Denote $\mathcal{L}(i(t)) = I(p)$ and $\mathcal{L}(u(t)) = U(p)$; the functions $I(p)$ and $U(p)$ are called *operator current* resp. *operator voltage*. Then from property VI. from Theorem 10 we have

$$\mathcal{L}\left(\frac{\mathrm{d}\, i(t)}{\mathrm{d}\, t}\right) = pI(p),$$

from VIII. of Theorem 10 we have

$$\mathcal{L}\left(\int_0^t i(\tau)\,\mathrm{d}\,\tau\right) = \frac{I(p)}{p}.$$

We rewrite the equation (2.35) into an operator form

$$LpI(p) + RI(p) + \frac{I(p)}{Cp} = U(p),$$

after adjustment

$$I(p) = \frac{U(p)}{Lp + R + \dfrac{1}{Cp}} = \frac{U(p)}{Z(p)}, \tag{2.36}$$

where $Z(p)$ is the operator impedance of the circuit. Formula (2.36) is called the *operator form of Ohm's law*.

To the end, the inverse Laplace transform then from (2.36) determines the circuit current i.

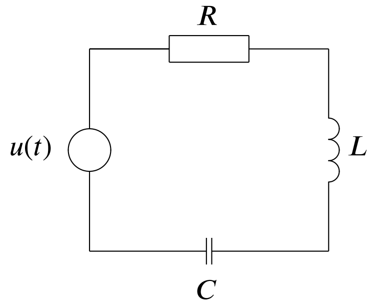

Figure 2.3: Oscillating circuit.

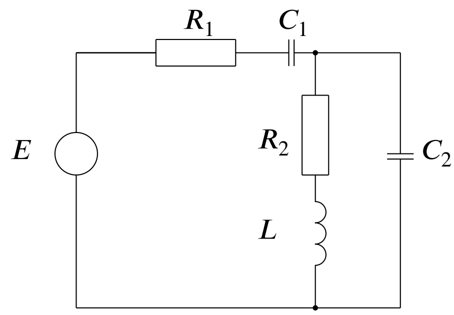

Figure 2.4: Oscillating circuit.

Example 36 Find the operator impedance and operator current flowing through the network shown in Figure 2.4.

Firstly, the following applies to operator impedances:

branch I. consists of resistance R_1 and capacitance C_1, and it holds that $Z_1 = R_1 + \dfrac{1}{C_1 p}$,
branch II. consists of resistance R_2 and inductance L, and it holds that $Z_2 = R_2 + Lp$,
branch III. consists of capacitance C_2, and it holds that $Z_3 = \dfrac{1}{C_2 p}$.

Branches II and III are connected in parallel, so their resulting impedance has the form

$$\frac{1}{Z_4} = \frac{1}{Z_2} + \frac{1}{Z_3}.$$

Furthermore, we can consider the circuit as a series-connected operator impedance Z_1 and Z_4. Hence,

$$Z = Z_1 + Z_4 = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}.$$

After the establishment and application of Kirchhof's second law $U(p) = E/p$, we

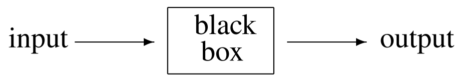

Figure 2.5: Control system as a black box.

get

$$I(p) = \frac{U(p)}{Z(p)} = \frac{E}{p} \frac{R_2 + Lp + \dfrac{1}{C_2 p}}{\left(R_1 + \dfrac{1}{C_1 p}\right)\left(R_2 + Lp + \dfrac{1}{C_2 p}\right) + (R_2 + Lp)\dfrac{1}{C_2 p}},$$

here in the considered oscillating circuit we consider the connected constant electromotive voltage $u = E$.

Tasks from regulatory systems

The control system can be imagined as a black box (see Figure 2.5) and its properties can be described by the reactions of the outputs to the input signals. The dynamic properties of control systems are determined by the relationships between output and input quantities. We will describe the dynamic properties of such systems in time dependence using linear differential equations with constant coefficients

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t),$$

where $a_i,\ b_j \in \mathbb{R}$ a $m \leq n$ is a condition of system feasibility.

The time shift of the signal can be described as a traffic delay

$$y(t) = u(t - T_d).$$

The transfer function of a given system is determined as the ratio of the image of the output quantity to the image of the input quantity with respect to the Laplace transform under the assumption of zero initial conditions.

$$y^{(n-1)}(0) = y^{(n-2)}(0) = \cdots = y'(0) = y(0) = 0.$$

The transfer function then has the form of a rational polynomial function

$$F(p) = \frac{P(p)}{Q(p)} = \frac{b_m(p - n_1)(p - n_2)\dots(p - n_m)}{a_n(p - k_1)(p - k_2)\dots(p - k_n)},$$

where k_i are the transmission poles and n_j are transmission zeros.

We use the impulse characteristic to describe the time dependence of a given control system, which can be obtained in response to the input signal in the form

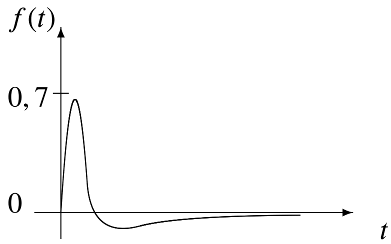

Figure 2.6: Impulse characteristics of the system.

of a Dirac pulse under zero initial conditions. The impulse characteristic $f(t)$ is obtained after the inverse Laplace transform

$$\mathcal{L}^{-1}(F(p)) = f(t).$$

Example 37 Find the impulse response of the transfer function

$$F(p) = \frac{5p+3}{p^3+6p^2+11p+6}.$$

We decompose the given function into partial fractions

$$F(p) = \frac{-1}{p+1} + \frac{7}{p+2} - \frac{6}{p+3}.$$

The inverse Laplace transform gives the impulse response (see Figure 2.6) for $t > 0$

$$f(t) = -e^{-t} + e^{-2t} - 6e^{-3t}.$$

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