An Introduction to Discrete Dynamical Systems

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Contents

Preface

This textbook provides an introduction to the theory of discrete dynamical systems and it is written, not only, for undergraduate students of applied mathematics and engineering. This text is based on talks that I presented at the Mathematical Institute of the Silesian University at Opava in years 2005–2011, Palacký University of Olomouc, VŠB – Technical University of Ostrava in years 2011–2022, and textbook for students [25].

Goals of this text

The main aim of this text is to present elementary and basic notions of modern theory of dynamical systems and to describe their properties. Goals of chapters are the following:

1 Dynamical modeling: We introduce fundamental models, i.e. population growth in one and two dimensions, like (im)migration, predator prey or overlapping generation systems. This chapter was mainly motivated by the book written by F.R. Marotto [30].

2 Elementary dynamics: We focus on preparatory behavior of discrete dynamical systems. That is, we define a dynamical system and describe essential concepts that will be used in the following text. This chapter was mainly motivated by the book written by R.L. Devaney [12].

3 Chaos: This chapter is devoted to the notion of chaos. Here we define a chaos in the sense of Devaney through transitivity, periodically dense property and sensitive dependence on initial conditions. We also define a chaos in the sense of Li and Yorke and we compare both this notions of chaos. This chapter was mainly motivated by the book written by S.N. Elaydi [14].

4 Fractals: Here we derive a definition of a fractal. We attain definitions of topological, box, similarity and fractal dimension. We also discuss iterated function systems yielding the Collage theorem. All notions and properties are demonstrated on a set of classical examples like the Koch curve and snowflake, Sierpinski gasket, Cantor set and Barnsley's fern. This chapter was mainly motivated by the book written by S.N. Elaydi [14].

5 Topological dynamics: Finally, we give a more general perspective on behaviour introduced in the previous chapters. These basic notions are constructed on more general spaces. We will observe further properties of discrete dynamical systems. This chapter was mainly motivated by the books written by R.L. Devaney [12] and P. Walters [40].

6 Simulations of dynamical properties: In this chapter there are given and explained source codes in Matlab that produces pictures through the text. We mainly focus on algorithms that will be used for standard dynamical systems (construction of trajectories, cobweb diagrams) and also for algorithms producing fractals (Koch curve and snowflake, Sierpinski gasket and Barnsley's fern). This chapter was mainly motivated by the book written by S. Lynch [27].

Prerequisites

For successful reading of this text linear high school algebra and multi variable calculus are required.

Apologies and ackowledgements

The author apologies himself for errors that might appear in the text, hence he will be appreciative for any comments or remarks that improves the text.

The author thanks to his students for pointing out mistakes in previous versions of this text (namely, Lukáš Kapera).

The author is also grateful for toleration of his wife during the time period of writing the text, without her responsiveness would this text never exist.

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Chapter 1

Dynamical modeling

The main tool used across many disciplines including epidemiology, ecology, economy, chemistry and social sciences is a difference equation.

If we denote by x_n the number of species in the n-th generation and the real model depends only on predecessor under some rule signed by f then we obtain *first order difference equation*

$$x_{n+1} = f(x_n). \tag{1.1}$$

For examples see the following Examples 1 and 2.

If the model depends not only on predecessor but many preceding generations than the *n-th order difference equation* can be defined by

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_2, x_1). \tag{1.2}$$

Samples of such phenomena are given in Examples 4, 5 and also by overlapping generation models 3.

1.1 Population growth models

1.1.1 One dimensional cases

We start our modeling by classical models coming from ecology, that are population models. We begin with a simple linear case and obtain one difference equation depending on time under iteration process. Later we discuss nonlinear cases and finally we construct predator prey models as a system of difference equations.

For the first model we have to take into account a growth process. That is, the number of births during any generation is proportional to that generation's population size. We have to assume that the number of deaths during each generation is also proportional to the generation's population size. The constant reflecting the birth is called the *birth rate* and denoted b and obviously $0 \leqslant b$. Analogously the constant reflecting death is called the *death rate* and denoted d and it is easy to see that $0 \leqslant d \leqslant 1$ since no more than 100% of a population can die.

Now, we can develop a *linear population model* as follows: Let us denote by P_n the population of the n-th generation. The number of births during that generation will be bP_n and the number of deaths dP_n. So, the number P_{n+1} of next $(n+1)$-th generation will be determined by adding the number of births and subtracting the number of death to P_n:

$$P_{n+1} = P_n + bP_n - dP_n = (1 + b - d)P_n.$$

For simplicity we absorb parameters of birth and death into one parameter $r = 1 + b - d$ called *growth rate* and we get

$$P_{n+1} = rP_n. \tag{1.3}$$

It seems that the above deduced model is seldom the case in the natural world. If members of the same class as that of the population under consideration, but who originated elsewhere, may regularly join the set being modeled, thereby contributing to an increase in its size — a *immigration* occur. In the second case a fixed number of the group may be regularly removed — a *migration* or *harvest*. We deduce

$$P_{n+1} = rP_n + k \tag{1.4}$$

where the constant k stands for the immigration if it is positive and for migration (harvest) if it is negative.

The following theorem is a useful tool for calculating a value of n-th generation without knowing the value of its predecessor, the proof is straight forward and is left to the reader.

Theorem 1 *The solution of the* (1.3) *model is given by*

$$P_{n+1} = r^{(n+1)}P_0$$

0	1	2	3	4
$P_0 = 1 \times 10^8$	$P_1 = 2 \times 10^8$	$P_2 = 4 \times 10^8$	$P_3 = 8 \times 10^8$	$P_4 = 16 \times 10^8$

Table 1.1: Table of values of the (1.5) model for the first six generations.

and the solution of the (1.4) *model is given by*

$$P_{n+1} = r^{(n+1)} P_0 + k \cdot \sum_{i=0}^{n} r^i.$$

Example 1 *Let us study the growth of the bacterial cultures in a yogurt. This is a basic example of Microbiology (see, e.g. [31]). It is known that each cell splits into two new cells. In this case, the birth rate is $b = 2$, the death rate is $d = 1$ and the growth rate equals to $r = 2$. So, we get the linear population model*

$$P_{n+1} = 2P_n. \tag{1.5}$$

One can observe that

$$\begin{aligned} P_1 &= 2P_0, \\ P_2 &= 2P_1 = 2^2 P_0, \end{aligned}$$

and it can be inductively proved that for any $n \in \mathbb{N}$

$$P_{n+1} = 2P_n = 2^{n+1} P_0.$$

The standard rate of alive microorganisms in 1 g *is* 100×10^6 *(e.g. Bifidobacterium or Lactobacillus acidophillus* [1]*). If we are given 1* g *at the beginning of the test, $P_0 = 100 \times 10^6$, then we can easily evaluate values of this model* (1.5) *for the first six generations with unite volume step, see Table 1.1.*

Example 2 *Let us calculate the time (number of unite volume steps) needed for getting 100* g *of alive microorganisms where 1* g *is given at the beginning. For this purpose we use the model determined in the previous example.*

To solve this problem we have to find n, such that

$$P_n = 100 \times 100 \times 10^6 = 10^{10}.$$

[1] The author would like to thank to Josef Hak M.D. for the helpful discussions on the topic of the Microbiology.

We know that $P_0 = 10^8$ *and* $P_n = 2^n\ P_0 = 2^n \times 10^8$ *that is*

$$\begin{aligned} 2^n \times 10^8 &= 10^{10}, \\ 2^n &= 10^2, \\ \log_2 2^n &= \log_2 10^2, \\ n &= 2\log_2 10 \approx 6.6439. \end{aligned}$$

Consequently, to get 10^{10} *alive microorganisms we need approximately 7 unit value steps.*

The foregoing example illustrates that linear models are not good for the simulation of phenomenon in the global case since:

1. the colony is growing if the growth rate is greater then one (so called *exponential growth*),
2. the colony is dying off if the growth rate is greater than zero and smaller than one (so called *exponential decay*).

We have to improve the linear model by other effects to get *non-linear models*. Non-linear models provide better explanation than linear ones for complex phenomena we sometimes observe in the world.

Now, let us improve our linear population model (1.3) by the principle of the *density dependence*. That is, the larger the population, the smaller its growth rate is likely to be. As it was pointed out, linear models do not take density dependence into account. One possibility how to construct a nonlinear model from linear one is to modify a growth rate r in (1.3). Let us replace r by a decreasing function of the population size P_n. We get a *non-linear model*

$$P_{n+1} = R(P_n)\,P_n \tag{1.6}$$

here, $R(P_n)$ represents the *growth rate function*. The simplest type of $R(P_n)$ is a linear function

$$R(P_n) = r\left(1 - \frac{P_n}{C}\right) \tag{1.7}$$

where r is the growth rate and C is a constant representing the *carrying capacity*, i.e., the largest population the environment can sustain. Substituting (1.7) into (1.6) we get

$$P_{n+1} = r\left(1 - \frac{P_n}{C}\right)\,P_n = \mu P_n(C - P_n) \tag{1.8}$$

so called logistic equation (see section 2.3 The quadratic family).

Another possibility how to implement the growth rate function by density dependence with fewer restrictions is to put

$$R(P_n) = re^{-P_n/N} \tag{1.9}$$

where r stands for growth rate and N is the population level that produces the maximum population. Whence we get

$$P_{n+1} = rP_n e^{-P_n/N}. \tag{1.10}$$

1.1.2 Two dimensional cases

In this section we introduce *predator prey models*, that is the population growth models of two species in which one, the prey, provides sustenance for the second one, the predator. If there is no dependence between the predators and preys, we can model them separately by

$$\begin{aligned} P_{n+1} &= r_1 P_n, \\ Q_{n+1} &= r_2 Q_n. \end{aligned} \tag{1.11}$$

Nevertheless, the predator consumes the prey. So the next generation of the prey population P_{n+1} will decline proportionally to the size of the present predator population Q_n. Analogously, the next generation of the predator population Q_{n+1} will increase proportionally to the size of the present prey population P_n. Combining this two facts we get the *linear predator prey model*

$$\begin{aligned} P_{n+1} &= r_1 P_n - s_1 Q_n, \\ Q_{n+1} &= s_2 P_n + r_2 Q_n. \end{aligned} \tag{1.12}$$

The model (1.12) could be improved by migration or immigration at a constant level. In this case we get

$$\begin{aligned} P_{n+1} &= r_1 P_n - s_1 Q_n + k_1, \\ Q_{n+1} &= s_2 P_n + r_2 Q_n + k_2. \end{aligned} \tag{1.13}$$

where r_1, r_2, s_1 and s_2 are non-negative predator prey constants, k_1 and k_2 are non-negative migration or immigration constants. Let us note, that the model (1.11) is a special case of (1.12), and the model (1.12) is a special case of (1.13).

Example 3 (Overlapping generations systems) *Let us suppose that the n-th generation of a species depends on two consecutive generations simultaneously. That is, P_{n+1} is dependent on both P_n and P_{n-1}, while using linear dependence we get*

$$P_{n+1} = rP_n + sP_{n-1} \tag{1.14}$$

As a real example of the (1.14) *one can use an asexual reproduction of haploid organisms on a multiplicative fitness landscape (see [32]).*

The model (1.14) *is of the second order since the difference between the highest and the lowest subscripts is* $2 = (n+1)-(n-1)$. *Let us put* $Q_n = P_{n-1}$ *so* $Q_{n+1} = P_n$ *and putting it into* (1.14) *we get a linear system*

$$\begin{aligned} P_{n+1} &= rP_n + sQ_n, \\ Q_{n+1} &= P_n. \end{aligned} \tag{1.15}$$

It is also worth noticing that the overlapping generation systems are used for economical pension schemes (see, e.g. [18], [19]).

The linear model (1.12) does not reflect the fact that the next generation of prey is increasing to a degree that is directly proportional to the number of contacts between predator and prey during the previous time step, and that the next predator population is decreased by a similar quantity. Hence, $s_1P_nQ_n$ should be subtracted from the next prey population and $r_2P_nQ_n$ should be added into next predator population. Consequently, we get a non-linear predator prey model

$$\begin{aligned} P_{n+1} &= r_1P_n - s_1P_nQ_n, \\ Q_{n+1} &= s_2P_n + r_2P_nQ_n. \end{aligned} \tag{1.16}$$

n	0	1	2	3	4	5	6	7	8
P_n	1.9	2.09	2.299	2.0484	1.4783	1.3122	1.9032	3.4157	1.8517
Q_n	1.1	1.1	0.891	0.7217	0.8876	1.4503	1.7947	0.5421	0.0228

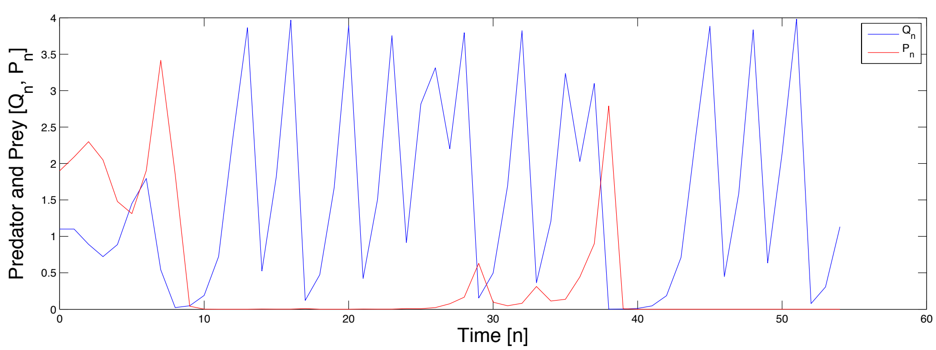
Table 1.2: Table of values of the 1.17 model for the first eight generations.

Example 4 *The following model was introduced in [2] and developed later in [5] and [18]. The model is defined as follows:*

$$\begin{aligned} P_{n+1} &= P_n(4 - P_n - Q_n), \\ Q_{n+1} &= P_n Q_n. \end{aligned} \tag{1.17}$$

Let us assume that we have $P_0 = 1.1$ units of predators and $Q_0 = 1.9$ unites of preys at the beginning. Then the values of the next generations could be calculated, see Table 1.2. The values of the species are in the Figure 1.1. So called time series graph *is two dimensional coordinate system where the horizontal axis stands for n and the vertical axis for the corresponding value of P_n.*

Figure 1.1: The values of the species of the (1.17) model.



1.2 Contagious disease models

1.2.1 One dimensional cases

Let I_n denote the number of infected individuals at a time n in a population of size N. Then it makes sense to say that the next number of infected I_{n+1} equals to the number currently infected I_n, minus the number of those who have recently recovered rI_n plus the number of new cases NC, here r stands for *recovery rate*. We get

$$I_{n+1} = I_n - rI_n + NC. \tag{1.18}$$

Obviously, $0 \leqslant r \leqslant 1$, if $r = 0$ then there is no recovery, only new cases appears; if $r = 1$ then in one time step all infected individuals become healthy and only new cases acts here. Now, the task is to determine those NC new cases.

To solve this problem we use generally accepted principle. The number of recent contacts between infected and susceptible individuals, and hence the number of new cases, are each directly proportional to the size of the infected population I_n multiplied by the size of the susceptible population $N - I - n$ in this case. Hence, we get

$$NC = kI_n(N - I_n) \tag{1.19}$$

where k is *infection constant* which means that it is difficult to get the disease if k is small, but if k is large then the illness is easily transmitted from one individual to another.

Combining (1.18) and (1.19) we finally get a *non-linear infection model* showing quadratic characteristic (compare with (1.8))

$$I_{n+1} = I_n - rI_n + kI_n(N - I_n) \tag{1.20}$$

It is also possible to make some additional assumption on new cases NC from epidemiological principles and get new cases in the form

$$NC = kI_n^2(N - I_n). \tag{1.21}$$

Now, putting (1.21) into (1.18) we get another non-linear infection model showing cubic characteristic

$$I_{n+1} = I_n - rI_n + kI_n^2(N - I_n). \tag{1.22}$$

In both cases (1.20) and (1.22) the immunity was not taken into consideration. This will be done in the next section.

1.2.2 Two dimensional cases

In a previous section the non-linear infection model (1.22) was investigated. This model does not reflect the immunity. Suppose now that temporary immunity is conferred after infection and recovery. This creates a new group of recovered and immune individuals R_n. Let us assume that the fraction t of the recovered population ($0 \leqslant t \leqslant 1$) lose immunity at each step. Since tR_n leaves the recovered and immune group at each step and rI_n enters it we have

$$R_{n+1} = R_n - tR_n + rI_n. \tag{1.23}$$

Observe that the number of susceptible individuals to the disease equals to $N - I_n - R_n$ at a time n.

Now, from the (1.18) we know that

$$I_{n+1} = I_n - rI_n + NC. \tag{1.24}$$

To determine new cases we have to take into account that the product of the population sizes of the two groups: I_n and $N - I_n - R_n$, hence

$$NC = kI_n(N - I_n - R_n). \tag{1.25}$$

Finally, putting (1.25) into (1.24) we get *non-linear infection recovery model*

$$\begin{aligned} I_{n+1} &= I_n - rI_n + kI_n(N - I_n - R_n), \\ R_{n+1} &= R_n - tR_n + rI_n. \end{aligned} \tag{1.26}$$

Example 5 *Let us simulate influenza in Ostrava while epidemic occurs. For this purpose we use the infection recovery model* (1.26)*. The epidemic is standardly defined in such a way that there are 2% ill inhabitants. Let us assume that*

- *there live 300 000 inhabitants in Ostrava,*
- *if someone recovers from influenza then he never becomes ill again,*
- *it takes 7 days to recovery from influenza,*
- *as a time unit we take one day,*

N	I_0	R_0	k	r	t
300 000	6000	0	$1/N$	$1/7$	0.

Table 1.3: Table of parameters of the (1.27) model.

- *the infection constant is reciprocal to the population rate.*

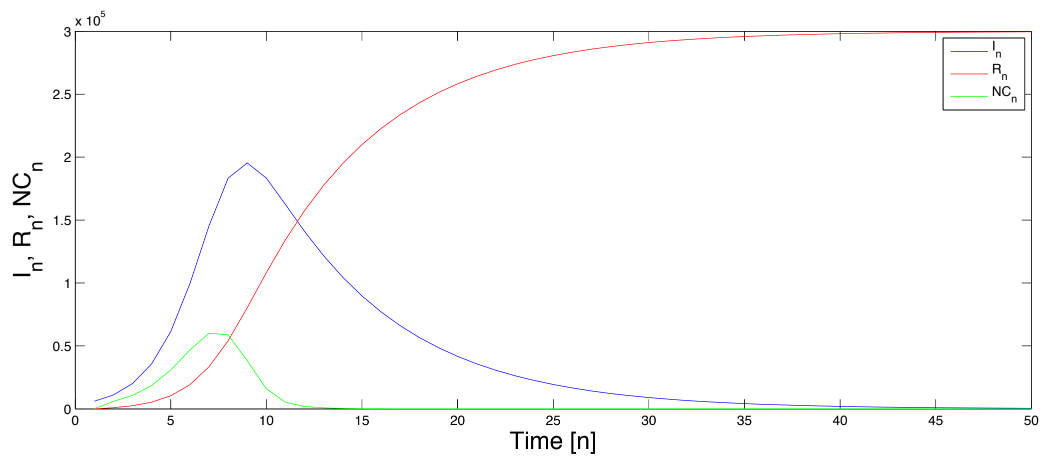
So we get parameters of the (1.26) *model written in the Table 1.3.*

Now, the model has the following form

$$\begin{aligned} I_{n+1} &= I_n - \frac{1}{7}I_n + \frac{1}{300\,000}I_n(300\,000 - I_n - R_n), \\ R_{n+1} &= R_n + \frac{1}{7}I_n. \end{aligned} \tag{1.27}$$

It follows from computational simulations that the maximal value of ill inhabitants is after 5 days. Then significant reduction occurs and after 60 days they are all healthy. Computational results are in Table 1.4 and graphs of I_n, R_n and NC_n are in Figure 1.2.

Figure 1.2: The values of the species of the (1.27) model.



n	0	1	2	3	4	5	6
I_n	6×10^3	1.1×10^4	2.0×10^4	3.5×10^4	6.1×10^4	9.9×10^4	1.4×10^5
R_n	0	8.5×10^2	2.4×10^3	5.3×10^3	1.0×10^4	1.9×10^4	3.3×10^4
NC_n	0	5.8×10^3	1.1×10^4	1.8×10^4	3.1×10^4	4.6×10^4	6.1×10^4

Table 1.4: Table of values of the (1.27) model for six days.

1.3 Exercises

1. Compute P_1–P_5 of the following linear models and find out their solutions:

   (a) $P_{n+1} = 1.2P_n + 100$ where $P_0 = 10$,
   (b) $P_{n+1} = 0.2P_n + 100$ where $P_0 = 10$,
   (c) $P_{n+1} = 1.3P_n - 10$ where $P_0 = 1000$,
   (d) $P_{n+1} = 0.3P_n - 10$ where $P_0 = 1000$.

2. Construct the linear population model that satisfies:

   (a) the population is initially 75 000,
   (b) the population increases 15% each generation.

3. Construct the linear population model that satisfies:

   (a) the population is initially 125 000,
   (b) the population birth rate is 7% and the death rate is 5%.

4. Construct the linear harvesting model that satisfies:

   (a) the population is initially 10 000,
   (b) the population increases 5% each generation,
   (c) the harvesting occurs at a constant rate of 150 each generation.

5. Construct the linear harvesting model that satisfies:

(a) P_0 =75 000,

(b) P_1 =77 000,

(c) P_2 =79 500.

6. Find out exact solutions of the difference equations:

   (a) $x_{n+1} = 1.5x_n$ where $x_0 = 0$,

   (b) $x_{n+1} = -1.75x_n$ where $x_0 = 3000$,

   (c) $x_{n+1} = -0.5x_n$ where $x_0 = 1$.

7. Assume that the population is growing by 5% per generation.

   (a) How many generations will it take for an initial population of 1000 to grow to 1600?

   (b) What growth rate will make this happen in four generations?

8. Assume an initial generation 150 000 growing 2% each generation and immigration at a constant rate 2500 per generation.

   (a) What is the 10th generation value?

   (b) What must have been the population five generations ago?

9. Identify r, s and N in the following infection model:

   (a) $I_{n+1} = 0.6I_n + 2.9I_n(1 - I_n/750)$,

   (b) $I_{n+1} = 3.6I_n(1 - 0.025I_n)$,

   (c) $I_{n+1} = 5I_n - 0.025I_n^2$.

10. Construct a disease model assuming that at each step rI_n recovers and the number of new cases is proportional to the product of:

    (a) I_n^2 and $(1 - I_n/N)^2$,

    (b) I_n^2 and $1 - I_n^2/N^2$,

    (c) I_n^2 and $e^{-I_n/N}$.

11. Construct the linear predator prey model with given parameters:

    (a) $r_1 = 1.3$, $r_2 = 0.9$, $s_1 = 0.3$, $s_2 = 0.6$ and $k_1 = k_2 = 0$,

(b) $r_1 = 4$, $r_2 = 0.9$, $s_1 = 0$, $s_2 = 0.4$ and $k_1 = k_2 = 10$,

(c) $r_1 = 1$, $r_2 = 1$, $P_0 = 2000$, $Q_0 = 1000$, $P_1 = 1800$, $Q_1 = 2400$ and $k_1 = k_2 = 0$.

12. Construct linear overlapping model with no (im)migration or harvesting:

(a) $P_{-1} = 100$, $P_0 = 200$, $P_1 = 300$ and $P_2 = 400$.

(b) $r = 1/2$, $P_0 = 400$, $P_1 = 800$ and $P_2 = 2400$.

13. Find out the solution of the (1.4) system and compare it with Theorem 1.

14. Find the general form of all solutions and find the unique solution that satisfies the given initial conditions:

(a)

$$\begin{aligned} P_{n+1} &= P_n - Q_n, \\ Q_{n+1} &= P_n + Q_n \end{aligned}$$

where $P_0 = 800$ and $Q_0 = 100$,

(b)

$$\begin{aligned} P_{n+1} &= 0.4P_n - 0.6Q_n, \\ Q_{n+1} &= 0.8P_n + 0.4Q_n \end{aligned}$$

where $P_0 = 500$ and $Q_0 = 200$.

15. Example 5 revised: Each parameter in the model (1.27) is constant, the influence of the type of the day is not taken into consideration. It is known that the number of infected humans is directly dependent on the number of working days. Take into account this fact and include it into the model in the dependence of weekends and holidays (so-called *non-autonomous dynamical systems* will be constructed).

Chapter 2

Elementary dynamics

The main aim of the theory of dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If the process is derived from differential equation whose independent variable is time, then behavior of the solution is continuously dependent on time. If the process is given by difference equation (or a map), then the theory studies properties of the iterations. We are now going to investigate discrete time situation.

A *(discrete) dynamical system* is an ordered pair (X, f) where X is a state space and $f : X \to X$ is a map (an action) which is into but not necessarily onto. Hence, the set X is *invariant* under f, that is $f(X) \subset X$[1]. There are additional assumptions on the state space and action while studying dynamical properties. Standardly, X is endowed by a metric d, that is the map $d : X \times X \to [0, \infty)$ satisfying the following conditions for each $x, y, z \in X$:

1. (non-negativity) $d(x, y) \geqslant 0$,
2. (identity) $d(x, y) = 0$ if and only if $x = y$,
3. (symmetry) $d(x, y) = d(y, x)$,
4. (triangle inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$.

So, the metric measures the distance between two points. The assumption given on a map f is continuity, that is f is continuous in any point of X which means that for any $\epsilon > 0$ there is δ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$.

[1] *strongly invariant* means $f(X) = X$

Example 6 *The Euclidean distance between points x and y is the length of the line segment connecting them. If we assume that $x, y \in \mathbb{R}$ then $d(x, y) = |x - y|$ and if $x, y \in \mathbb{R}^2$ then $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Analogously could be defined the Euclidean distance between points in $\mathbb{R}^n$.*

In the previous section we constructed several difference equations that are closely related to the notion of dynamical system. In Example 4 the investigated model consists of two species, predators and preys, so the state space $\triangle$ is a subset of $\mathbb{R}^2$ and a derived map $F : \triangle \to \triangle$ is given by $F(x, y) = (x(4 - x - y), xy)$, it is not difficult to verify that $\triangle$ is a triangle with vertices $(0, 0)$, $(0, 4)$ and $(4, 0)$. As a consequence of this process we can now focus on properties of a dynamical system (X, f) where X is a (compact) metric space and f is a continuous map on X. Recall, that a set $X \subset \mathbb{R}^n$ is *compact* if it is closed and bounded, so the unit closed interval I and also the unit circle $\mathbb{S}^1$ are compact sets.

Firstly we have to know what an iteration process is. Given f, then its *iterations at a point x* are points $f^n(x)$ where $f^0(x) = \mathrm{id}(x)$ (id is the identity map) and for $n > 0$ $f^n(x)$ stands for n fold composition of f

$$f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x).$$

Definition 1 *The* forward orbit *of $x \in X$ with respect to f is the set*

$$\mathrm{Orb}_f^+(x) = \{f^n(x),\ n \in \mathbb{N} \cup \{0\}\}.$$

The backward orbit *of $x \in X$ with respect to f is the set*

$$\mathrm{Orb}_f^-(x) = \{f^{-n}(x),\ n \in \mathbb{N}\}.$$

The full orbit *of $x \in X$ with respect to f is the set*

$$\mathrm{Orb}_f(x) = \mathrm{Orb}_f^+(x) \cup \mathrm{Orb}_f^-(x).$$

Definition 2 *The point $x \in X$ is a* fixed point *of the map f if $f(x) = x$. The point $x \in X$ is a* periodic point *of period n of the map f if $f^n(x) = x$ and $f^m(x) \neq x$ for any $1 \leqslant m < n$.*

The set of all fixed points of the map f is denoted by $\mathrm{Fix}(f)$, the set of all periodic points with period n by $\mathrm{Per}_n(f)$ and $\mathrm{Per}(f)$ denotes the set of all periodic points of the map f, that is

$$\mathrm{Per}(f) = \bigcup_{n \in \mathbb{N}} \mathrm{Per}_n(f).$$

Example 7 *Let us define dynamical system (X, f) in such a way that $X = \mathbb{R}$ and $f(x) = x^3$. It is easy to see, that $\mathrm{Fix}(f) = \{-1, 0, 1\}$ and that there are no other periodic points, so $\mathrm{Fix}(f) = \mathrm{Per}(f)$. Hence, $\mathrm{Orb}_f^+(0) = \{0\}$, $\mathrm{Orb}_f^+(1) = \{1\}$ and also $\mathrm{Orb}_f^+(-1) = \{-1\}$ since those points are fixed. Now, $\mathrm{Orb}_f^+(1.01) = \{1.01, 1.01^3, 1.01^9, 1.01^{27} \ldots\}$, the point 1.01 tends to the infinity under f, see* cobweb diagram *in Figure 2.1a. On the other hand $\mathrm{Orb}_f^+(0.99) = \{0.99, 0.99^3, 0.99^9, 0.99^{27} \ldots\}$ and the point 0.99 tends to 0, again see the cobweb diagram in Figure 2.1b. Consequently, $X = \mathbb{R}$ is invariant under f as well as $\mathrm{Fix}(f)$. So, invariant set can contain proper subset that is again invariant.*

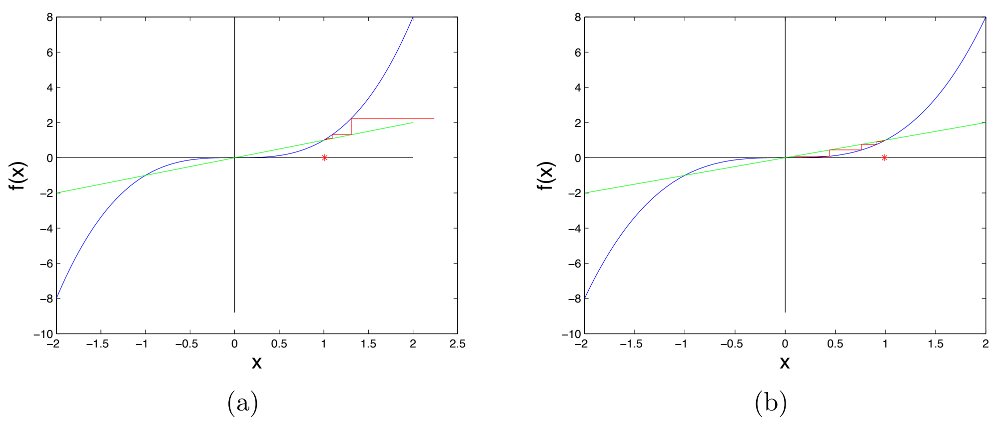


Figure 2.1: Cobweb diagram of $f(x) = x^3$ for (a) $x = 1.01$ and (b) $x = 0.99$.

Example 8 *Put $\lambda \in \mathbb{R}$, $R_\lambda(x) = x + 2\pi\lambda$ and $X = \mathbb{S}^1$ be a unit circle. The map R_λ now denotes the rotation on the circle where x stands for the rotation angle in a counterclockwise direction. We have to distinguish between two cases. If λ is rational number then all points are periodic. If λ is irrational number then there are no periodic points, see the following theorem.*

For the following theorem we need the notion of a *dense set*. Roughly speaking a set A is dense in I if any open interval $U \subset I$ intersects A, formal definition is given in Definition 7. This definition could be also extended in a natural way to the n-th dimensional cube or a circle.

Theorem 2 *Each orbit of the irrational rotation of the unit circle is dense in the unit circle.*

Proof: Let us identify a point from $\mathbb{S}^1$ by an angle α. We are going to prove that $\mathrm{Orb}_R^+(\alpha)$ is dense in $\mathbb{S}^1$, that is for any arbitrarily small open subset Arc of $\mathbb{S}^1$ there is $n \in \mathbb{N}$ such that $R_\lambda^n(\alpha) \in Arc$.

Firstly, any two points of the orbit of the point α are distinct, for if $R_\lambda^n(\alpha) = R_\lambda^m(\alpha)$ we would have $(m-n)\lambda \in \mathbb{Z}$, hence, $m = n$. Any infinite set of points on $\mathbb{S}^1$ must have a limit point. Thus, given any $\epsilon > 0$, there must be integers $n, m \in N$ such that $|R_\lambda^n(\alpha) - R_\lambda^m(\alpha)| < \epsilon$. Let $k = m - n$, then $|R_\lambda^k(\alpha) - \alpha| < \epsilon$.

Secondly, R_λ preserves length of arcs in $\mathbb{S}^1$. Consequently, R_λ^k maps the arc of endpoints λ and $R_\lambda^k(\alpha)$ to the arc of endpoints $R_\lambda^k(\alpha)$ and $R_\lambda^{2k}(\alpha)$ with the same length, that is $|R_\lambda^k(\alpha) - R_\lambda^{2k}(\alpha)| < \epsilon$. Hence, the points $\alpha, R_\lambda^k(\alpha), R_\lambda^{2k}(\alpha), R_\lambda^{3k}(\alpha) \ldots$ form a partition of $\mathbb{S}^1$ into arcs of the length less than ϵ. Ending the proof, the assertion follows from the fact that ϵ was arbitrarily chosen. $\square$

2.1 One dimensional stability

The classical task of dynamical systems is stability. In this section we discuss stability of one dimensional maps, that is $X \subset \mathbb{R}$.

Definition 3 *The point $x \in Fix(f)$ is* hyperbolic *if $|f'(x)| \neq 1$. A hyperbolic fixed point x is* attracting *provided there is an open set $x \in U$ such that for $y \in U \backslash \{x\}$ we have $\lim_{n\to\infty} f^n(y) = x$. We say that a hyperbolic point x is* repelling*, if there is an open set $x \in U$ such that for $y \in U \backslash \{x\}$, there is n_y such that $f^{n_y}(y) \notin U$.*

Theorem 3 *Let $(\mathbb{R}, f)$ be a dynamical system and $x \in \mathrm{Fix}(f)$. Assume that f is differentiable at x.*

1. *If $|f'(x)| < 1$ then the point x is attracting.*

2. If $|f'(x)| > 1$ then the point x is repelling.

Proof: We are going to prove the first part, the second one could be proved analogously.

Since $|f'(x)| < 1$, there is an interval $[x-\epsilon, x+\epsilon]$ and λ with $0 < \lambda < 1$ such that $|f'(y)| \leqslant \lambda$ for any $y \in [x-\epsilon, x+\epsilon]$. Now, by the Mean Value Theorem for $y \in [x-\epsilon, x+\epsilon]$ there is z between y and x such that

$$|f(y) - x| = |f(y) - f(x)| = |f'(z)| \cdot |y - x| \leqslant \lambda|y - x| < |y - x|.$$

Thus, $f(y)$ is closer to x than y, so $f(y) \in [x-\epsilon, x+\epsilon]$ and we can repeat the argument, hence, by induction we get

$$|f^j(y) - x| \leqslant \lambda^j|y - x|.$$

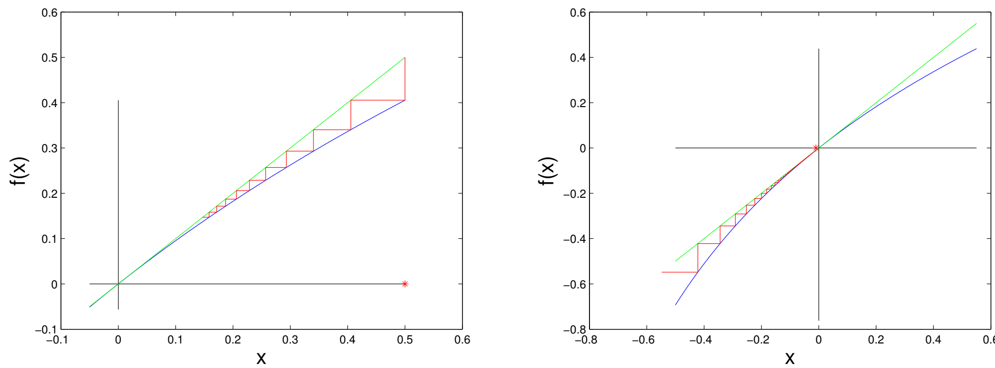
This shows that $f^j(x) \in [x-\epsilon, x+\epsilon]$ for any $j \geqslant 0$. Since $\lambda^j|y-x|$ goes to zero, $f^j(y)$ converges to x as j tends to the infinity, proving that the fixed point x is attracting. □

Let us note that the above definition and theorem formulated for fixed point could be extended for any periodic point using n-th derivative of the map in the suitable point.

Example 9 *Let us continue with the Example 7. There are three fixed points for $f(x) = x^3$ and it is easy to calculate derivative $f'(x) = 3x^2$. Hence, by Theorem 3 we have that both 1 and -1 are repelling, since $f'(\pm 1) = 3 > 1$ and 0 is attracting fixed point since $f'(0) = 0 < 1$.*

The following example illustrates that the condition $|f'(x)| < 1$ from Theorem 3 is essential and if this condition is not fulfilled than we have to analyze local stability individually.

Example 10 *Let $f(x) = \ln(x+1)$. It is easy to see that $\mathrm{Fix}(f) = \{0\}$ and that $f'(x) = 1/(x+1)$. So, $f'(0) = 1$ and we can not apply Theorem 3. Moreover, this point is not attracting nor repelling since any point from $(0, \infty)$ tends to 0 and any point from $(-1, 0)$ tends to $-\infty$, see Figure 2.2 for a cobweb diagram. Hence, this is a saddle point, compare it with the definition in the next section.*



(a) (b)

Figure 2.2: Cobweb diagram of $f(x) = \ln(x+1)$ for (a) $x = 0.1$ and (b) $x = -0.01$.

2.2 Higher dimensional stability

Let us introduce analogous phenomena derived in previous section for a higher dimensional dynamical systems.

A fixed point x for $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if $Df(x)$ has no eigenvalues on the unit circle, where $Df(x)$ is the Jacobian matrix of f at the point x. Such a hyperbolic point x is

1. a *sink* fixed point if all eigenvalues of $Df(x)$ are less than one in absolute value,

2. a *source* fixed point if all eigenvalues of $Df(x)$ are greater than one in absolute value,

3. a *saddle* fixed point otherwise, i.e., if some eigenvalues of $Df(x)$ are less and some larger than one in absolute value.

Proposition 1 ([12]) *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a sink fixed point x. Then there is an open set containing x in which all points tend to x under forward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *stable set* of x and is denoted by $W^s(x)$.

Proposition 2 ([12]) *Supposing that $f : \mathbb{R}^n \to \mathbb{R}^n$ has a source fixed point x. Then there is an open set containing x in which all points tend to x under backward iteration of f.*

The largest such open set in $\mathbb{R}^n$ is called the *unstable set* of x and is denoted by $W^u(x)$.

Example 11 *Let us return to Example 4, here a dynamical system $(\triangle, F)$ is investigated, where $F : \triangle \to \triangle$ is given by $F(x, y) = (x(4-x-y), xy)$ and $\triangle$ is a triangle with vertices $(0, 0)$, $(0, 4)$ and $(4, 0)$. Putting $x(4-x-y) = x$ and $xy = y$ one can easily compute, that $\mathrm{Fix}(F) = \{(0, 0), (1, 2), (3, 0)\}$. Now let us verify whether these fixed points are stable or not and let us find suitable stability regions. Firstly,*

$$DF(x, y) = \begin{pmatrix} 4 - 2x - y & -x \\ y & x \end{pmatrix}.$$

Now, by calculating characteristic polynomials we get eigenvalues. So,

$$\det(DF(0, 0) - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \lambda(\lambda - 4),$$

hence, $\lambda_1 = 0$ and $\lambda_2 = 4$ and corresponding eigenvectors are $v_1 = (0, 1)$ and $v_2 = (1, 0)$. So, the fixed point $(0, 0)$ is a saddle and has stable set with basin v_1, that is $W^s(0, 0) = [\![(0, 1)]\!]$ and $W^u(0, 0) = [\![(1, 0)]\!]$. Which means, that a point that is close to the origin is falling down in the direction of the vector v_1 and leaving small neighborhood of the origin in the v_2 direction under iterations.

Analogously could be calculated that $(1, 2)$ is a source and $W^s(0, 0) = [\![\ ]\!]$, $W^u(0, 0) = [\![(1, 0), (0, 1)]\!]$. Finally, the fixed point $(3, 0)$ is a source.

2.3 The quadratic family

The main aim of this section is to study one specific example, that is the quadratic family given by the map

$$F_\mu(x) = \mu x(1 - x) \tag{2.1}$$

defined on the unit closed interval $I = [0, 1]$ and we focus on parameters μ in the interval $[0, 4]$, graphs of F_μ and their iterations for several parameters μ are given in Figure 2.3.

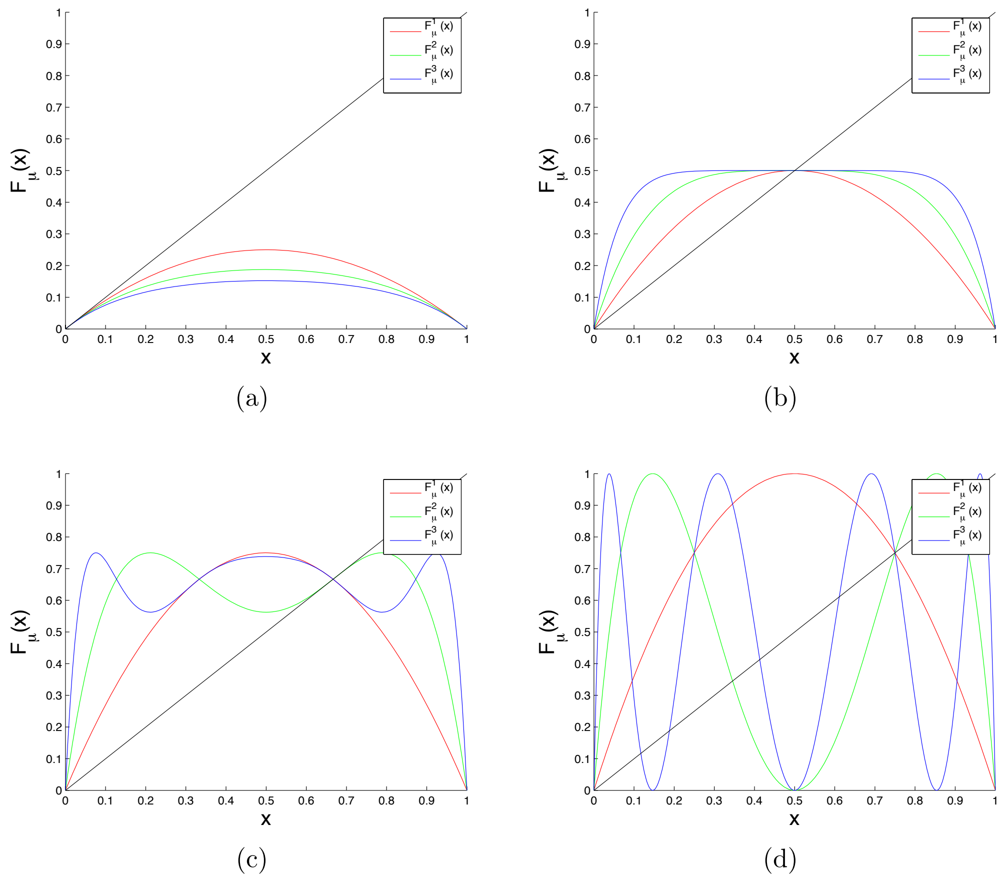


Figure 2.3: Morphology of F_μ for some values of the parameter μ. Case (a) corresponds to $\mu = 1$, case (b) corresponds to $\mu = 2$, case (c) corresponds to $\mu = 3$ and case (d) corresponds to $\mu = 4$.

Lemma 1 *For $F_\mu(x) = \mu x(1 - x)$ defined on I it holds:*

1. $F_\mu(0) = 0$ *and* $F_\mu(p_\mu) = p_\mu$ *where* $p_\mu = (\mu - 1)/\mu$,
2. *if* $1 < \mu \leqslant 4$ *then* $0 < p_\mu < 1$,

3. *if* $\mu = 1$ *then* $\mathrm{Fix}(F_1) = \{0\}$,
4. *if* $0 < \mu < 1$ *then* $p_\mu \notin I$.

Proof:

1. We have to solve the equation $F_\mu(x) = x$, that is $\mu x(1-x) = x$. Trivial solution is $x_0 = 0$, hence, $F_\mu(0) = 0$ and the second one is $p_\mu = (\mu-1)/\mu$, so $F_\mu(p_\mu) = p_\mu$.
2. If $1 < \mu \leqslant 4$ then from classical analysis of the function $P(\mu) = (\mu-1)/\mu$ one gets $0 < p_\mu < 1$.
3. If $\mu = 1$ then $p_\mu = (1-1)/1 = 0$ and $\mathrm{Fix}(F_1) = \{0\}$.
4. If $0 < \mu < 1$ then the analysis of the function $P(\mu) = (\mu-1)/\mu$ yields $p_\mu \notin I$.

□

The following lemma follows directly from graphical analysis, we give also analytic proof.

Lemma 2 *Let* $0 < \mu < 1$, *then for any* $x \in I$ *it holds*

$$\lim_{n\to\infty} F_\mu^n(x) = 0.$$

Proof: Obviously, the assertion trivially follows for $x \in \{0, 1\}$. Let us observe that for given $0 < \mu < 1$ and any $x \in (0,1)$ it holds

$$\mu x(1-x) < x.$$

Consequently, $F_\mu(x)$ is closer to zero than x hence

$$\lim_{n\to\infty} F_\mu^n(x) = 0$$

ending the proof. □

From the last two conditions of Lemma 1 and Lemma 2 it follows that it is essential to pick $\mu > 1$. Now, let us justify the condition given on the domain of (2.1).

Lemma 3 *Let $\mu > 1$, then*

1. *if $x < 0$ then $\lim_{n\to\infty} F_\mu^n(x) = -\infty$,*
2. *if $x > 1$ then $\lim_{n\to\infty} F_\mu^n(x) = -\infty$.*

Proof: Firstly assume that $x < 0$, then $\mu x(1-x) < x$. Hence, $F_\mu(x) < x$ and $F_\mu^n(x)$ is a decreasing sequence of points. This sequence is not converging to a point $p \in \mathbb{R}$, since $F_\mu^{n+1}(x) \to F_\mu(p) < p$, whereas, $F_\mu^n(x) \to p$. Consequently, $\lim_{n\to\infty} F_\mu^n(x) = -\infty$ ending the proof of the first part.

Secondly assume $x > 1$. Then $F_\mu(x) < 0$ and again by the first part $\lim_{n\to\infty} F_\mu^n(x) = -\infty$. □

From now we analyze dynamical properties of (2.1) with the respect to μ. Proof of the following lemma is left to the reader as an exercise using Theorem 3.

Lemma 4

1. *The fixed point p_μ is attracting for F_μ if $\mu \in (1,3)$,*
2. *the fixed point p_μ is repelling for F_μ if $\mu \in (3,4]$,*
3. *the fixed point 0 is attracting for F_μ if $\mu \in (0,1)$,*
4. *the fixed point 0 is repelling for F_μ if $\mu \in (1,4]$.*

Lemma 5 *Let $1 < \mu < 3$ and $0 < x < 1$ then $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.*

Proof: Let us firstly prove the case when $\mu \in (1,2)$. Let $x \in (0,1/2]$, then it is easy to see that for $x \neq p_\mu$

$$|F_\mu(x) - p_\mu| < |x - p_\mu|.$$

So, $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$. If $x \in (1/2,1)$ then obviously $F_\mu(x) \in (0,1/2]$ and using previous arguments we again get $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.

Let us now discuss the case when $\mu \in (2,3)$. In this case $p_\mu \in (1/2,1)$. Now, let us pick a point $a \in (0,1/2)$ such that $F_\mu(a) = p_\mu$. We are going to consider three cases depending whether x belongs to the interval $(0,a)$, $[a,p_\mu]$ or $(p_\mu,1)$. Easily $[a,p_\mu]$ is mapped inside $[1/2,p_\mu]$, more precisely $F_\mu^2([a,p_\mu]) \subset [1/2,p_\mu]$. Hence, for any $x \in [a,p_\mu]$ is $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.

Now, if $x < a$ then there is $k \in \mathbb{N}$ such that $F_\mu^{n+k}(x) \in [a, p_\mu]$, hence, using previous step we again have $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$. Easily, F_μ maps the interval $(p_\mu, 1)$ onto $(0, p_\mu)$, so for any $x \in (p_\mu, 1)$ we get $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$, closing the proof of this part, since $(0,1) = (0,a) \cup [a, p_\mu] \cup (p_\mu, 1)$.

To the end of the proof it remains to discuss the case $\mu = 2$ which is easy to check, and is left to the reader. □

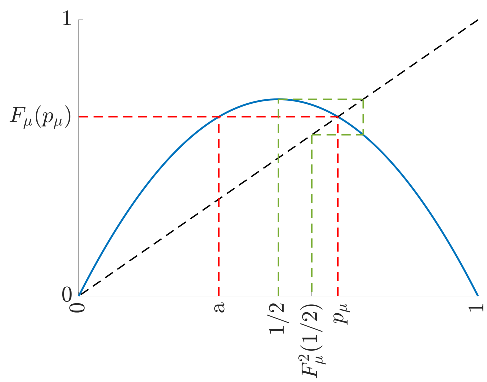


Figure 2.4: Illustration to the proof of Lemma 5 for $\mu = 2.85$.

Remark 1 *Point out that proof of Lemma 5 gives more information than point 1. of Lemma 4, that is interval of attractivity, the maximal set U for which Definition 3 holds, was located.*

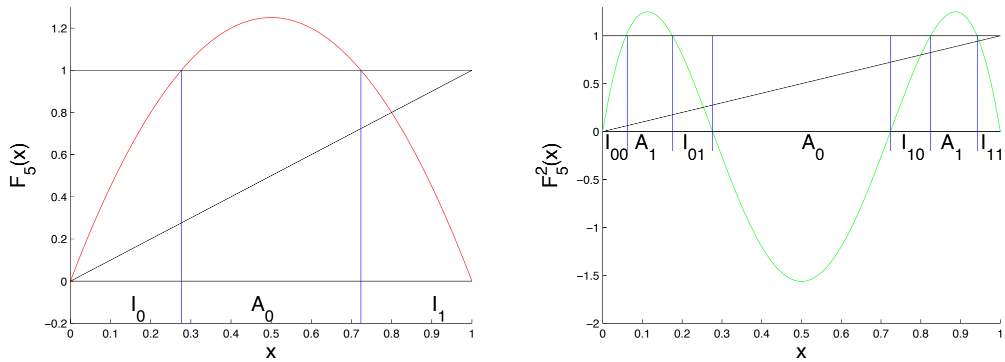
Hence, the map F_μ has exactly two fixed points in I for $\mu \in (1, 4]$. By Lemma 5 the behavior of F_μ is very pure for $\mu \in [0, 3)$, any point tends to a fixed point. The situation will change dramatically if we increase μ above 3. Now, let us discuss the situation for $\mu > 4$, the following construction is depicted on Figure 2.5.

It is easy to see that the map F_μ has maximum at the point $1/2$ with the value $\mu/4$. Assuming $\mu > 4$ we have $\mu/4 > 1$, so the values of F_μ are out of I. Let us denote the set

$$A_0 = \{x \in I : \; F_\mu(x) > 1\}.$$

Clearly, A_0 is an open set with center $1/2$ and for any $x \in A_0$ it holds

$$\lim_{n\to\infty} F_\mu^n(x) = -\infty,$$



(a) (b)

Figure 2.5: Morphology of F_μ for some values of the parameter μ. Case (a) corresponds to the construction of A_0, I_0 and I_1; case (b) corresponds to the construction of A_1 and I_{00}, I_{01}, I_{10}, I_{11}.

and if $x \in I \backslash A_0$ then $F_\mu(x) \in I$ but not necessarily $F_\mu^2(x) \in I$. So let A_1 be the set of such points from I that they leave I under second iteration, that is

$$A_1 = \{x \in I : \; F_\mu(x) \in A_0\}.$$

If $x \in A_1$ then $F_\mu^2(x) > 1$ and again $\lim_{n\to\infty} F_\mu^n(x) = -\infty$. Now, we can inductively define

$$A_n = \{x \in I : \; F_\mu^n(x) \in A_0\}.$$

Then is

$$A_n = \{x \in I : \; F_\mu^i(x) \in I \text{ for } i \leqslant n \text{ but } F_\mu^{n+1}(x) \notin I\},$$

so the set A_n consists of all points which escape from I at the $(n+1)$-st iteration and they tend to $-\infty$. Now, we know the ultimate fate of any point which lies in A_n for some n, it therefore remains only to analyze the behavior of those points which never escape form I, i.e., the set of points which lie in the set

$$\Lambda = I \backslash \bigcup_{n=0}^{\infty} A_n.$$

The task now is to describe this set more precisely and later answer the question: what are their behaviors.

The set A_0 is open connected interval so the set $I\backslash A_0$ contains two components, closed intervals, denoted by I_0 and I_1 where the subscripts denote: 0 left hand side and 1 right hand side, see Figure 4.5a. Now, the set $I\backslash(A_0 \cup A_1)$ consists of four closed intervals, I_{00} and I_{01} are subsets of I_0 and I_{10} and I_{11} are subsets of I_1, see Figure 4.5b. Inductively, the set $I\backslash(A_0 \cup A_1 \cup \cdots \cup A_n)$ consists of $2^{(n+1)}$ closed intervals and obviously each of them is mapped by F_μ^{n+1} monotonically onto I. So, the structure of Λ seems to be quite complicated.

Definition 4 *A set C is a* Cantor set *if and only if it is perfect and totally disconnected. A set is* totally disconnected *if it contains no interval; a set is* perfect *if it is closed and without isolated points.*

Example 12 (The Cantor Middle-Thirds Set C) *This is a classical construction of a Cantor set. Pick unit closed interval I and in a first step of construction remove the middle third, that is open interval $(1/3, 2/3)$. Now, remove from what remains two middle thirds, that are intervals $(1/3^2, 2/3^2)$ and $(7/3^2, 8/3^2)$. Continue with removing the middle thirds in this fashion. So, in the n-th step one removes 2^{n-1} intervals of the length $1/3^n$. Finally, the set C has the following form*

$$C = I\backslash \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right).$$

This procedure is analogous to the construction of the set Λ and this set C is an example of fractal, see chapter 4 Fractals.

At the end of this section we formulate the following theorem whose proof goes beyond this text and could be found e.g. in [12].

Theorem 4 *Let $\mu > 4$. Then the set Λ is a Cantor set.*

2.4 The symbolic dynamics

In this section we are going to investigate one special dynamical system that is very useful and frequently used for construction of counterexamples.

Let Σ_2 be a set of all sequences over two point alphabet, that is

$$\Sigma_2 = \{s = s_0 s_1 s_2 \cdots : \ s_i \in \{0, 1\}, \ i \in \mathbb{N} \cup \{0\}\}.$$

This set is called *shift space* and it is standardly endowed by the following metric: for any $s, t \in \Sigma_2$

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}. \tag{2.2}$$

Lemma 6

1. d is a metric on Σ_2.

2. Let $s, t \in \Sigma_2$ and $s_i = t_i$ for $0 \leqslant i \leqslant n$, then $d(s,t) \leqslant 1/2^n$.

3. If $d(s,t) < 1/2^n$ then $s_i = t_i$ for $0 \leqslant i \leqslant n$.

Proof:

1. We are going to verify four conditions form the definition of a metric given at the beginning of this section. It is easy to see that $d(s,t) \geqslant 0$ for any $s, t \in \Sigma_2$ and $d(s,t) = 0$ if and only if $s_i = t_i$ for any i, so $s = t$. Now, $d(s,t) = d(t,s)$ since $|s_i - t_i| = |t_i - s_i|$ for any i. Finally, let $r, s, t \in \Sigma_2$, then for any i it holds $|s_i - r_i| + |r_i - t_i| \geqslant |s_i - t_i|$ and we deduce $d(s,r) + d(r,t) \geqslant d(s,t)$.

2. If $s_i = t_i$ for $i \leqslant n$, then

$$d(s,t) = \sum_{i=0}^{n} \frac{|s_i - s_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leqslant \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

3. Now, if $s_k \neq t_k$ for some $k \leqslant n$, then

$$d(s,t) \geqslant \frac{1}{2^k} \geqslant \frac{1}{2^n}.$$

So, if $d(s,t) < 1/2^n$ then $s_i = t_i$ for $0 \leqslant i \leqslant n$ ending the proof.

□

Remark 2 *Point out that* $\max_{s,t \in \Sigma_2} d(s,t) = \sum_{i=0}^{\infty} 1/2^i = 2$ *showing* $\mathrm{diam}(\Sigma_2) = 2$. *Here,* $\mathrm{diam}(X)$ *stands for the diameter of the space X defined by* $\mathrm{diam}(X) = \sup_{x,y \in X} d(x,y)$, *where d is the metric on X.*

Now, let us define a dynamical system (Σ_2, σ) by defining σ. The map σ is called a shift map and is defined by

$$\sigma : \Sigma_2 \to \Sigma_2,\ \sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots). \tag{2.3}$$

Lemma 7 *The shift map is continuous.*

Proof: For the proof we follow the so called $\epsilon\delta$ definition. Let $\epsilon > 0$ and $s = s_0 s_1 s_2 \dots$. Now, pick n such that $1/2^n < \epsilon$ and let $\delta = 1/2^{n+1}$. If now $t \in \Sigma_2$ is such that $d(s,t) < \delta$, then by Lemma 6 it follows that $s_i = t_i$ for $i \leqslant n+1$. Hence, the i^{th} entires of $\sigma(s)$ and $\sigma(t)$ agree for $i \leqslant n$. Therefore $d(\sigma(s), \sigma(t)) \leqslant 1/2^n < \epsilon$ ending the proof. $\square$

Remark 3 *This note is addressed to readers with basic knowledge of topology. Here, elegant alternative proof of Lemma 7 is given.*

A basis for the topology of Σ_2 is the family of cylinder sets

$$C_t[s_0, \dots, s_k] = \{x \in \Sigma_2 :\ x_t = s_0, \dots, x_{t+k} = s_k\}.$$

Any cylinder is clopen set. Hence, using known equivalence of continuity, that is the preimage of any open set is open, Lemma 7 directly follows. For more details on topological properties of shifts see e.g. [44].

Lemma 8

1. *The set* $\mathrm{Per}_n(\sigma)$ *contains exactly* 2^n *points.*

2. *The set* $\mathrm{Per}(\sigma)$ *is dense in* Σ_2.

3. *There is a point with dense orbit in* Σ_2 *for* σ.

Proof:

1. If a point s is periodic of period n for σ, then it has the form

$$s = (s_0 s_1 s_2 \dots s_{n-1}\, s_0 s_1 s_2 \dots s_{n-1}\, \dots).$$

   Hence, there are 2^n periodic points of period n for σ each being a concatenation of finite blocks of zeros and ones each of the same length n.

2. For the proof we have to construct a sequence α^n of points in Σ_2 that is converging to the given point $s \in \Sigma_2$. For the given point $s \in \Sigma_2$ we construct a sequence in the following form

$$\alpha^n = (s_0 s_1 s_2 \dots s_n \, s_0 s_1 s_2 \dots s_n \, \dots).$$

So, α^n is constructed in such a way that it repeat sequence whose entires agree with s up to n-th position. Now, by Lemma 6 $d(\alpha^n, s) \leqslant 1/2^n$, so α^n tends to s for $n \to \infty$.

3. Put

$$s^\star = (\underbrace{\;01\;}_{1\text{blocks}} \; \underbrace{00\,01\,10\,11}_{2\text{blocks}} \; \underbrace{000\,001\,010\,\dots\,111}_{3\text{blocks}} \;\dots).$$

Now applying arguments used in the previous point of this proof we get that $\mathrm{Orb}_\sigma(s^\star)$ is dense in Σ_2 ending the proof.

□

2.5 Topological conjugacy

The goal of this section is to detect whether two dynamical systems have the same dynamical properties, that is they have the same types and number of periodic points etc. This tool is topological conjugacy, see Figure 2.6.

Definition 5 *Let (X, f) and (Y, g) be two dynamical systems. They are said to be* topological conjugated *if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism is called a* topological conjugacy*.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Figure 2.6: Diagram of topological conjugacy.

Next lemma is a natural consequence of the foregoing definition, proof of this statement is left to the reader as an exercise. As a hint an observation $g^n = h \circ f^n \circ h^{-1}$ can be recommended that is derived from

$$g^n = \underbrace{h \circ f \circ \underbrace{h^{-1} \circ h}_{\text{id}} \circ f \circ h^{-1} \cdots h \circ f \circ h^{-1}}_{n\text{-fold composition of } h \circ f \circ h^{-1}} = h \circ f^n \circ h^{-1}.$$

Lemma 9 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$.*

1. *If $x \in \mathrm{Per}_n(f)$ then $h(x) \in \mathrm{Per}_n(g)$.*
2. *The cardinality of $\mathrm{Per}_n(f)$ equals to $\mathrm{Per}_n(g)$.*
3. *If $\mathrm{Per}_n(f)$ is dense in X then $\mathrm{Per}_n(g)$ is dense in Y.*
4. *If x has a dense orbit in X then $h(x)$ has a dense orbit in Y.*

Remark 4 *Let us remark that the conjugacy does not have to exist and if it exists then it is not necessarily unique, see 2.6 Exercise 6. Moreover, conjugacy does not preserve differentiability, that is there are conjugated maps such that one of which is smooth and the other does not have derivative in a point, see 3.5 Exercise 3.*

Remark 5 *Note, the topological conjugacy defines an* equivalence *relation, that is the topological conjugacy is reflexive, symmetric and transitive. See Figure 2.7.*

Example 13 *Let us verify that the family F_μ is conjugated to the family g_a defined by*

$$g_a(x) = ax^2 - 1.$$

The task is to find out the conjugacy with respect to the parameter.

This conjugacy have to preserve fixed points. The fixed points of F_μ are 0 and p_μ, while $x_\pm = [1 \pm (1+4a)^{1/2}]/2a$ are fixed for g_a. Let us note that:

1. *$g_a(-x_+) = x_+$ and $F_\mu(1) = 0$,*
2. *the critical points of F_μ and g_a are $1/2$ and 0 respectively.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h_1"'] \arrow[dd, bend right, "h = h_2 \circ h_1"'] & X \arrow[d, "h_1"] \arrow[dd, bend left, "h = h_2 \circ h_1"] \\
Y \arrow[r, "g"] \arrow[d, "h_2"'] & Y \arrow[d, "h_2"] \\
Z \arrow[r, "w"] & Z
\end{tikzcd}
$$

Figure 2.7: Transitivity diagram of topological conjugacy.

Assume that the conjugacy has the form

$$h(x) = mx + b.$$

Since $-x_+ < x_- < x_+$ *and* $0 < p_\mu < 1$ *the following must be fulfilled*

$$
\begin{aligned}
h(-x_+) &= 1, \\
h(x_-) &= p_\mu, \\
h(x_+) &= 0, \\
h(0) &= 1/2.
\end{aligned}
$$

Substituting in h *we get*

$$
\begin{aligned}
m(-x_+) + b &= 1, \\
m(x_-) + b &= 1 - 1/\mu, \\
m(x_+) + b &= 0, \\
m \cdot 0 + b &= 1/2.
\end{aligned}
$$

From the last equation we obtain that $b = 1/2$. *Subtracting the first equation from the second one we get* $m(1/a) = -1/\mu$ *or* $m = -a/\mu$. *Substituting these values in the third equation we get* $-[1 + (1 + 4a)^{1/2}]/2\mu = -1/2$, $\mu = 1 + (1 + 4a)^{1/2}$ *or* $4a = \mu^2 - 2\mu$. *The last two expressions give necessary conditions for the maps to be conjugated:*

$$\mu = 1 + (1 + 4a)^{1/2} \text{ or } 4a = \mu^2 - 2\mu$$

and

$$h(x) = 1/2 - ax/\mu.$$

Now, we can verify that h is desired conjugacy:

$$\begin{aligned} F_\mu \circ h(x) &= F_\mu(1/2 - ax/\mu) \\ &= \mu/4 - a^2x^2/\mu \end{aligned}$$

and

$$\begin{aligned} h \circ g_a(x) &= h(ax^2 - 1) \\ &= a/\mu + 1/2 - a^2x^2/\mu. \end{aligned}$$

Consequently, these two quantities are equal since $4a = \mu^2 - 2\mu$, that shows the family F_μ is conjugated to the family g_a when the parameters are correctly related.

Now, let us observe that F_μ defined in Section 2.3 is conjugated with the shift constructed in Section 2.4. We use the same notation as in these sections.

Definition 6 *The* itinerary *of $x \in I$ is a sequence $S(x) = s_0s_1s_2\ldots$ where*

$$\begin{aligned} s_j &= 0 \text{ if } F_\mu^j(x) \in I_0, \\ s_j &= 1 \text{ if } F_\mu^j(x) \in I_1. \end{aligned}$$

Next auxiliary result will be useful in the sequel, its proof could be found in e.g. [12].

Lemma 10 *If $\mu > 2 + \sqrt{5}$ then $S : \Lambda \to \Sigma_2$ is a homeomorphism.*

Lemma 11 *The map F_μ is conjugated to σ if $\mu > 2 + \sqrt{5}$, that is $S \circ F_\mu = \sigma \circ S$.*

Proof: A point $x \in \Lambda$ could be defined uniquely by the nested sequence of intervals

$$\bigcap_{n \geqslant 0} I_{s_0s_1s_2\ldots s_n\ldots}$$

determined by the itinerary $S(x)$. Now

$$I_{s_0s_1s_2\ldots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \cdots \cap F_\mu^{-n}(I_{s_n})$$

so that $F_\mu(I_{s_0 s_1 s_2 \ldots s_n})$ could be written in the form

$$I_{s_1} \cap F_\mu^{-1}(I_{s_2}) \cap \cdots \cap F_\mu^{-n+1}(I_{s_n}) = I_{s_1 s_2 \ldots s_n}$$

since $F_\mu(I_{s_0}) = I$. Hence

$$\begin{aligned} S \circ F_\mu(x) &= S \circ F_\mu\left(\cap_{n=0}^{\infty} I_{s_0 s_1 s_2 \ldots s_n}\right) \\ &= S\left(\cap_{n=1}^{\infty} I_{s_0 s_1 s_2 \ldots s_n}\right) \\ &= s_1 s_2 s_3 \ldots \\ &= \sigma \circ S(x) \end{aligned}$$

ending the proof. □

Theorem 5 *If $\mu > 2 + \sqrt{5}$ and $F_\mu(x) = \mu x(1-x)$, then:*

1. *the cardinality of* $\mathrm{Per}_n(F_\mu)$ *is* 2^n,
2. $\mathrm{Per}(F_\mu)$ *is dense in* Λ,
3. F_μ *has a dense orbit in* Λ.

Proof: By Lemma 11 F_μ and σ are conjugated. Consequently, by Lemma 9 and Lemma 8 one gets the assertion. □

2.6 Exercises

1. Consider the map $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = -x^2 + x + 2$.Calculate all periodic points of period two for this map. Draw the graph of $f(x)$ and mark positions of all period two points. Include cobweb diagrams for all period two orbits and illustrate the stability of the fixed points by cobweb plots.

2. Classify analytically the stability of all fixed points of the map f from 2.6 Exercise 1.

3. Classify analytically the stability of all fixed points of the Hénon map $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $H(x, y) = (y + 1 - x^2, x)$.

4. Draw a cobweb diagram for a one-dimensional map of your choice showing a prime period three orbit and an eventually periodic orbit.

5. Show that for the map $B_3(x) = 3x \mod 1$, $x \in [0,1)$, the number of periodic orbits of period n is $3^n - 1$, as follows. Draw $B_3(x)$ and its second iterate. Identify $\mathrm{Fix}(B_3)$ and $\mathrm{Per}_2(B_3)$ in your drawings and calculate the corresponding periodic points analytically. On this basis, argue for the result for general n.

6. Let $f(x) = x + 2$ and $g(x) = x + 3$ be defined on $\mathbb{R}$. Then, for any $b \in R$ and $h_b : \mathbb{R} \to \mathbb{R}$ defined by $h_b(x) = 3/2\, x + b$ verify, that f and g are topologically conjugated by the conjugacy h_b.

7. Show that the logistic map F_μ is topologically conjugated with $G_a(x) = ax^2 + bx + c$, where $\mu \in (0,1]$ and $a \neq 0$, via the conjugacy $h : [0,1] \to [(-\mu - b)/2a, (\mu - b)/2a]$ defined by $h(x) = -\mu/a\, x + (\mu - b)/2a$.

8. Prove that F_4 is topologically conjugated to $f(x) = 2x^2 - 1$ defined on the interval $[-1,1]$. (Hint: Use linear conjugacy $h : [0,1] \to [-1,1]$ defined by $h(x) = -2x + 1$.)

9. Calculate the distance between the following two points in the shift space:

   (a) $s = 000\cdots = \overline{0}$ and $t = 111\cdots = \overline{1}$,

   (b) $s = 0101\cdots = \overline{01}$ and $t = 1010\cdots = \overline{10}$.

10. Prove Lemmas 4, 9 and 10.

Chapter 3

Chaos

Chaos theory studies the behavior of dynamical systems that are sensitive to initial conditions and nonlinear. Small differences in initial conditions yield widely diverging outcomes for chaotic systems, rendering long-term prediction impossible in general. Chaotic behavior can be observed in many natural systems, such as weather, chemical reactions or some ecological systems.

The above mentioned properties are described more precisely in the foregoing sections using one specific example, the Tent map. The Tent map T is defined on the unit closed interval $[0,1]$ by the following formula

$$T(x) = \begin{cases} 2x & \text{if } 0 \leqslant x < 1/2, \\ 2(1-x) & \text{if } 1/2 \leqslant x \leqslant 1. \end{cases}$$

which could be equivalently rewritten

$$T(x) = 1 - |2x - 1|.$$

It is easy to verify that the Tent map is conjugated to the Logistic map via the conjugacy $h(x) = \sin^2(\pi x/2)$, hence they have similar dynamical behavior.

3.1 Density of periodic points

The main aim of this section is considering a set of all periodic points of the Tent map and a proof that it is dense on I.

Definition 7 *Let I be an interval in $\mathbb{R}$. A set A is said to be* dense in I *if for every $x \in I$ any open interval containing x intersects A. In other words, for each $\delta > 0$, it holds true*

$$(x - \delta, x + \delta) \cap A \neq \varnothing.$$

Example 14 *The set of all rational numbers $\mathbb{Q}$ is dense in the set of all real numbers $\mathbb{R}$. To verify this we write a point $x \in \mathbb{R}$ in the following way*

$$x = \sum_{n=0}^{\infty} \frac{d_n}{10^n}$$

where $d_n \in \{0, 1, 2, \ldots, 9\}$. Let $\delta > 0$. Then there is a positive integer m such that $10^{-m} < \delta$. Consider now the rational number

$$y = \sum_{n=0}^{m} \frac{d_n}{10^n}.$$

Then

$$|x - y| = \sum_{n=m+1}^{\infty} \frac{d_n}{10^n} \leqslant \sum_{n=m+1}^{\infty} \frac{9}{10^n} = \frac{9/10^{m+2}}{1 - 1/10} = 1/10^m.$$

Consequently, $|x - y| < \delta$ and hence $\mathbb{Q}$ is dense in $\mathbb{R}$.

The following lemmas will be used in the proof of the goal of this section. We say that a point x is *eventually periodic under f* if there are $m \in \mathbb{N}$ and a periodic point p such that $f^m(x) = p$. Roughly speaking, eventually periodic point is a point that is mapped on a periodic point. It is easy to see that points $1/3, 1/4, 3/4$ are eventually fixed, points $1/5, 3/5$ are eventually 2-periodic of the Tent map.

Lemma 12 *A point $p \in I$ is eventually periodic under T if and only if it is a rational number in I.*

Proof: Let $p = r/s$ be in its reduced form. Assume now that $s = 2k + 1$ is an odd integer. Then $T^n(r/s) =$ (even integer)$/s$ for all $n \in \mathbb{N}$. Moreover, there are exactly k numbers in the interval $[0, 1]$ in the form (eveninteger)$/s$, namely, $2/s$, $4/s \ldots$, $(2k)/s$. So, the orbit of the point p has at most k elements and p is eventually periodic. Assume now that $s = 2k$, then for some positive integer m either $T^m(p) =$ integer/ odd integer, which was discussed, or $T^m(p) = 1$ and hence $T^{n+m}(p) = 0$ for any $n \in \mathbb{N}$, hence p is eventually periodic ending the proof of necessary condition.

Conversely, assume that p is eventually periodic of T. Then $T^n(p) = t_n + 2^n p$ for some integer t_n. The point p is eventually periodic so $T^n(p) = T^{n+k}(p)$ for some positive integer k. Thus

$$t_{k+n} \pm 2^{k+n}p = t_n \pm 2^n p$$

or

$$p = \frac{t_{k+n} - t_n}{\pm 2^n \mp 2^{n+k}}$$

which proves that p is rational. □

Lemma 13 *Let $p = r/s$ be a rational number in I. Then the point p is periodic if and only if r is even and s is odd integer.*

Proof: Let $p = r/s \in (0,1)$ where r is even and s is odd integer. Now by Lemma 12 it follows that p is eventually periodic point of the map T. Therefore there is a least nonnegative integer m and a least positive integer $n > m$ such that $T^m(p) = T^n(p)$. If $m = 0$ the point p is periodic with period n. If $m > 0$ then

$$T^{-1}\left(\frac{r}{s}\right) = \frac{\text{even integer}}{\text{odd integer}}.$$

Thus

$$T^m\left(\frac{r}{s}\right) = \begin{cases} 2ei/s = 4i/s & \text{if } 0 \leqslant T^{-1}\left(\frac{r}{s}\right) \leqslant 1/2, \\ 2(1 - ei/s) = 4i/s + 2 & \text{if } 1/2 < T^{-1}\left(\frac{r}{s}\right) \leqslant 1. \end{cases}$$

where ei is an even integer and i is integer. Hence, for $T^m(r/s)$ to be equal to $T^n(r/s)$ we must either have both $T^{m-1}(r/s)$ and $T^{n-1}(r/s)$ in the interval $[0, 1/2]$ or have both in the interval $(1/2, 1]$. Without lost of generality we assume now that $T^{m-1}(r/s)$ and $T^{n-1}(r/s)$ are in the interval $[0, 1/2]$. Hence

$$2T^{m-1}(r/s) = T^m(r/s) = T^n(r/s) = 2T^{n-1}(r/s).$$

Consequently, $T^{m-1}(r/s) = T^{n-1}(r/s)$ which contradicts the minimality of m and n. Therefore $m = 0$ and p is a periodic point with period n for T.

The converse implication could be proved analogously to the previous techniques. □

Theorem 6 *The Tent map T has a dense set of periodic points.*

Proof: Let $J = (a, b)$ be an open interval in I where $t = b - a$. Choose now an odd integer s such that $s > 2/t$ and consider the set

$$A = \{1/s, 2/s, \dots (s-1)/s\}.$$

We observe now that for any successive numbers r/s and $(r+1)/s$ in A it holds

$$\frac{r+1}{s} - \frac{r}{s} = \frac{1}{s} < \frac{t}{s}.$$

This implies that there are two successive numbers m/s and $(m+1)/s$ in A belonging into J. Now, one of m, $m+1$ is an even integer and the interval J contains a point in the form $c =$ (even integer)/(odd integer). By Lemma 13 the point c is periodic. Consequently the set of periodic points of T is dense in I. □

Let us note that any irrational rotation on the unit circle does not have dense set of periodic points, moreover there are no periodic points at all. To see this apply Theorem 2.

Lemma 14 *Any rational rotation of the unit circle has a dense set of periodic points. Moreover,* $\mathrm{Per}(R_\lambda) = \mathbb{S}^1$.

Proof: Let $\lambda = p/q$ be in its reduced form. Then

$$R_\lambda^q(\alpha) = \alpha + 2\pi\lambda q = \alpha + 2\pi p = \alpha$$

for any $\alpha \in \mathbb{S}^1$. So any point $\alpha \in \mathbb{S}^1$ is periodic with period p ending the proof. □

To the end of this section we give a lemma that could be used for flipping dense set of periodic points from the Tent map to the Logistic map, the proof is left to the reader as an exerciese.

Lemma 15 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$. Then f has dense set of periodic points if and only if g has.*

Theorem 7 *The Logistic map F_4 has a dense set of periodic points.*

Proof: The Tent map and the Logistic map are conjugated by the conjugacy $h(x) = \sin^2(\pi x/2)$. Now, applying Theorem 6 and Lemma 15 one gets the assertion, ending the proof. □

3.2 Transitivity

Definition 8 *Let f be a continuous map on an interval $I \subset \mathbb{R}$. The map f is* topologically transitive *if for any pair of nonempty open intervals U and V there is $n \in \mathbb{N}$ such that*

$$f^n(U) \cap V \neq \varnothing.$$

Theorem 8 *For a continuous onto map f defined on the closed interval I the following are equivalent:*

1. *f is topologically transitive,*
2. *f has a dense orbit.*

Proof: Let us firstly assume, that f is topologically transitive. For each n is the interval I covered by finitely many intervals of the length $1/n$, denote them by $U_1, U_2, \ldots, U_n$. For each k the set

$$G_k = \bigcup_{n=1}^{\infty} f^{-n}(U_k)$$

is open and dense in I and by [34] there is a point $x \in I$ which is contained in G_k for all k. Since the orbit of x intersects each U_k, this orbit is dense.

Conversely, assume that f has a point x_0 whose orbit is dense. Let U and V be two non-empty open intervals in I. Then there are $m \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $n > m$, $f^m(x_0) \in U$ and $f^n(x_0) \in V$. Then $f^{n-m}(U) \cap V \neq \varnothing$. Thus f is topologically transitive map, ending the proof. $\square$

Remark 6 *The Theorem 8 remains valid if we replace the space I by compact metric space X without isolated points.*

Let us note that the statement of Theorem 8 is not valid in general.

Example 15 *Let $X = \{a, b\}$ be a set endowed with discrete topology structure, that is all subsets of X are open. Now define a map on X in such a way that $f(a) = a$ and $f(b) = a$. Now, $\mathrm{Orb}_f^+(b) = X$, hence we have a point with dense orbit. On the other side, put $U = \{a\}$ and $V = \{b\}$, then for any $n \in \mathbb{N}$ it holds $f^n(U) = U$. Consequently, there is no n such that $f^n(U)$ will intersect V, the map f is not topological transitive.*

As in the previous section we give a lemma which shows that conjugacy preserves transitivity. Again, the proof is left to the reader as an exercise.

Lemma 16 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$. Then f is transitive if and only if g is.*

Theorem 9 *The Tent map, Logistic map, shift map and irrational rotation are topological transitive maps.*

Proof: There are several ways how to prove the assertion. It seems that the easiest way is to show that each of the mentioned maps has dense orbit using Theorem 8 and Remark 6. It follows that the Tent map, Logistic map, shift map and irrational rotation are topological transitive maps applying Theorem 5, Lemma 9, Lemma 8 and Theorem 2, respectively. □

Remark 7 *The rational rotation of the unit circle is not topologically transitive, since all points from the unit circle have the same period, see Lemma 14.*

3.3 Sensitive dependence on initial conditions

Definition 9 *A continuous map f on an interval I has* sensitive dependence *on initial conditions if there is ϵ such that for any $x_0 \in I$ there is $y_0 \in (x_0 - \delta, x_0 + \delta)$ and $k \in \mathbb{N}$ such that*

$$|f^k(x_0) - f^k(y_0)| \geqslant \epsilon.$$

The number ϵ is called a sensitive constant *of f.*

Example 16 *The simplest map possessing sensitive dependence is the linear map $f(x) = cx$ for $c > 1$. It is easy to observe that for initial points x_0 and $x_0 + \delta$ one gets*

$$f^n(x_0 + \delta) - f^n(x_0) = c^n(x_0 + \delta) - c^n x_0 = c^n \delta.$$

Since $c > 1$ the distance $|f^n(x_0 + \delta) - f^n(x_0)|$ will increase to the infinity as n tends to the infinity, regardless of how small δ is. Let us point out, that this map has sensitive dependence on initial conditions but has no other dynamical properties. This map is not topological transitive and does not have dense set of periodic points.

As in the previous sections we give technical lemma that makes obvious that conjugacy preserves sensitive dependence on initial conditions, again, proof is left to the reader as an exercise.

Lemma 17 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$. Then f has sensitive dependence on initial conditions if and only if g has.*

Now, let us consider a point x_0 and its neighboring point $x_0+\delta$. The error err_n we did replacing the original point by its neighbor in the n-th iteration defined by

$$\mathrm{err}_n = |f^n(x_0+\delta) - f^n(x_0)|$$

and the relative error by

$$\left|\frac{\mathrm{err}_n}{\delta}\right| = \frac{|f^n(x_0+\delta) - f^n(x_0)|}{\delta}.$$

If the map f has sensitive dependence on initial conditions we suppose the relative error to grow exponentially with n and thus

$$e^{n\lambda} = \lim_{\delta\to 0} \frac{|f^n(x_0+\delta) - f^n(x_0)|}{\delta} = \left|\frac{\mathrm{d}}{\mathrm{dx}} f^n(x_0)\right| = |f'(x_0)f'(x_1)\dots f'(x_{n-1})|.$$

Hence

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|$$

which motivates us to define the Lyapunov exponent of a map f with respect to the initial point x_0.

Definition 10 *The* Lyapunov exponent $\lambda(x_0)$ *of an orbit* $\mathrm{Orb}^+(x_0)$ *of an interval map $f : [0,1] \to [0,1]$ is defined as the number*

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \ln |f'(x_k)|, \tag{3.1}$$

if the limit exists.

The notions of positive Lyapunov exponent and sensitive dependence on initial conditions play a prominent role in chaotic dynamical systems. Indeed, it is a popular practice to use the numerical value of a Lyapunov exponent as the quantitative measure of sensitivity or the lack thereof. A firm mathematical basis for this practice is often not clear.

The Lyapunov exponent is a property of the orbit $\mathrm{Orb}^+(x_0)$ since it is easily seen that if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \ln |f'(x_k)|$$

exists then for $m > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \ln |f'(x_{k+m})|$$

exists also and the limits are equal.

The following theorem will be very useful while determining sensitive dependence on initial conditions, the proof is out of the scope of this text and can be found in [23] .

Theorem 10 ([23]) *Suppose $f : [0,1] \to [0,1]$ is differentiable on I, that the orbit $\mathrm{Orb}^+(x_0)$ satisfies*

$$\inf_{n\geqslant 0} |f'(x_n)| > 0$$

and that the Lyapunov exponent $\lambda(x_0) > 0$ exists as a limit. Then the orbit $\mathrm{Orb}^+(x_0)$ exhibits sensitive dependence on initial conditions.

Remark 8

1. *Let us point out that the converse implication to Theorem 10 is not valid, the contra example was given in Example 16.*

2. *It is possible to replace the added condition on the derivative in Theorem 10, which means that the orbit stays away from critical points, with a reasonable strengthening of the definition of Lyapunov exponent.*

Theorem 11 *The Tent map, Logistic map and shift map posses sensitive dependence on initial conditions.*

Proof: Let $x_0 \in I \backslash PO$ where PO is the set of all pre-images of zero. Since $|T'(y)| = 2$ where $y \in (0, 1/2) \cup (1/2, 1)$ we have

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln 2 \approx 0.6931.$$

Applying Theorem 10 we get that the Tent map has sensitive dependence on initial conditions.

Now, by Theorem 17 Logistic map and shift map are also sensitive on initial conditions. □

Lemma 18 *The rotation on the unit circle fails to be sensitive on initial conditions.*

Proof: Any rotation of the unit circle $R_\lambda(\alpha) = \alpha + 2\pi\lambda$ has $|R'_\lambda(\alpha)| = 1$ for any $\alpha \in \mathbb{S}^1$. Hence,

$$\lambda(\alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln 1 = 0.$$

Consequently, any rotation of the unit circle fails to be sensitive to initial conditions. □

3.4 The notion of chaos

Definition 11 ([12]) *A dynamical system (X, f) is said to be* chaotic in the sense of Devaney *if:*

1. *f is topologically transitive,*
2. *the set* $\mathrm{Per}(f)$ *is dense in X.*

It was originally required on f to have sensitive dependence on initial conditions, but later on it was proved in [6] that this condition is superfluous.

Theorem 12 ([6]) *If $f : X \to X$ is topologically transitive and has a dense set of periodic points then f has sensitive dependence on initial conditions.*

To the end, the following theorem shows that it is enough to verify topological transitivity of a continuous map on the interval to be chaotic in the sense of Devaney.

Theorem 13 ([38]) *If f is topologically transitive on the interval I (not necessarily bounded) then the set* $\mathrm{Per}(f)$ *is dense in I.*

Lemma 19 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$. Then f is chaotic in the sense of Devaney if and only if g is.*

Proof: The assertion directly follows from Lemmas 15, 16 and 17. □

Theorem 14 *The Tent map, Logistic map and shift map are chaotic in the sense of Devaney.*

Proof: It easily follows from Theorems 6, 7, 8 and 9. □

Lemma 20 *The rotation on the unit circle fails to be chaotic in the sense of Devaney.*

Proof: Firstly, if the rotation is irrational than it has no periodic points. Secondly, if the rotation is rational it is not transitive, see Remark 7. □

The above discussed feeling of chaos became famous for the notion of sensitive dependence on initial conditions. Nevertheless it was not the first concept of chaos. The first notion of chaos was given by [26], defined by:

Definition 12 ([26]) *A dynamical system (X, f) is said to be* chaotic in the sense of Li and Yorke *if there is an uncountable set $S \subset X$ such that for any $x \neq y \in S$*

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

As in foregoing sections the following lemma shows that Li and Yorke chaos is preserved by conjugacy, proof is also left to the reader as an exercise.

Lemma 21 *Let (X, f) and (Y, g) be two dynamical systems conjugated by conjugacy $h : X \to Y$. Then f is chaotic in the sense of Li and Yorke if and only if g is.*

Theorem 15 *The Tent map, Logistic map and shift map are chaotic in the sense of Li and Yorke.*

Proof: Using Lemma 21 it is enough to show that (Σ_2, σ) is chaotic in the sense of Li and Yorke. Let us pick the following two points:

$$s = 0000\ldots,$$

$$t = 010010001000010000001\ldots$$

Here, t is concatenation of increasing blocks of zeros and single ones. Now, it is easy to compute that

$$\limsup_{n\to\infty} d(\sigma^n(s), \sigma^n(t)) > 0,$$

$$\liminf_{n\to\infty} d(\sigma^n(s), \sigma^n(t)) = 0.$$

We constructed Li and Yorke pair and it remains to add more points to s and t to get uncountable scrambled set. This construction is possible since the shift space is uncountable and it is easy to control distances in the shift space finishing the proof. $\square$

Lemma 22 *The rotation on the unit circle fails to be chaotic in the sense of Li and Yorke.*

Proof: As it was pointed out in the proof of Theorem 2 each rotation on the unit circle preserves the length of an arbitrarily chosen arc. Hence, the distance between each two points remains constant under iterations. Consequently, limit of the sequence of distances of iterations of two points exists and is finite, ending the proof. $\square$

The notion of chaos was (and still is) studied by many authors. There are many different types of notions of chaos and relations between them are not understood well. Some of them were solved and discussed in [24]. The following two theorems show the relationship between the chaos in the sense of Li and Yorke and Devaney one, moreover Devaney chaos is stronger then Li Yorke one.

Theorem 16 ([20]) *Let (X, f) be a dynamical system such that f is chaotic in the sense of Devaney. Then f is chaotic in the sense of Li and Yorke.*

Theorem 17 ([10]) *There is a dynamical system (X, f) which is chaotic in the sense of Li and Yorke, but not in the sense of Devaney.*

3.5 Exercises

1. Prove that $D : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $D(\phi) = 2\phi$ is transitive.

2. Prove that $D : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $D(\phi) = 2\phi$ has dense set of periodic points, and find out whether D is chaotic in the sense of Devaney or Li and Yorke.

3. Prove the fact, that Tent map is conjugated to the Logistic map.

4. Verify that points $1/3, 1/4, 3/4$ are eventually fixed, points $1/5, 3/5$ are eventually 2-periodic for the Tent map.

5. Prove the following statements:

   (a) Let T be the Tent map. Let $I_{n,k}$ denote the diadic interval $[(k-1)/2^n, k/2^n]$ for any $n \in N$ and $k = \{1, 2, 3 \ldots 2^n\}$. Then, T^n restricted to $I_{n,k}$ is a linear homeomorphism onto I. That is, the interval I is mapped onto $I_{n,k}$ by T^n.

   (b) Using Exercise 5a prove: The set of periodic points is dense for the Tent map.

   (c) Using Exercise 5a prove: The set of eventually periodic points which are not periodic is dense for T.

   (d) Using Exercise 5a prove: The set of eventually fixed points which are not fixed points is dense for T.

   (e) Using Exercise 5a prove: For any $n \in N$, the set $\{x \in [0, 1] : T^n(x) = x\}$ has 2^n elements.

6. Calculate the Lyapunov exponents for all fixed points and for all prime period two orbits of the following maps:

   (a) $f(x) = -x^2 + x + 2$,

   (b) $f(x) = 2x^2 - 5x$.

7. Calculate the Lyapunov exponents for all fixed points of the map D defined in 3.5 Exercise 1.

8. Prove Lemma 15, Lemma 16, Lemma 17 and Lemma 21.

9. Determine whether the Baker map $B_3(x) = 3x \mod 1$, $x \in [0, 1)$ is chaotic in the sense of Devaney and Li and Yorke, respectively.

10. Discuss whether the map $F : \triangle \to \triangle$ given by $F(x, y) = (x(4 - x - y), xy)$ where $\triangle$ is a triangle with vertices $(0, 0)$, $(0, 4)$ and $(4, 0)$ is chaotic in the sense of Devaney and Li and Yorke, respectively.

Chapter 4

Fractals

Intuitively, a *fractal* is a set which is self-similar under magnification, a correct definition will be given later in this chapter. Before inquiring into the mathematical basis, we introduce some well known examples.

The Koch curve

The *initiator* of the Koch curve is a straight line. The *generator* is obtained by partitioning the initiator into three equal segments. Then remove the middle third and replace it with an equilateral triangle, see Figure 4.1.

Put the length of the initiator equal 1. Then the generator consists of four line segments each of length 1/3. So the total length of the generator is 4/3. In the second step each of the four line segments acts as an initiator which is replaced by the corresponding reduced generator. The newly constructed curve contains 16 segments each of the length $1/3^2$ and the length of the whole curve is $(4/3)^2$. Now, proceed with described construction infinitely, the limiting curve is called the *Koch curve*. This curve has the following properties:

- is not differentiable anywhere (i.e. it has no tangent line),
- its length is ∞ since the n-th step line has the length $(4/3)^n$,
- it is undoubtedly self-similar since every part is itself miniature of the whole curve.

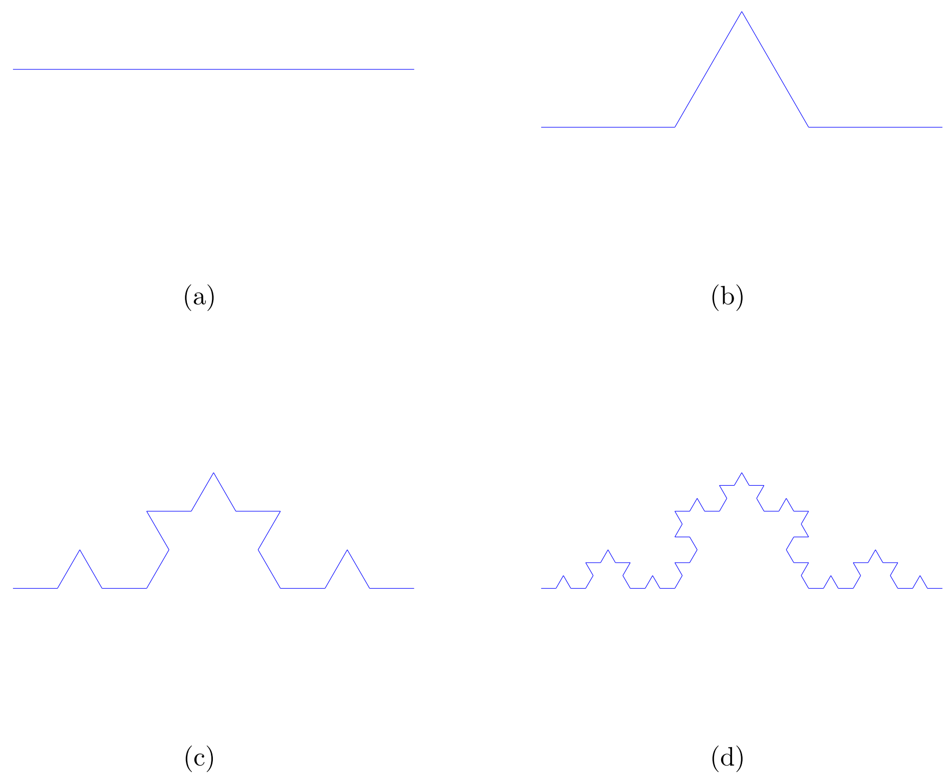


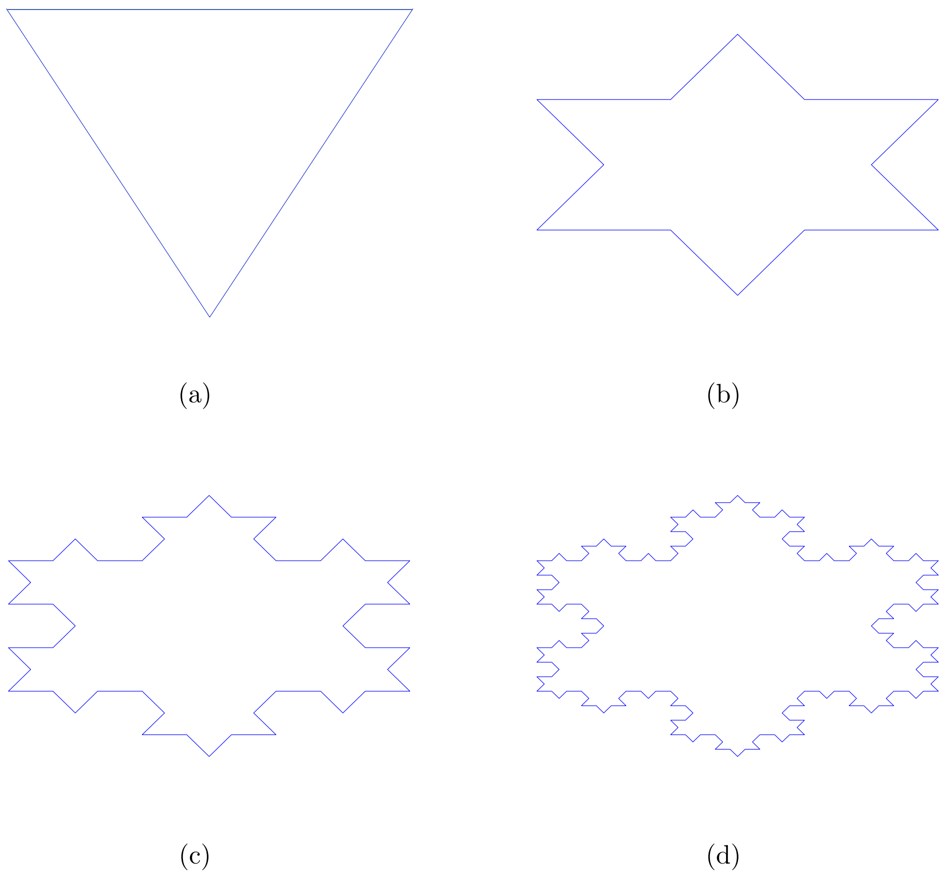
Figure 4.1: Construction of the Koch curve. Case (a) corresponds to the initiator, case (b) corresponds to the first step Koch curve, case (c) corresponds to the second step and case (d) corresponds to the third step.

The Koch snowflake

The *initiator* of the Koch snowflake is an equilateral triangle. The *generator* is obtained by partitioning each side of the initiator into three equal segments. Then remove the middle third of each segment and replace it with an equilateral triangle, see Figure 4.2.

Now, the second step will be analogous as in the previous example and the limiting curve is called the *Koch snowflake*. This curve has the following properties:

- is not differentiable anywhere (i.e. it has no tangent line),
- its length is ∞,
- the area bounded by this curve is finite, but we can never wrap a length of a string around its boundary.



(a) (b) (c) (d)

Figure 4.2: Construction of the Koch snowflake. Case (a) corresponds to the initiator, case (b) corresponds to the first step Koch snowflake, case (c) corresponds to the second step and case (d) corresponds to the third step.

The Sierpinski Gasket

The *initiator* of the Sierpinski gasket is an equilateral triangle with sides of unit length, thought of as a solid object. The *generator* is obtained by partitioning of the initiator into four equal equilateral triangles (connect midpoints of the three sides of the triangle) each with the side length $1/2$. Then remove the middle one, see Figure 4.3. Now repeat this procedure with new initiator, the three triangles of side length $1/2$. We get 3^2 new triangles of side length $(1/2)^2$. So, in the n-th step we get 3^n triangles with the side length $(1/2)^n$. The limiting object is the *Sierpinski gasket*. It has the following properties:

- is self-similar,
- has zero area:

  Denote by T the area of the original initiator. In the first step we remove $(1/4)\,T$, in the second $3(1/4)^2\,T$ and in the n-th step $3^{n-1}(1/4)^n\,T$. So, the remover area equals to

  $$(1/4)^1\,T + 3(1/4)^2\,T + \cdots + 3^{n-1}(1/4)^n\,T + \cdots = T.$$

  Consequently, the Sieprinski gasket has zero area.

4.1 Dimension of a fractal

The main aim of this section is to introduce the rigorous definition of fractals. This definition is due to B. Mandelbrot [31] who constructed so called fractal dimension and compared it with the topological one, his concept corresponds to the notion of capacity used by Kolmogorov [22].

More precisely, a *fractal* is a geometrical object that is linked to the, at least one, property [1]:

- a fractal dimension is greater then topological dimension (see Definition 13),
- self-similarity,
- it is an attractor of IFS (iterated function system, see Definition 16).

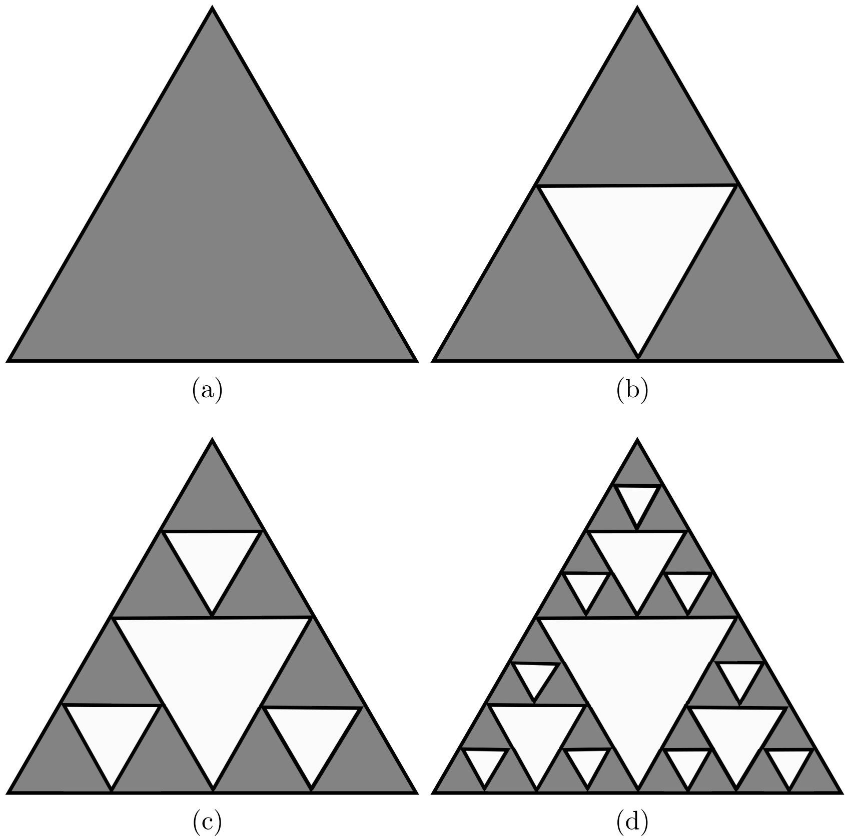


Figure 4.3: Construction of the Sierpinski gasket. Case (a) corresponds to the initiator, case (b) corresponds to the first step Sierpinski gasket, case (c) corresponds to the second step and case (d) corresponds to the third step.

This properties are not coherent, that means that there are examples of geometrical object having at least one, (exactly one or two) of properties afore mentioned. For example, it is possible to construct a geometrical object that can be generated as an attractor of IFS, its fractal dimension is greater then topological one, but is not self-similar, see Example 18. The relation of the three properties can be depicted, as shown in Figure 4.4, by a Venn diagram. Point out, that the classical concept of a fractal fulfills all three properties, and as a folklore it is assumed of having these properties.

The *topological dimension* intuitively gives us that a line has the topological dimension one, a solid square two and finally a solid cube three.

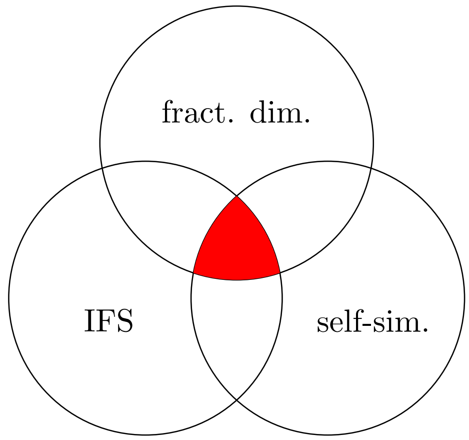


Figure 4.4: Venn diagram of three properties defining a fractal.

Definition 13 *A set S has the* topological dimension 0 *if every point in S has an arbitrary small neighborhood whose boundary does not intersect S. A set S has the* topological dimension $d > 0$, *if every point in S has an arbitrarily small neighborhood whose boundary intersects S in a set of topological dimension $d - 1$ and d is the least positive integer for which this holds. The topological dimension of a set S is denoted by* $\mathrm{D}_{\mathrm{top}}(S)$.

It is easy to see that the set of all integers, all rational numbers or the set of all irrational numbers have zero topological dimension. Any solid circle or square in the plane has the topological dimension 2. Nevertheless a circle line in the plane has the topological dimension 1.

Now the topological dimension of both, Koch curve and snowflake again equals to one.

Suppose now that the line segment of the unit length is divided into N equal subsegments with the scaling quotient h. Obviously, $Nh = 1$, which means $N = (1/h)^1$. If we have a solid square in the plane and we divide it into N equal sub-squares with the scaling ratio h, we get $Nh^2 = 1$. Now on divide a solid cube into N equal sub-cubes with the scaling quotient h then again $Nh^3 = 1$. If we extend our construction into higher dimensions we get $Nh^d = 1$, it means $N = (1/h)^d$. The exponent corresponds to the dimension of an object. Note that

$$d = \frac{\ln(N)}{\ln(1/h)}.$$

The relationship may be also understood in the opposite way. Observe that if we magnify a line 3 times its length is 3 times greater. If a square is three times magnified its area is 3^2 times greater. Magnifying a cube three times its volume rises 3^3 times.

Let us apply this approach to the Sierpiski gasket. It consists of its three half sided copies. Hence, in this case $N = 3$ and $h = 1/2$. We obtain

$$D = \frac{\ln(N)}{\ln(1/h)} = \frac{\ln(3)}{\ln(2)}.$$

We can define the *similarity dimension*

$$\mathrm{D}_{\mathrm{sim}}(S) = \frac{\ln(N)}{\ln(1/h)}.$$

Example 17

The Koch snowflake *It is easy to observe that the Koch snowflake* Ks *has the initiator consisting of three line segments each made up of four line segments and with the scaling quotient* $1/3$*, that is* $N = 4$ *and* $h = 1/3$*. So,* $\mathrm{D}_{\mathrm{sim}}(Ks) = \ln 4/\ln 3 \approx 1.26$ *while* $\mathrm{D}_{\mathrm{top}}(Ks) = 1$*.*

The Cantor set *The first fractal of this text, the Cantor set, has* $N = 2$ *and* $h = 1/3$*. So,* $\mathrm{D}_{\mathrm{sim}}(C) = \ln 2/\ln 3 \approx 0.63$ *while* $\mathrm{D}_{\mathrm{top}}(C) = 0$*.*

The similarity dimension may be also applied to only statistically self-similar objects.

Example 18

A non self-similar fractal NSF *As an initiator pick a unit solid square. As generator: divide initiator into nine equal squares and delete randomly one of them. Now, replace each of remaining subsquares with the scaled generator. The limiting object is a fractal, but is not self-similar. Easily,* $N = 8$ *and* $h = 1/3$*, so*

$$\mathrm{D}_{\mathrm{sim}}(NSF) = \lim_{n\to\infty} \frac{\ln(N(h))}{\ln(1/h)} = \lim_{n\to\infty} \frac{\ln(8^n)}{\ln(3^n)} = \ln 8/\ln 3 \approx 1.89$$

while $\mathrm{D}_{\mathrm{top}}(NSF) = 1$*.*

Note that there exist a great deal of approaches to the fractal dimension (see e.g. [15]). Nevertheless, for any self-similar set S the fractal dimension $\mathrm{D}_{\mathrm{frac}}(S)$ corresponds to the similarity dimension $\mathrm{D}_{\mathrm{sim}}(S)$.

Definition 14 *A set $S \subset \mathbb{R}^n$ is a* fractal *if*

$$\mathrm{D}_{\mathrm{top}}(S) < \mathrm{D}_{\mathrm{frac}}(S).$$

Example 19 *As it was computed before, the Koch curve, the Koch snowflake, the Cantor set and the Sierpinski gasket are fractals as defined in 14.*

4.2 Iterated function systems

The main purpose of this section is to formulate mathematical background for correct construction of fractals.

We start with a *linear map* $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

1. $F(x + y) = F(x) + F(y)$ and $x, y \in \mathbb{R}^2$,
2. $F(cx) = cF(x)$ for any $c \in \mathbb{R}$ and $x \in \mathbb{R}^2$.

It is well known that each linear map may be represented by a matrix M:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So, the map F can be written in the form

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

Moreover, a linear map F is called *affine* if it can be represented in the form:

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

Assume now, that (X, d) is a metric space. A map $F : X \to X$ is called a *contraction* if there is a factor $0 < \alpha < 1$, called *contraction factor*, such that for any $x, y \in X$

$$d(F(x), F(y)) \leqslant \alpha d(x, y).$$

It is easily seen that for any affine map represented by 2×2 matrix M it holds

$$d(F(x), F(y)) = d(Mx, My) \leqslant ||M|| d(x, y).$$

The map represented by this matrix M is contraction if $||M|| < 1$ which appears when all eigenvalues are in absolute values less then one, i.e. $|\lambda_i| < 1$ for each i. Here

$$d(Mx, My) = |Mx - My|.$$

Let us recall that

$$||M|| = \sup_{|x| \leqslant 1} \{|Mx|\}$$

where $|(x_1, x_2)| = |x_1| + |x_2|$, $x = (x_1, x_2)^{\mathrm{T}}$ and $y = (y_1, y_2)^{\mathrm{T}}$.

Let us also remind similitudes. A map $F : X \to X$ is called a *similitude* if there is a factor $0 < \alpha < 1$, such that for any $x, y \in X$

$$d(F(x), F(y)) = \alpha d(x, y).$$

Note that in $\mathbb{R}^2$ a similitude is an affine contraction where the matrix M is in addition orthogonal. It plays a key role in calculation of similarity dimension.

Let us proceed to the class of closed bounded subsets of $\mathbb{R}^n$ "where fractals really live". Let $(\mathbb{R}^n, d_{\mathbb{R}^n})$ be a metric space with the metric $d_{\mathbb{R}^n}$ and H denotes the set of all closed and bounded subsets of $\mathbb{R}^n$. Now, we would like to endow the space H with a suitable metric. Naturally, for $A, B \in H$ and any $a \in A$ one defines

$$d(a, B) = \inf\{d_{\mathbb{R}^n}(a, b) : \, b \in B\}$$

and the distance between A an B could be defined

$$d(A, B) = \sup\{d(a, B) : \, a \in A\}.$$

Unfortunately, d is not a metric. If we put $A = \{(x, y) \in \mathbb{R}^2 : \, (x+1)^2 + y^2 \leqslant 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : \, (x - 3)^2 + y^2 \leqslant 4\}$ then we can easily compute that $d(A, B) = 3$ and $d(B, A) = 5$. Consequently, $d(A, B) \neq d(B, A)$ so d does not preserve the axiom of commutativity. This problem is solved in the following definition.

Definition 15 *Let $A, B \in H$. Then the* Hausdorff distance d_H *between the sets A and B is defined*

$$d_H(A, B) = \max\{d(A, B), d(B, A)\}.$$

It is easy to verify, that d_H is really a metric, proof is left to the reader as exercise. Returning to our foregoing counterexample $d_H(A, B) = 5$. Hence, (H, d_H) is a metric space.

The following lemma is a very important observation needed in the Theorem 18. For the formulation we need the completeness of the metric space, that is each Cauchy sequence is convergent in this space (for correct definition see e.g. [14]).

Lemma 23 *The (H, d_H) is a complete metric space.*

Theorem 18 *Let (X, d) be a complete metric space and $F : X \to X$ be a contraction with the contraction factor c. Then F has a unique fixed point $p \in X$. Moreover, this fixed point p is a unique global attractor, that is for any $x \in X$ it holds*

$$F^n(x) \xrightarrow[n\to\infty]{} p.$$

We are ready to introduce iterated function systems.

Definition 16 *Let $F_1, F_2, \ldots, F_N$ be a family of contractions on $\mathbb{R}^k$.Then the system*

$$\{\mathbb{R}^k : F_i,\, i = 1, 2, \ldots, N\}$$

is called an iterated function system*, briefly* IFS*.*

Let F_i be a family of contractions in $\mathbb{R}^n$ for $i = 1, 2, \ldots, N$ as above. Define a map $F : H \to H$ by

$$F(S) = F_1(S) \cup F_2(S) \cup \cdots \cup F_N(S),$$

where

$$F_i(S) = \bigcup_{x\in S} F_i(x),\; i = 1,\, 2,\, \ldots,\, N,$$

for each $S \in H$.

The following lemma shows that the union map F is again a contraction while all maps in the family are contractions, proof is left to the reader and could be found in [14].

Lemma 24 *Let $F_1, F_2, \ldots, F_N$ be a family of contractions on $\mathbb{R}^k$ with contraction factors α_i. Then the union $F = \bigcup_{i=1}^N F_i$ is again a contraction with a contraction factor $\alpha = \max_{i=1,2,\ldots N}\{\alpha_i\}$.*

According to Theorem 18 and the previous lemma, for any set $S \in H$, the sequence $F^n(S)$ converges to the closed and bounded set A_F, while n tends to the infinity, which is called the *attractor* for F. Hence, the attractor depends only on the space and the contractions comprising the IFS. See the following theorem, its entire proof could be found in [14].

Theorem 19 *Let $F_1, F_2, \ldots, F_N$ be a family of contractions on $\mathbb{R}^k$ then there is a unique attractor $A_F \in H$ for the union map $F = \bigcup_{i=1}^{N} F_i$. Moreover, for any $B \in H$ the sequence $F^n(B)$ converges to A_F in the Hausdorff metric while n tends to the infinity.*

This attractor A_F is an invariant set since $F(A_F) = A_F$. Moreover it follows that

$$A_F = F(A_F) = F_1(A_F) \cup F_2(A_F) \cup \cdots \cup F_N(A_F).$$

This means that the attractor consists of its contracted copies, hence it is self-similar.

Definition 17 *If a set S is an attractor of an iterated function system then it is called* self-similar.

The definition of self-similarity is not unique, further conditions are often assumed, for example any two tiles $F_i(A_F)$, $F_j(A_F)$ of the attractor are supposed not to have a significant overlap or all the contractions need to be similitudes. These two additional conditions are necessary for the calculation of the similarity dimension of attractors (see [21], [15]). It is obvious that the attractors in our examples are generated by similitudes and their tiles do not overlap significantly. Hence their similarity dimension may be calculated easily.

Now, we return to examples of IFSs.

IFS for the Koch curve

We will derive the contractions comprising the IFS related to the Koch curve from the initiator and generator. The initiator is the unit closed interval $[0, 1]$ and the generator consists of four segments K_i each of length $1/3$ so we have to define four contractions F_i respectively.

$\boxed{F_1}$ We obtain K_1 scaling the initiator by a factor 1/3, hence

$$F_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$\boxed{F_2}$ We obtain K_2 contracting the initiator by a factor 1/3 and rotating in a counterclockwise direction by an angle $\pi/3$ and then translating by a vector $(1/3, 0)^T$. Hence,

$$\begin{aligned} F_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1/3\cos(\pi/3) & -1/3\sin(\pi/3) \\ 1/3\sin(\pi/3) & 1/3\cos(\pi/3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \\ &\quad + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1/6 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}. \end{aligned}$$

$\boxed{F_3}$ We get K_3 contracting the initiator by a factor 1/3 and rotating in a clockwise direction by an angle $\pi/3$ and then translating by a vector $(1/2, \sqrt{3}/6)^T$. Hence,

$$\begin{aligned} F_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1/3\cos(-\pi/3) & -1/3\sin(-\pi/3) \\ 1/3\sin(-\pi/3) & 1/3\cos(-\pi/3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \\ &\quad + \begin{pmatrix} 1/2 \\ \sqrt{3}/6 \end{pmatrix} = \\ &= \begin{pmatrix} 1/6 & \sqrt{3}/6 \\ -\sqrt{3}/6 & 1/6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ \sqrt{3}/6 \end{pmatrix}. \end{aligned}$$

$\boxed{F_4}$ We get K_4 contracting the initiator by a factor 1/3 and translating by a vector $(2/3, 0)^T$. Hence,

$$F_4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.$$

Consequently, the Koch curve is obtained as a limit of $\mathrm{Orb}_F^+(S)$, where $S \in H$ and F is the union map related to the IFS $\{\mathbb{R}^2 : F_1, F_2, F_3, F_4\}$. Note that we can choose any closed bounded set $S \in H$ not only $S = I$ according to Theorem 19.

IFS for the Sierpinski gasket

The initiator is now a solid equilateral triangle Tr and the family of contractions is given by

$\boxed{F_1}$

$$F_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$\boxed{F_2}$

$$F_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

$\boxed{F_3}$

$$F_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix}.$$

Consequently, the IFS is $\{\mathbb{R}^2 : F_1, F_2, F_3\}$ and we get the Sierpinski gasket

$$G = \lim_{n\to\infty} F^n(S),\ S \in H.$$

4.3 The collage theorem

The following theorem was developed in [9] as a core of the patented IFS compression algorithm.

Theorem 20 (The collage theorem) *Let $\{S : F = \bigcup_{i=1}^{N} F_i\}$ be an iterated function system with contraction factors $\alpha_1, \alpha_2, \ldots, \alpha_N$ and*

$$\alpha = \max_{1 \leqslant i \leqslant N} \{\alpha_i\}$$

for which A_F is the attractor. If for any $\epsilon > 0$

$$d_H(S, \bigcup_{i=1}^{N} F_i(S)) < \epsilon,$$

then

$$d_H(S, A_F) < \frac{\epsilon}{1-\alpha}.$$

Proof: The proof employs Theorem 18. Let $F = \bigcup_{i=1}^{N} F_i$. Then $\lim_{n\to\infty} F^n(S) = A_F$. Hence

$$\begin{aligned} d_H(S, A_F) &= d_H(S, \lim_{n\to\infty} F^n(S)) = \lim_{n\to\infty} d_H(S, F^n(S)) \leqslant \\ &\leqslant \lim_{n\to\infty}(d_H(S, F(S)) + d_H(F(S), F^2(S)) + \\ &\quad + \cdots + d_H(F^{n-1}(S), F^n(S))) \leqslant \\ &\leqslant \lim_{n\to\infty} d_H(S, F(S)) \left(1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1}\right) \leqslant \frac{\epsilon}{1-\alpha}. \end{aligned}$$

□

4.4 Exercises

1. Calculatethe topological and fractal dimension of fractals generated by generators given in Figure 4.5.
2. Draw the third, fourth and fifth iterations of the fractals generated by generators given in Figure 4.5.
3. Find the IFS for initiators and generators given in Figure 4.5.

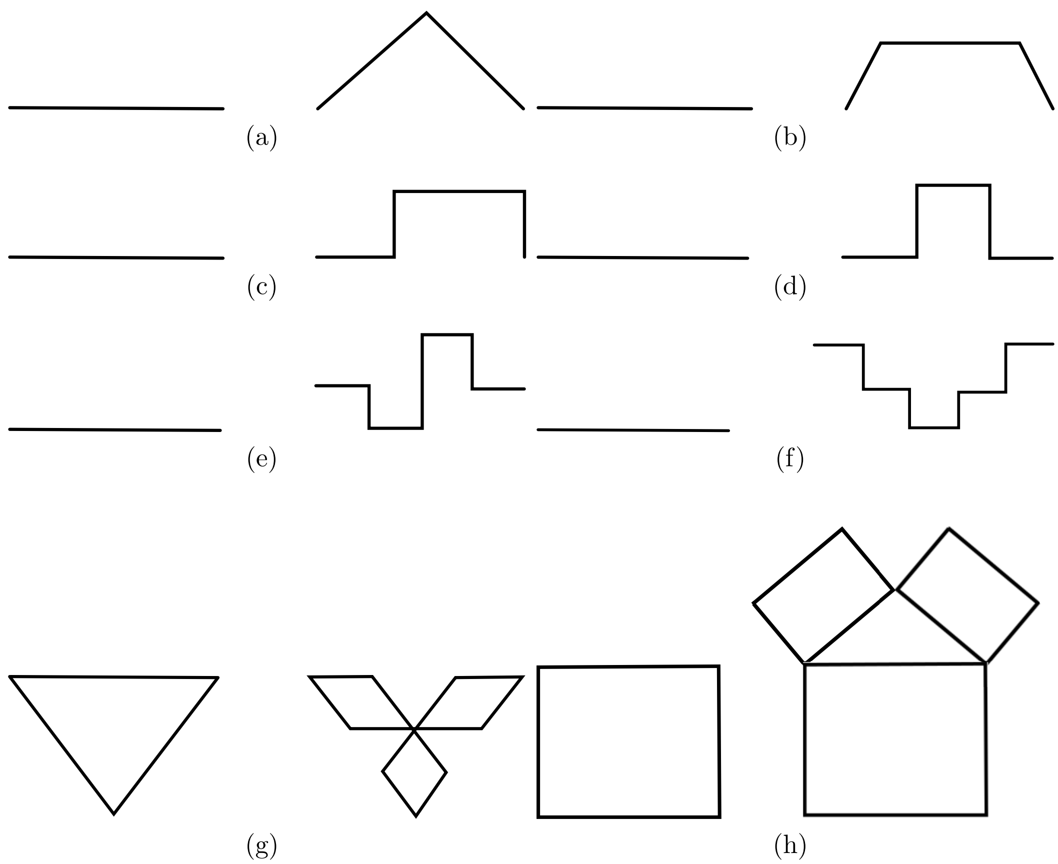


Figure 4.5: Initiators and generators for examples given in 4.4.

Chapter 5

Topological dynamics

5.1 Fixed point Property

Let (X, f) be a (discrete) dynamical system, that is X is a compact metric space and $f : X \to X$ is a continuous map. The space X have *the fixed point property* if for every continuous f the (X, f) has a fixed point (for more see e.g. [42]).

Theorem 21 (Brower) *Every compact convex space has a fixed point property.*

This theorem means that for given space, e.g. n- dimensional cube, any continuous map has at least one fixed point (fore details see e.g. [43]).

Let us pint out that both conditions from the Brower's fixed point theorem are essential and could not be excluded.

5.1.1 Period order

Let us consider the following ordering of natural numbers $\mathbb{N}$:

$$\begin{array}{ccccccccccccc}
3 & \prec & 5 & \prec & 7 & \prec & 9 & \prec & \dots & \prec & (2n+1)\cdot 2^0 & \prec & \dots \\
3\cdot 2 & \prec & 5\cdot 2 & \prec & 7\cdot 2 & \prec & 9\cdot 2 & \prec & \dots & \prec & (2n+1)\cdot 2^1 & \prec & \dots \\
3\cdot 2^2 & \prec & 5\cdot 2^2 & \prec & 7\cdot 2^2 & \prec & 9\cdot 2^2 & \prec & \dots & \prec & (2n+1)\cdot 2^2 & \prec & \dots \\
3\cdot 2^3 & \prec & 5\cdot 2^3 & \prec & 7\cdot 2^3 & \prec & 9\cdot 2^3 & \prec & \dots & \prec & (2n+1)\cdot 2^3 & \prec & \dots \\
 & \vdots & & & & & & & & & & & \\
\dots & \prec & 2^n & \prec & \dots & \prec & 2^3 & \prec & 2^2 & \prec & 2 & \prec & 1
\end{array}$$

It consists of:

- the odd numbers in increasing order,
- 2 times the odds in increasing order,
- 4 times the odds in increasing order,
- 8 times the odds, etc.,
- at the end we put the powers of two in decreasing order.

Theorem 22 (Sharkovskii [36]) *Let $(\mathbb{R}, f)$ be a discrete dynamical system. If there is a periodic point for f with the period n, then there is a periodic point of the map f with the period m, for every $m \in \mathbb{N}$ such that $n \prec m$.*

It is worthy to note that the it is not possible to extend the above stated theorem on general compact metric spaces. Even in one-dimensional case counter examples can be easily constructed, e.g. rotation R_λ on the unit circle $\mathbb{S}^1$ for $\lambda = 1/3$ has only three cycles. Fortunately, the statement of Theorem 22 can be preserved having additional assumption (and also in other space types):

Theorem 23 (Block [3]) *Let f be a continuous map on the circle $\mathbb{S}^1$. Suppose f has a fixed point and a periodic point of period $n > 1$. Then (at least) one of the following holds.*

(i) *For every integer m with $n < m$, there is periodic point of f with period m.*

(ii) *For every integer m with $n \prec m$, there is periodic point of f with period m.*

The period structure of a dynamical system is not "self-standing", it has connection to its chaos behavior:

Theorem 24 (Li-Yorke [26]) *Any continuous map on the interval I having point of period three is chaotic in the sense of Li and Yorke.*

5.1.2 Period doubling

Let us consider the quadratic family $F_\mu(x) = \mu x(1-x)$ on the unit closed interval $[0,1]$. It was pointed out that for $\mu \in (0,3]$ the map F_μ has only one fixed points. It is possible to verify that for $\mu \in (3, 1+\sqrt{6}]$ the map F_μ has a two cycle (two points with period 2). It was proved that for the quadratic family as μ increases new periods are added to the list of periods appearing and never disappear once they have occurred, see e.g. [33].

Let μ_n be the infimum of the parameter values $\mu > 0$ for which F_μ has a point of period 2^n. By the Sharkovskii theorem $\mu_n \leqslant \mu_{n+1}$. Let μ_∞ be the limiting value of the μ_n as n tends to the infinity. This sequence of bifurcations is often called the *period doubling route to choas*.

A natural question is to ask is the rate of convergence of the parameter μ_n to μ_∞. In general, we want to define a quantity which measures the geometric rate of convergence to the limiting value. Feigenbaum (1978) calculated the rate of convergence by means of the limit

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}.$$

This value δ is called *Feigenbaum constant*.

The Feigenbaum constant of the quadratic family equals to

$$\delta = 3.569945672\ldots.$$

5.2 Topological dynamics

The main aim of this chapter is focussed on topological dynamics of discrete dynamical systems. Namely, omega limit sets, recurrence, minimality and transitivity.

5.2.1 Omega limit sets

Let (X, f) be a discrete dynamical systems, recall that it is assumed that X is a compact metric space and f is continuous. Now we can define *omega limit set of x under f*

$$\omega_f(x) = \{y \in X : \exists n_i \nearrow \infty \wedge f^{n_i}(x) \to y\}$$

or equivalently

$$\omega_f(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) : k > n\}}.$$

Let us denote by $C(X)$ the set of all continuous maps on X, for simplicity.

Lemma 25 *Let $f \in C(X)$ and $x \in X$ then*

1. $\omega_f(x) \neq \emptyset$,
2. $\omega_f(x)$ *is a closed set,*
3. $\omega_f(x) = f(\omega_f(x))$.

Proof:

1. It directly follows form the fact that X is a compact space.
2. Let us denote y_k arbitrary sequence from $\omega_f(x)$ such that $y_k \to y \in X$ for $k \geqslant 1$. The aim is to show that $y \in \omega_f(x)$. For every $j \geqslant 1$ let us pick k_j such that $d(y_{k_j}, y) < 1/2j$. Let us pick n_j such that $d(f^{n_j}(x), y_{k_j}) < 1/2j$ and $n_j < n_{j+1}$. Then $d(f^{n_j}(x), y) < 1/j$ and $y \in \omega_f(x)$.
3. Obviously $\omega_f(x) \supset f(\omega_f(x))$. On contrary, let us assume that $y \in \omega_f(x)$ and $f^{n_i}(x) \to y$. Then $\{f^{n_i - 1}(x)\}$ has a convergent subsequence, hence $f^{n_{i_j} - 1}(x) \to z \in X$. Then $f^{n_{i_j}}(x) \to f(z)$ a $f(z) = y$. Consequently $z \in \omega_f(x)$ and $\omega_f(x) = f(\omega_f(x))$.

$\square$

Theorem 25 ([37]) *Every countable omega limit set contains a cycle.*

Proof: Let W_0 be a countable omega limit set. W_0 contains an isolated point x_1. If there will be no such point in W_0 then W_0 will be perfect, hence uncountable. Let us put $W_1 = \omega_f(x_1)$, obviously $W_1 \subset W_0$. So we can define a transfinite sequence W_β. Then $\bigcap_{\alpha<\beta} W_\beta$ contains a cycle. □

Example 20 *The omega limit sets could be finite, infinite countable and also uncountable, in general.*

- *The first case corresponds to a cycle.*

- *Let us construct x_1, resp. x_2 from Σ_2 such that $\omega_\sigma(x_1)$ is finite and $\omega_\sigma(x_2)$ is countable infinit. As x_1 one can pick arbitrary periodic sequence with period k. Its omega limit set is finite and contains exactly k points. If we put $x_1 = \overline{101}$ the*

  $$\omega_\sigma(x_1) = \{\overline{101}, \overline{011}, \overline{110}\}.$$

  Now, let us construct a point whose omega limit set is countably infinite. Let us put

  $$x_2 = 10\ 100\ 1000\ 10000\ 100000\ldots.$$

  It follows that

  $$\omega_\sigma(x_2) = \{\overline{0}\} \cup \{\underbrace{00\ldots0}_{k}1\overline{0}\ |k = 0, 1, 2, \ldots\}.$$

- *For an example of uncountable omega limit set it suffices to pick irrational rotation.*

Theorem 26 *Let $f \in C(X)$ be topologically conjugated with $g \in C(Y)$ by a conjugacy h. Then $h(\omega_f(x)) = \omega_g(h(x))$.*

Proof: Let us denote be A' the set of all accumulation points of the set A. Then $\omega_g(h(x)) = \omega_{h\circ f\circ h^{-1}}(h(x)) = (\{(h \circ f \circ h^{-1})^n(h(x))\}_{n=0}^\infty)' = (\{(h \circ f)^n(x)\}_{n=0}^\infty)' = h(\{(f)^n(x)\}_{n=0}^\infty)' = h(\omega_f(x))$. □

5.2.2 Recurrence and minimality

The point $x \in X$ is *recurrent* for the map f if for every open neighborhood U of the point x there is $n \geqslant 1$ such that $f^n(x) \in U$. The set of all recurrent points of f is denoted by $\mathrm{Rec}(f)$.

Let us note that it is possible to define the recurrent point, equivalently, analogously to the definition of omega limit set:

$$\exists n_k \nearrow \infty : f^{n_k}(x) \to x \tag{5.1}$$

Theorem 27 *Let $f \in C(X)$ then:*

1. $x \in \mathrm{Rec}(f) \Longleftrightarrow x \in \omega_f(x)$,
2. $\mathrm{Rec}(f) \neq \varnothing$.

Proof: 1. The assertion follows directly form the definitions of recurrent point and omega limit set.

2. Let $\mathcal{F}$ be a systems of all nonempty closed invariant sets such that $Y \subset X$. This system $\mathcal{F}$ is obviously nonempty, and is ordered by the ordering "$\subset$". Now, by Zorn's lemma the system $\mathcal{F}$ has a minimal element Y_0. We show that the point $x \in Y_0$ is recurrent. Since Y_0 and invariant under f, it follows $\overline{\mathrm{Orb}_f^+(x)} \subset Y_0$. The set $\overline{\mathrm{Orb}_f^+(x)}$ is also closed and invariant. From the minimality of Y_0 it follows $Y_0 = Y$. That means that for every neighborhood of the point x contains some $f^n(x)$ for $n \geqslant 1$. Consequently $\mathrm{Rec}(f) \neq \varnothing$. (It is possible to prove this theorem without using Zorn's lemma, see e.g. [16].)

□

Example 21 *The easiest example or recurrent points are periodic ones. As nontrivial example one can think about points from Σ_2 that have dense orbit. The construction of nontrivial recurrent points will be given in 23.*

Definition 18 *The set $M \subset X$ is called minimal under the map f if it is*

1. *nonempty and closed,*
2. *invariant and*
3. *has no proper subset with the previous two properties.*

Lemma 26 *The nonempty set M is minimal if and only if $\omega_f(x) = M$ for every $x \in M$.*

Proof: If M is minimal for every $x \in M$ then $\omega_f(x)$ is nonempty, closed, and invariant (see Lemma 25). So $M = \omega_f(x)$.

On the other hand, if $\omega_f(x) = M$ for every $x \in M$, then M is nonempty, closed, and invariant (see Lemma 25). Let us assume that there is nonempty closed and invariant set $N \subset M$. If $y \in N$ then $M = \omega_f(y) \subseteq N$ and $M = N$. □

Theorem 28 *Let f be a homeomorphism on X. Then the following statements are equivalent:*

1. *X is minimal under f,*
2. *the only closed sets $E \subset X$ with the property $f(E) = E$ are $\emptyset$ and X,*
3. *for every nonempty open set $U \subset X$it holds*

$$\bigcup_{n=-\infty}^{\infty} f^n(U) = X.$$

Proof: Let us prove the following implications:

1. $\Rightarrow$ 2. Let us assume that X is minimal under f, $E \neq \emptyset$ is closed and $f(E) = E$. If $x \in E$ then $\mathrm{Orb}_f^+(x) \subset E$ and $X = \overline{\mathrm{Orb}_f^+(x)} \subset E$. So, $X = E$.

2. $\Rightarrow$ 3. If U is nonempty and closed, then $E = X \setminus \bigcup_{n=-\infty}^{\infty} f^n(U)$ is closed and $f(E) = E$. Since $E \neq X$it follows $E = \emptyset$.

3. $\Rightarrow$ 1. Let $x \in X$ and U be a nonempty and closed subset of X. Then there is $n \in \mathbb{Z}$ such that $x \in f^n(U)$. Then $f^{-n}(x) \in U$ and $\overline{\mathrm{Orb}_f^+(x)} = X$. □

Theorem 29 *For every homeomorphism there is a minimal set.*

Proof: Let $\mathcal{F}$ be a system of all nonempty closed sets $Y \subset X$. The systems $\mathcal{F}$ is obviously nonempty ($X \in \mathcal{F}$), let us order this family by the ordering "$\subset$". By Zorn's lemma $\mathcal{F}$ has a minimal element Y_0, which is nonempty, closed and invariant set Y_0 that is minimal. □

The above given proof uses the Axion of choice that can cause some unpleasant reactions of some mathematicians, hence an alternative proof is desirable without using equivalence of the Zorn's lemma. Such proof can be found, e.g., in [39].

It is worthy to note, that each minimal set is finite (corresponds to the cycle) or uncountable. Hence, there is no countable infinite minimal set, see Theorem 25.

Definition 19 *The point $x \in X$ is uniformly recurrent under the map f if $x \in \omega_f(x)$ and $\omega_f(x)$ is minimal. The set of all uniformly recurrent points of f is denoted by* $\mathrm{UR}(f)$.

Example 22 *The sequence $s^\star \in \Sigma_2$ constructed in the section Symbolic dynamics, with dense orbit, is obviously recurrent but not uniformly recurrent since its omega limit set is not minimal.*

Example 23 *Let us construct uniformly recurrent point in Σ_2. Let us firstly decompose the set of all positive integer into infinite subsets of infinite pairwise disjoint sets:*

$$\mathcal{N} = \{N_n = 2^n(1 + 2\mathbb{N}_0), n \in \mathbb{N}_0\}.$$

Now, pick $A \subset \Sigma_2$ the set of all sequences containing infinitely many ones and infinitely many zeros. This set A is obviously uncountable. Now, let us define a map that makes "concatenations" of each point from A in such a way to have uniformly recurrent point. So, $\varphi : A \to \Sigma_2$ define by: $\varphi(x) = y_1y_2y_3\dots$ where $y_k = x_s$ if $k \in N_s$. Hence, $\varphi(x) = x_1x_2x_1x_3x_1x_2x_1x_4x_1x_2x_1x_3x_1\dots$. The point $\varphi(x)$ is uniformly recurrent every its block is eventually periodic. Moreover, $\omega_\sigma(\varphi(x))$ is uncountable set.

The map φ is injective, so the set $\varphi(A)$ is also uncountable and every its point is uniformly recurrent with uncountable omega limit set.

The first property of the following theorem directly follows form the definition of the uniformly recurrent point and the second one can be shown analogously as in the case of recurrent point.

Theorem 30 *Let $f \in C(X)$ then:*

1. *if M is minimal under f then every point from M is uniformly recurrent,*
2. $\mathrm{UR}(f) \neq \emptyset$.

5.2.3 Transitivity

Definition 20 *The map $f \in C(X)$ is (onesided) transitive, if there is $x \in X$ such that $\overline{\mathrm{Orb}_f^+(x)} = X$. Moreover, if f is homeomorphism, we call f as (twosided) transitive, if there is $x \in X$ such that $\overline{\mathrm{Orb}_f(x)} = X$.*

Theorem 31 *Let $f \in C(X)$ and $f(X) = X$. Ten the following are equivalent:*

1. *$f \in C(X)$is (onesided) transitive,*
2. *if E is closed subset of X and $E \subset f^{-1}(E)$, then $E = X$ or E nowhere dense set,*
3. *for every nonempty sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$,*
4. *the set of all x such that $\overline{\mathrm{Orb}_f^+(x)} = X$, is dense and G_δ.*

Proof: Let us prove the following implications:

1. $\Rightarrow$ 2. Let us assume that $\overline{\mathrm{Orb}_f^+(x_0)} = X$ and $E \subset f^{-1}(E)$, where $E \subset X$ is nonempty closed set. Let us alos assume that $U \subset E$ is open set. Then there is $p \in \mathbb{N}$ such that $f^p(x_0) \in U$ and $\{f^p(x_0), n > p\} \subset E$. Then $\{x_0, f(x_0), \ldots f^{p-1}(x_0)\} \cap E = X$ and $f(\{x_0, f(x_0), \ldots f^{p-1}(x_0)\} \cap E) = f(X)$. So $\{f(x_0), \ldots f^{p-1}(x_0)\} \cap E = X$. After $p-2$ iterations one gets $E = X$.

2. $\Rightarrow$ 3. If $U, V \subset X$ are nonempty and closed sets then $\bigcup_{n=1}^{\infty} f^{-n}(U) \subset X$ jis open. Then $f(\bigcup_{n=1}^{\infty} f^{-n}(U)) \subset \bigcup_{n=1}^{\infty} f^{-n}(U)$ and by the second assumption the set $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X. Hence $f^n(U) \cap V \neq \varnothing$.

3. $\Rightarrow$ 4. Let use denote $\{U_n\}_{n=1}^{\infty}$ base of the topology on X. Then the set of all points whose orbit is dense in X can be written as $\bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} f^{-m}(U_n)$. Then by the third assumption $\bigcap_{m=0}^{\infty} f^{-m}(U_n)$ is dense and G_δ.

4. $\Rightarrow$ 1. Is trivial. □

Definition 21 *Let $f \in C(X)$, then the point x is wandering, if there is open neighborhood U of the point x such that for $n \geqslant 0$ the sets $f^{-n}(U)$ are pairwise disjoint. A point x is nonwandering, if it is not wandering. The set of all non-vangering points of f is denoted by $\Omega(f)$ (that is $\Omega(f) = \{x \in X : \forall\,\text{open set } U \text{ of the point } x\ \exists n \geqslant 1 : f^{-n}(U) \cap U \neq \varnothing\}$).*

Theorem 32 *For $f \in C(X)$ it holds:*

1. *$\Omega(f)$ is closed,*

2. *$\bigcup_{x\in X}\omega_f(x) \subset \Omega(f)$, ($\Omega(f) \neq \varnothing$),*

3. *$\mathrm{Per}(f) \subset \Omega(f)$,*

4. *$f(\Omega(f)) \subset \Omega(f)$, moreover if f is homeomorphism then $f(\Omega(f)) = \Omega(f)$,*

5. *if E is minimal for f, then $E \subset \Omega(f)$.*

Proof: 1. It follows from the definition of $\Omega(f)$ that $X\backslash\Omega(f)$ is open set.

2. Let $x \in X$ and $y \in \omega_f(x)$. We are going to show, that $y \in \Omega(f)$. Let V be a neighborhood of y. We would like to find $n \geqslant 1$ such that $f^{-n}(V) \cap V \neq \varnothing$, we are looking for $n \geqslant 1$ and $z \in V$ such that $f^n(z) \in V$. We know that $f^{n_i}(x) \to y$ for some sequence $\{n_i\}$. Let us pick $n_{i_0} < n_{i_1}$ such that $f^{n_0}(x) \in V$ and $f^{n_1}(x) \in V$. Finally, put $n = n_{i_1} - n_{i_0}$ and $z = f^{n_0}(x)$.

3. If $f^n(x) = x$, $n > 0$, and U is a neighborhood of x, then $x \in f^{-n}(U) \cap U$.

4. Let $x \in \Omega(f)$ and V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is the neighborhood of the point x, se there is $n > 0$ such that $f^{-(n+1)}(V) \cap f^{-1}(V) \neq \varnothing$. Hence, $f^{-n}(V) \cap V \neq \varnothing$ and $f(x) \in \Omega(f)$. Moreover, f is homeomorphism, so $\Omega(f) = \Omega(f^{-1})$and hence $f^{-1}\Omega(f) \subset \Omega(f)$.

5. Follows directly from (2).

□

Definition 22 *For $f \in C(X)$ we say that:*

1. *f is transitive, if for every open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$,*

2. *f is totaly transitive if for every $n \in \mathbb{N}$ the map f^n is transitive,*

3. *f is weakly mixing, if $f \times f : X \times X \to X \times X$ is transitive,*

4. *f is mixing if for every nonempty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that for every $N > n$ it holds $f^N(U) \cap V \neq \varnothing$.*

Let us point out that transitivity is equivalent with onesided transitivity, see Theorem 31.

Note, that the property of weakly mixing is possible to define analogously to the transitivity, that is f is *weakly mixing* if for every three nonempty open sets $U, V, W \subset X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap W \neq \varnothing$ and $f^n(V) \cap W \neq \varnothing$. So it obviously follows:

Theorem 33 *For $f \in C(X)$ the following holds:*
f is mixing $\Rightarrow$ f is weakly mixing $\Rightarrow$ f is totally transitive $\Rightarrow$ f is transitive.

Remark 9 *To the end, let us note that the above mentioned dynamical properties are also observable on invariant subsets of the state space. More precisely it is possible to find portions of a given dynamical system on which the property holds but is not working on its union.*

For example, the family of logistic maps $f_\mu : [0,1] \to [0,1]$*, where* $0 \leqslant \mu \leqslant 4$*, by*

$$f_\mu(x) = \mu x(1-x)$$

can be demonstrated. The interval $I_\mu = [f_\mu^2(1/2), f_\mu(1/2)]$ is called the core *of f_μ, when $\mu \in (2,4]$, see Figure 5.1. For the choice of parameter $\mu \in [0,2]$ the interval I_μ is still well defined but does not have such nice properties. Namely, $I_2 = \{1/2\}$, $I_0 = \{0\}$ and for $\mu \in (0,2)$ it is not invariant under f_μ, hence not remarkable in these cases. The core I_μ is strongly invariant, that is $f_\mu(I_\mu) = I_\mu$, and every point from $(0,1)$ is attracted to I_μ. The dynamics on the core can be very rich. For example, in [8] the authors show that for the family of tent maps the dynamics on the core is topologically exact for some range of parameters, which, generally speaking, means that most rich dynamical behavior is present in the core. In the case of logistic maps, the calculations are much harder and spectrum of possible dynamical behaviors is richer. However, it is known that for some parameters the dynamics on the core of logistic map is the same (in the sense of topological conjugacy) as on the core of tent map with slope corresponding to μ (e.g. see [11]).*

5.2.4 Exercieses

Prove the following sttements:

a) The map $f \in C(x)$ is transitive iff there is $x \in X$ with dense orbit.

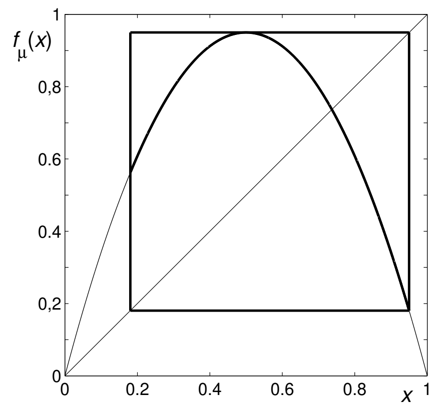


Figure 5.1: Graph of f_μ for $\mu = 3.8$ and the graph restricted to the core I_μ (bounded by box).

b) The map f is called *bitransitive* if f^2 is transitive. If $f \in C(I)$ is bitransitive, then f^n is transitive for every $n \in \mathbb{N}$. Find a countra example for the opposite implication.

c) The set of all periodic points of transitive map $f \in C(I)$ is dense in I.

d) Prove Theorem 33 and find countra examples to given implications.

e) The Tent map T is transitive.

Chapter 6

Simulations of dynamical properties

This chapter is devoted to the Matlab® commands that are used for illustrating of dynamical properties. The scripts and functions listed in this chapter are very easy and they are written without using stronger tools like Simulink®. For better results of simulations it will be useful to use some stronger tools like Simulink® and it is left to the reader to improve given source codes and discuss their flanks. Given Listings could be found in a different forms on the Internet and it is also possible to rewrite them into Mathematica® or Maple®.

6.1 Elementary tools

Time series

Time series could be simulated in several ways in Matlab®, here we use the `for` loop. In Listing 6.1 there are commands of the 1.17 model that was given in Example 4. In the following Listing 6.2 there are provided commands for a script of the influenza model that was introduced in Example 5.

Iterations and Cobweb pot

For simulations of iterations at a point and cobweb plot we use a function `handle` in Matlab® that provides a means of calling a function indirectly. In Listing 6.3 there are given commands for iterations where `iterates` function is defined, this function has three parameters. If we pick `fcn = @(x) 4*x.*(1-x)` as a function that will iterate our initial point `x0 = 0.1` and

Listing 6.1: Iteration of the point

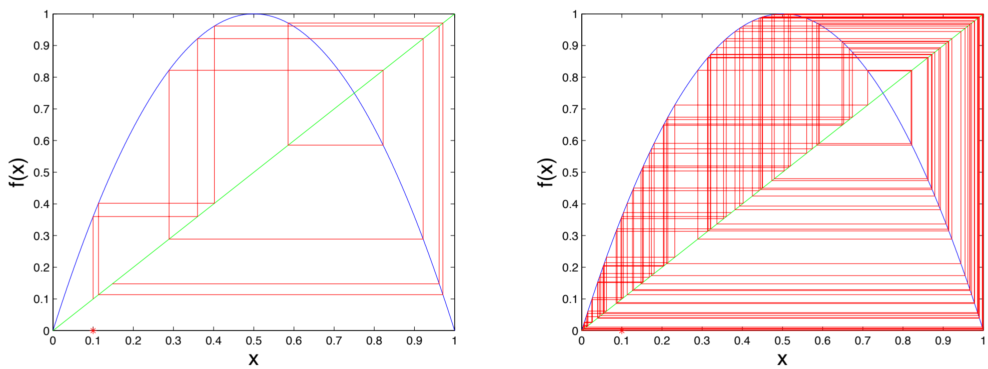
```
N=55; % the number of iterates
format long e;

X=zeros(N,1);
Y=zeros(N,1);
t=zeros(N,1);

X(1)=1.1; % x-coordinate of the initial point
Y(1)=1.9; % y-coordinate of the initial point

for i=1:N-1;

  X(i+1)=X(i)*(4-X(i)-Y(i));
  Y(i+1)=X(i)*Y(i);

  t(i+1)=i;
end


plot(t(1:end),X(1:end),t(1:end),Y(1:end),'r');
xlabel('Time [n]','FontSize',18);
ylabel('Predator and Prey [Q_n, P_n]','FontSize',18);
legend('Q_n','P_n');
```

Listing 6.2: Iteration of the point

```
NN=50; % the number of iterates

I=zeros(NN,1);
R=zeros(NN,1);
NC=zeros(NN,1);

t=zeros(NN,1);

format long e;

N=300000; % initial parameters of the model
I(1)=0.02*N;
R(1)=0;
NC(1)=0;

for i=1:NN-1;

  nc=(1/(N)).*I(i)*(N-I(i)-R(i));
  I(i+1)=I(i)-1/7*I(i)+nc;
  R(i+1)=R(i)+1/7*I(i);
  NC(i+1)=(1/(N)).*I(i)*(N-I(i)-R(i));

  t(i+1)=i;
end

plot(t(1:end),I(1:end),...
  ...t(1:end),R(1:end),'r',t(1:end),NC(1:end),'g');
xlabel('Time [n]','FontSize',18);
ylabel('I_n, R_n, NC_n','FontSize',18);
legend('I_n','R_n','NC_n');
```

we iterate it `N = 10` times, we get a vector 0.1, 0.36, 0.9216, 0.28901376, 0.82193922612265, 0.585420538734197, 0.970813326249438, 0.113339247303761, 0.401973849297512, 0.961563495113813, 0.147836559913285.

Now, in Listing 6.4 there are commands for a cobweb plot, the function `cobweb` is defined. This commands use previously defined `iterates` function and extra parameters `xmin` and `xmax` that define range of the plot. We put additionally `xmin = 0` and `xmax = 1` for previously given parameters and we get cobweb plots given in Figure 6.1.



(a) (b)

Figure 6.1: Cobweb plots of the map $f(x) = 4x(1-4)$ at a point 0.1. Case (a) corresponds to $N = 10$ and case (b) corresponds to $N = 100$.

6.2 Chaos control

Lyapunov exponent

In Listing 6.5 there are commands needed for calculating Lyapunov exponents defined in section 3.3 of the Logistic family introduced in section 2.3. Graph of dependence of the Lyapunov exponent on parameter μ is given in Figure 6.2.

Bifurcation diagrams

For complete understanding of the chaotic behavior the bifurcation diagram of the Logistic family is given. Algorithm is evident from Listing 6.6 and the situation is depicted in Figure 6.2.

Listing 6.3: Iteration of the point

```
function Y=iterates(fcn,x0,N)

%%
% fcn is the name of the function,
% x0 is the starting point,
% N is the number of iterates.
%%

Y=[x0];
x=x0;
for i=1:N
    y=feval(fcn,x);
    Y=[Y y];
    x=y;
end;
```

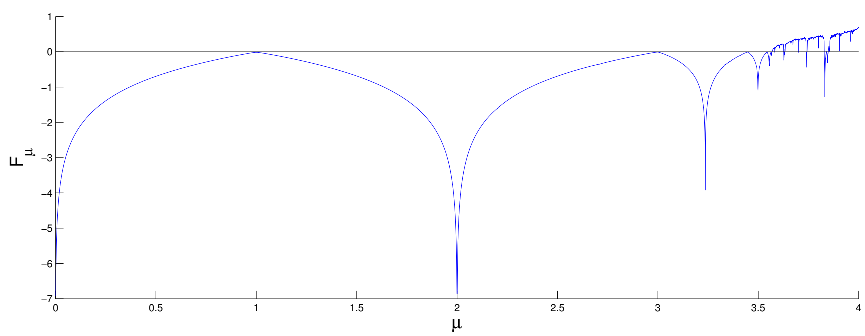


Figure 6.2: Lyapunov exponents of the Logistic family.

6.3 Fractals

Koch curve

The Listing 6.7 was downloaded from [13] and could be used for the performance of the Koch curve, see Figure 4.1.

Koch snowflake The next Listing 6.8 was downloaded from [28] and is used for the Koch snowflake, the mathematical background here needs complex analysis, see Figure 4.2.

Listing 6.4: Cobweb plot

```
function cobweb(fcn,x0,N,xmin,xmax)

%%
% fcn is the name of the function,
% x0 is the initial point
% N is the number of iterates,
% xmin and xmax give the range of x-values to be plotted.
%%

dx=(xmax-xmin)/1000;
x=xmin:dx:xmax;
y=feval(fcn,x);

plot(x,y,'b',[xmin xmax],[0 0],'k',...
...[0 0],[min(y)-.1*abs(min(y)) max(y)],'k',...
...[xmin xmax],[xmin xmax],'g');
xlabel('x','FontSize',18);
ylabel('f(x)','FontSize',18);
hold on
Y=iterates(fcn,x0,N);
YY(1)=Y(1);
for i=1:N
  XX(2*i-1)=Y(i);
      XX(2*i)=Y(i);
      YY(2*i)=Y(i+1);
      YY(2*i+1)=Y(i+1);
end;
XX(2*N+1)=Y(N+1);
plot(XX,YY,'r',x0,0,'r*');
```

Listing 6.5: Lyapunov exponents for the Logistic family

```
clear all;
format long e;
itermax=499;
LE=[];

for mu=0.001:0.001:4
x=0.1;
x0=x;
for n=1:itermax
  xn=mu*x0*(1-x0);
  x=[x xn];
  x0=xn;
end

LExp=sum(log(abs(mu*(1-2*x))))/itermax;
LE=[LE, LExp];
end
```

Listing 6.6: Bifurcation diagram for the Logistic family

```
itermax=250;
finlits=75;
finits=itermax-(finlits-1);

for mu=0:0.001:4
x=0.4;
x0=x;
for n=2:itermax
xn=mu*x0*(1-x0);
x=[x xn];
x0=xn;
end
plot(mu*ones(finlits),x(finits:itermax))
hold on
end
```

Listing 6.7: Koch curve

```
function []=koch(n)

%%
% KOCH: Plots 'Koch Curve' Fractal koch(n) plots the 'Koch Curve' Frac-
% tal after n iterations e.g koch(4) plots the Koch Curve after 4 ite-
% rations. (be patient for n>8, depending on Computer speed) The 'kline'
% local function generates the Koch Curve co-ordinates using recursive
% calls, while the 'plotline' local fnc is used to plot the Koch Curve.
% Copyright (c) 2000 by Salman Durrani (dsalman@wol.net.pk)
%%

if (n==0)
   x=[0;1];
   y=[0;0];
   line(x,y,'Color','b');
   axis equal
   set(gca,'Visible','off')
else
   levelcontrol=10^n;
   L=levelcontrol/(3^n);
   l=ceil(L);
   kline(0,0,levelcontrol,0,l);
   axis equal
   set(gca,'Visible','off')
   set(gcf,'Name','Koch Curve')
end
%------------------------------------------------------------------
function kline(x1,y1,x5,y5,limit)
length=sqrt((x5-x1)^2+(y5-y1)^2);
if(length>limit)
   x2=(2*x1+x5)/3;
   y2=(2*y1+y5)/3;
   x3=(x1+x5)/2-(y5-y1)/(2.0*sqrt(3.0));
   y3=(y1+y5)/2+(x5-x1)/(2.0*sqrt(3.0));
   x4=(2*x5+x1)/3;
   y4=(2*y5+y1)/3;
   % recursive calls
   kline(x1,y1,x2,y2,limit);
   kline(x2,y2,x3,y3,limit);
   kline(x3,y3,x4,y4,limit);
   kline(x4,y4,x5,y5,limit);
else
   plotline(x1,y1,x5,y5);
end
%------------------------------------------------------------------
function plotline(a1,b1,a2,b2)
x=[a1;a2];
y=[b1;b2];
line(x,y);
%------------------------------------------------------------------
```

Listing 6.8: Koch snowflake

```
function z = snowflake(n,a)

%%
%SNOWFLAKE Koch Snowflake Curve
%Z = SNOWFLAKE(N,A) is a closed curve in the complex plane
%with 3*2^N+1 points. N is a nonnegative integer and A is a
%complex number with |A| < 1 and |1-A| < 1.
%Default is A = 1/2 + i*sqrt(3)/6.
% % Examples
%plot(snowflake(10)), axis equal
%plot(snowflake(10,0.45+0.35i)), axis equal
%Author: Jonas Lundgren <splinefit@gmail.com> 2010
%%

if nargin < 1, n = 0; end
if nargin < 2, a = 1/2 + sqrt(-3)/6; end
% Constants
b = 1 - a;
c = 1/2 + sqrt(-3)/2;
d = 1 - c;
% Generate point sequence
z = 1;
for k = 1:n
    z = conj(z);
    z = [a*z; b*z+a];
end
% Close snowflake
z = [0; z; 1-c*z; 1-c-d*z];
```

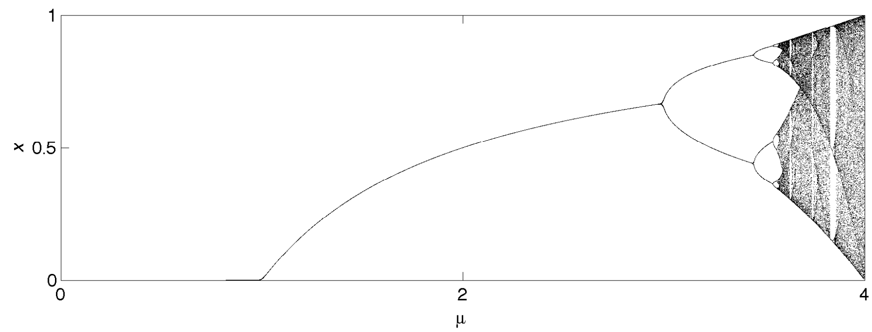


Figure 6.3: Bifurcation diagram of the Logistic family.

6.4 Exercises

1. Implement algorithms given in this section using programming tools (programming languages) you wish, e.g. Maple, Python or Lisp.
2. Implement the Sierpinski gasket. Use the Pascal triangle analogy to assemble the structure.

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