Approximation and numerical realization of 3D quasistatic contact problems with Coulomb friction.

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Abstract

This paper deals with the full discretization of quasistatic 3D Signorini problems with local Coulomb friction and a coefficient of friction which may depend on the solution. After a time discretization we obtain a system of static contact problems with Coulomb friction. Each of these problems is solved by the T-FETI domain decomposition method used in auxiliary contact problems with Tresca friction. Numerical experiments show the efficiency of the proposed method.

Key words: contact problems, Coulomb friction, domain decomposition methods

1. Introduction

Contact problems represent a special branch of mechanics of solids which analyzes the behavior of loaded, deformable bodies being in a mutual contact. Due to non-penetration and friction conditions, the resulting problems are highly non-linear. Both, unilateral and friction phenomena depend on time. Therefore contact problems with friction should be defined as a dynamical

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process. If however applied forces vary slowly in time, inertia of the system can be neglected and one can use a quasistatic approximation. This, together with a dependence of friction coefficients on the solution, typically arises in geomechanics (modelling of a movement of tectonic plates, prediction of earthquakes). Applied forces are functions of time, the Coulomb law of friction uses the velocity of the tangential contact displacements and the coefficient of Coulomb friction itself may depend on the modulus of the previously mentioned velocities (see [6]). This paper extends results of [14] from two (2D) to three (3D) space dimensions. Let us note that this extension is not straightforward since the friction conditions are more involved in 3D.

Using a three–step finite–difference approximation of the time derivative we arrive at a sequence of static contact problems with (local) Coulomb friction whose solutions are defined by using a fixed point approach. Thus the method of successive approximations turns out to be a natural tool for the numerical solution. Each iterative step is represented by an auxiliary contact problem with a given slip bound (Tresca model of friction). Since its mathematical formulation is given by a variational inequality of the second kind (see [8, 11]), we obtain (after a finite element discretization) a convex, but non-smooth constrained minimization problem for a discrete total potential energy function. Using the method of Lagrange multipliers, the non-penetration conditions are released and the non-smooth frictional term is transformed into a smooth one. Eliminating the displacement field from this formulation we obtain a new formulation solely in terms of Lagrange multipliers which represent the discrete contact stresses. In this way we arrive at a minimization problem for a quadratic function subject to separated simple and quadratic inequality constraints. The numerical minimization may be performed by the algorithm proposed in [15] and analyzed in [16]. To increase the efficiency of the whole computational process we use a variant of the FETI domain decomposition method ([7]) which introduces (additionally to the original setting) also new Lagrange multipliers by means of which the solutions on the individual sub-domains are glued together. The resulting minimization problem is solved by the augmented Lagrangian method ([4]) in which the algorithm from [16] is used repeatedly. To solve static contact problems with Coulomb friction, the augmented Lagrangian method is combined together with the method of successive approximations in one inexact iterative loop.

Let us mention main benefits of the variant ([2]) of the FETI called the total FETI (T–FETI). Unlike to the original FETI the satisfaction of

the Dirichlet boundary conditions is enforced also by Lagrange multipliers. Therefore all subdomains can be treated as floating bodies with six rigid body modes. Thus the kernel spaces of the stiffness matrices may be identified directly without any computation and, moreover, the Moore-Penrose inverse is easily available. Since the global stiffness matrix exhibits the block diagonal structure, one can handle it in parallel. Finally, under additional assumptions on the used finite element partitions, the spectrum of all blocks and, consequently of the whole stiffness matrix, lies within an interval in $\mathbb{R}^1_+$ which does not depend on the mesh norms ([12]). It is well–known that convergence of conjugate gradient type methods depends on the spectrum of the Hessian ([5, 9, 16]). Therefore the number of iterations needed to get a solution with a given accuracy can be independent of the mesh norms, as well. This property is known as the *scalability* of the method. See [1, 3, 4] for more details on scalable algorithms.

The paper is organized as follows: Section 2 deals with the formulation of quasistatic contact problems with Coulomb friction and introduces their three–step time discretization. Next two sections present main ideas of the used methods in the continuous setting: Section 3 describes the fixed–point approach for static contact problems with Coulomb friction while Section 4 gives the T–FETI domain decomposition formulation of contact problems with a given slip bound. In Section 5 we analyze an algebraic representation of the discrete problems and mention optimization algorithms used in computations. Finally Section 6 shows results of numerical experiments.

2. Formulation of the problem

Let us consider an elastic body represented by a domain $\Omega \subset \mathbb{R}^3$ whose Lipschitz boundary $\partial\Omega$ is decomposed as follows: $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_p \cup \overline{\Gamma}_c$, where different boundary conditions will be prescribed. The body is subject to volume forces of density $\mathbf{F} := (F_i)_{i=1}^3$ and surface tractions of density $\mathbf{P} := (P_i)_{i=1}^3$ on Γ_p both acting in an interval $[0, T_0]$, $T_0 > 0$ given. On Γ_u the body is fixed, while along Γ_c is supported by a rigid foundation S. For the sake of simplicity of our presentation we shall suppose that S is represented by the half–space and there is no gap between Γ_c and S. On Γ_c the non–penetration and friction conditions will be prescribed. Friction will be involved into the mathematical model by using the local Coulomb law with a coefficient of friction depending on the solution. Our aim is to find an equilibrium state of Ω.

By a *classical solution* we mean a sufficiently smooth displacement field $\mathbf{u} : \Omega \times [0, T_0] \mapsto \mathbb{R}^3$ satisfying the following equations and boundary conditions:

(equilibrium equations)

$$\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad in\ \Omega \times (0, T_0)\,, \quad i = 1, 2, 3\ )^1 \tag{2.1}$$

(Hooke's law)

$$\left.\begin{aligned} \tau_{ij} &= c_{ijkl}\varepsilon_{kl}(\mathbf{u}), \quad i, j, k, l = 1, 2, 3 \\ \varepsilon_{kl}(\mathbf{u}) &= \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \end{aligned}\right\} \tag{2.2}$$

(compatibility of stresses on Γ_p)

$$\tau_{ij}\nu_j = P_i \quad on\ \Gamma_p \times (0, T_0)\,, \quad i = 1, 2, 3 \tag{2.3}$$

(zero displacements on Γ_u)

$$u_i = 0 \qquad on\ \Gamma_u \times (0, T_0)\,, \quad i = 1, 2, 3 \tag{2.4}$$

(unilateral conditions on Γ_c)

$$u_\nu \leq 0, \quad \tau_\nu(\mathbf{u}) \leq 0, \quad u_\nu \tau_\nu(\mathbf{u}) = 0 \quad on\ \Gamma_c \times (0, T_0) \tag{2.5}$$

(Coulomb's law of friction on Γ_c) $\quad (\forall x \in \Gamma_c)$:

$$\left.\begin{aligned} \dot{\mathbf{u}}_T(\mathbf{x}) = \mathbf{0} \quad &\Rightarrow \quad \|\boldsymbol{\tau}_T(\mathbf{x})\| \leq -\mathcal{F}(\mathbf{x}, 0)\tau_\nu(\mathbf{u}(\mathbf{x})) \\ \dot{\mathbf{u}}_T(\mathbf{x}) \neq \mathbf{0} \quad &\Rightarrow \quad \boldsymbol{\tau}_T(\mathbf{x}) \quad = \mathcal{F}(\mathbf{x}, \|\dot{\mathbf{u}}_T(\mathbf{x})\|)\tau_\nu(\mathbf{u}(\mathbf{x}))\frac{\dot{\mathbf{u}}_T(\mathbf{x})}{\|\dot{\mathbf{u}}_T(\mathbf{x})\|} \end{aligned}\right\} \tag{2.6}$$

(initial condition)

$$\mathbf{u}(0) = \mathbf{u}_0 \quad in\ \Omega\ . \tag{2.7}$$

We use classical notation of the linear elasticity: $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^3$ is a stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{i,j=1}^3$ a linearized strain tensor corresponding to a displacement vector $\mathbf{u}$, $c_{ijkl} \in L^\infty(\Omega)$, $i,j,k,l = 1,2,3$ are coefficients of a linear Hooke's law (2.2). We shall suppose that they satisfy the following symmetry and ellipticity conditions:

$$\left.\begin{aligned} c_{ijkl} = c_{jikl} = c_{klij} \qquad & a.e.\ in\ \Omega\,,\ i,j,k,l = 1,2,3 \\ \exists \alpha > 0:\ c_{ijkl}\xi_{ij}\xi_{kl} \geq \alpha \xi_{ij}\xi_{ij} \qquad & a.e.\ in\ \Omega\ \forall \xi_{ij} = \xi_{ji} \in \mathbb{R}^1\ . \end{aligned}\right\} \tag{2.8}$$

Further, $\boldsymbol{\nu} = (\nu_i)_{i=1}^3$ is the unit normal vector to $\partial\Omega$, $u_\nu := u_i\nu_i$, $\mathbf{u}_T := \mathbf{u} - u_\nu \boldsymbol{\nu}$ is the normal and tangential component of a displacement vector $\mathbf{u}$. Analogously, $\tau_\nu := \tau_{ij}\nu_i\nu_j$, $\boldsymbol{\tau}_T := \tau_{ij}\nu_j - \tau_\nu \boldsymbol{\nu}$ is the normal and tangential component of a stress vector $(\tau_{ij}\nu_j)_{i=1}^3$ on $\partial\Omega$. Finally, the symbol $\mathcal{F}$ in (2.6) stands for the coefficient of friction which depends on the spatial variable $\mathbf{x} \in \Gamma_c$ and on the euclidean norm $\|\dot{\mathbf{u}}_T\|$ of the velocity $\dot{\mathbf{u}}_T := \frac{\partial}{\partial t}\mathbf{u}_T$ on Γ_c.

To define the weak solution we introduce the following sets of functions:

$$\begin{aligned} V &= \{v \in H^1(\Omega) \mid v = 0 \quad on\ \Gamma_u\}\,, \quad \mathbb{V} = V \times V \times V \\ \mathbb{K} &= \{\mathbf{v} \in \mathbb{V} \mid v_\nu \leq 0 \quad a.e.\ on\ \Gamma_c\} \\ H^{1/2}(\Gamma_c) &= V_{|\Gamma_c} \quad (space\ of\ traces\ on\ \Gamma_c\ of\ functions\ from\ V) \\ H^{-1/2}(\Gamma_c) &= (H^{1/2}(\Gamma_c))' \quad (the\ dual\ of\ H^{1/2}(\Gamma_c)) \\ H^{-1/2}_-(\Gamma_c) & \quad (cone\ of\ all\ non-positive\ elements\ of\ H^{-1/2}(\Gamma_c)). \end{aligned}$$

Next we shall suppose that Γ_c is sufficiently smooth so that $v_\nu \in H^{1/2}(\Gamma_c)$ for every $\mathbf{v} \in \mathbb{V}$.

By the *weak formulation* of the quasistatic contact problem with local Coulomb law of friction and the coefficient of friction $\mathcal{F}$ which depends on the solution we mean the following problem formulated in terms of the displacement vector $\mathbf{u}$ and the normal contact stress λ_ν:

$$\left.\begin{aligned} & Find\ \mathbf{u} \in W^{1,2}(0,T_0,\mathbb{V}),\ \lambda_\nu \in W^{1,2}(0,T_0,H^{-1/2}(\Gamma_c))\ such\ that \\ & \mathbf{u}(t) \in \mathbb{K} \quad for\ a.a.\ t \in [0,T_0], \quad \mathbf{u}(0) = \mathbf{u}_0 \quad in\ \Omega \\ & a(\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t)) + j(\lambda_\nu(t), \mathbf{v}) - j(\lambda_\nu(t), \dot{\mathbf{u}}(t)) \geq L(t)(\mathbf{v} - \dot{\mathbf{u}}(t)) + \\ & \qquad + \langle \lambda_\nu(t), v_\nu - \dot{u}_\nu(t)\rangle \quad \forall \mathbf{v} \in \mathbb{V}\ and\ a.a.\ t \in [0,T_0] \\ & \langle \lambda_\nu(t), z_\nu - u_\nu(t)\rangle \geq 0 \qquad \forall \mathbf{z} \in \mathbb{K}\ and\ a.a.\ t \in [0,T_0]\ , \end{aligned}\right\} \tag{$\mathcal{P}$}$$

[1] here and in what follows the summation convention is used.

where

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega c_{ijkl}\varepsilon_{ij}(\mathbf{u})\varepsilon_{kl}(\mathbf{v})\ d\mathbf{x}\ ,$$

$$L(t)(\mathbf{v}) := \int_\Omega F_i(t)v_i\ d\mathbf{x} + \int_{\Gamma_p} P_i(t)v_i\ ds$$

$$j(\lambda_\nu,\mathbf{v}) := -\langle \mathcal{F}(\|\dot{\mathbf{u}}_T\|)\lambda_\nu, \|\mathbf{v}_T\|\rangle$$

and $\langle\ ,\ \rangle$ is the duality pairing between $H^{-1/2}(\Gamma_c)$ and $H^{1/2}(\Gamma_c)$. Using Green's formula with an appropriate choice of the test functions $\mathbf{v}$ one can show that the classical and weak formulations are formally equivalent. In addition, λ_ν is equal to the normal contact stress on Γ_c.

We shall suppose that $\mathbf{F} \in W^{1,2}(0,T_0,(L^2(\Omega))^3)$, $\mathbf{P} \in W^{1,2}(0,T_0,(L^2(\Gamma_p))^3)$ and the initial displacement $\mathbf{u}_0 \in \mathbb{K}$ satisfies the compability condition

$$a(\mathbf{u}_0,\mathbf{v}-\mathbf{u}_0) + j(\lambda_{0\nu},\mathbf{v}-\mathbf{u}_0) \geq L(0)(\mathbf{v}-\mathbf{u}_0) \quad \forall \mathbf{v} \in \mathbb{K}\ ,$$

where $\lambda_{0\nu} = \tau_\nu(\mathbf{u}_0)_{|\Gamma_c}$. The mathematical analysis of $(\mathcal{P})$ in the case when $\mathcal{F} := \mathcal{F}(\mathbf{x})$ depends *only* on the spatial variable $\mathbf{x}$ has been already done in [17]. The more general case of $\mathcal{F} := \mathcal{F}(\mathbf{x},\|\dot{\mathbf{u}}_T\|)$ is analyzed in [6].

Now we replace problem $(\mathcal{P})$ by a sequence of static problems arising from its time discretization. Let $\Delta t = T_0/n$, $n \in \mathbb{N}$, be a time step, $t_i = i\Delta t$, $i = 0,\dots,n$ and denote $\mathbf{u}^i := \mathbf{u}(t_i)$, $\mathbf{F}^i := \mathbf{F}(t_i)$, etc. Assume that $(\mathbf{u}^{i+1},\lambda_\nu^{i+1}) \in \mathbb{K}\times H^{-1/2}(\Gamma_c)$ is a solution of $(\mathcal{P})$ at $t = t_{i+1}$:

$$\left.\begin{aligned} &a(\mathbf{u}^{i+1},\mathbf{v}-\dot{\mathbf{u}}^{i+1}) - \langle \mathcal{F}(\|\dot{\mathbf{u}}_T^{i+1}\|)\lambda_\nu^{i+1},\|\mathbf{v}_T\|\rangle + \langle \mathcal{F}(\|\dot{\mathbf{u}}_T^{i+1}\|)\lambda_\nu^{i+1},\|\dot{\mathbf{u}}_T^{i+1}\|\rangle \\ &\qquad \geq L^{i+1}(\mathbf{v}-\dot{\mathbf{u}}^{i+1}) + \langle \lambda_\nu^{i+1}, v_\nu - \dot{u}_\nu^{i+1}\rangle \qquad \forall \mathbf{v} \in \mathbb{V} \\ &\langle \lambda_\nu^{i+1}, z_\nu - u_\nu^{i+1}\rangle \geq 0 \qquad \forall \mathbf{z} \in \mathbb{K}\ , \end{aligned}\right\} (\mathcal{P})_{i+1}$$

where $\dot{\mathbf{u}}^{i+1} := \frac{\partial}{\partial t}\mathbf{u}_{|t=t_{i+1}}$ using also the definition of the frictional term j.

Next we replace $\dot{\mathbf{u}}^{i+1}$ by a three–step finite difference:

$$\dot{\mathbf{u}}^{i+1} \approx \frac{\Delta^{i+1}\mathbf{u}}{\Delta t}\ , \tag{2.9}$$

where

$$\Delta^{i+1}\mathbf{u} := \alpha\mathbf{u}^{i+1} + \beta\mathbf{u}^i + \gamma\mathbf{u}^{i-1}\,, \qquad \alpha,\beta,\gamma \in \mathbb{R}^1\,, \alpha > 0\ .$$

Inserting (2.9) into the first inequality in $(\mathcal{P})_{i+1}$ we get ($\forall \mathbf{v} \in \mathbb{V}$):

$$
\begin{aligned}
& a(\mathbf{u}^{i+1}, \mathbf{v} - \frac{\alpha \mathbf{u}^{i+1} + \beta \mathbf{u}^i + \gamma \mathbf{u}^{i-1}}{\Delta t}) - \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{v}_T\| \rangle \\
& \quad + \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \frac{\|\alpha \mathbf{u}_T^{i+1} + \beta \mathbf{u}_T^i + \gamma \mathbf{u}_T^{i-1}\|}{\Delta t} \rangle \\
& \quad \geq L^{i+1}(\mathbf{v} - \frac{\alpha \mathbf{u}^{i+1} + \beta \mathbf{u}^i + \gamma \mathbf{u}^{i-1}}{\Delta t}) + \langle \lambda_\nu^{i+1}, v_\nu - \frac{\alpha u_\nu^{i+1} + \beta u_\nu^i + \gamma u_\nu^{i-1}}{\Delta t} \rangle \,.
\end{aligned}
$$

Multiplying this inequality by Δt and dividing by α we obtain:

$$
\begin{aligned}
& a(\mathbf{u}^{i+1}, \frac{\Delta t}{\alpha}\mathbf{v} - \mathbf{u}^{i+1} - \frac{\beta}{\alpha}\mathbf{u}^i - \frac{\gamma}{\alpha}\mathbf{u}^{i-1}) - \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \frac{\Delta t}{\alpha}\|\mathbf{v}_T\| \rangle \\
& \quad + \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{u}_T^{i+1} + \frac{\beta}{\alpha}\mathbf{u}_T^i + \frac{\gamma}{\alpha}\mathbf{u}_T^{i-1}\| \rangle \\
& \geq L^{i+1}(\frac{\Delta t}{\alpha}\mathbf{v} - \mathbf{u}^{i+1} - \frac{\beta}{\alpha}\mathbf{u}^i - \frac{\gamma}{\alpha}\mathbf{u}^{i-1}) \\
& \quad + \langle \lambda_\nu^{i+1}, \frac{\Delta t}{\alpha}v_\nu - u_\nu^{i+1} - \frac{\beta}{\alpha}u_\nu^i - \frac{\gamma}{\alpha}u_\nu^{i-1} \rangle \quad \forall \mathbf{v} \in \mathbb{V} \,.
\end{aligned}
$$

Setting $\mathbf{w} := \frac{\Delta t}{\alpha}\mathbf{v} - \frac{\beta}{\alpha}\mathbf{u}^i - \frac{\gamma}{\alpha}\mathbf{u}^{i-1} \in \mathbb{V}$ we arrive at

$$
\left.
\begin{aligned}
& a(\mathbf{u}^{i+1}, \mathbf{w} - \mathbf{u}^{i+1}) - \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{w}_T + \frac{\beta}{\alpha}\mathbf{u}_T^i + \frac{\gamma}{\alpha}\mathbf{u}_T^{i-1}\| \rangle \\
& \quad + \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{u}_T^{i+1} + \frac{\beta}{\alpha}\mathbf{u}_T^i + \frac{\gamma}{\alpha}\mathbf{u}_T^{i-1}\| \rangle \\
& \quad \geq L^{i+1}(\mathbf{w} - \mathbf{u}^{i+1}) + \langle \lambda_\nu^{i+1}, w_\nu - u_\nu^{i+1} \rangle \quad \forall \mathbf{w} \in \mathbb{V} \,.
\end{aligned}
\right\} \tag{2.10}
$$

If we restrict ourselves to test functions $\mathbf{w} \in \mathbb{K}$ in (2.10), the last term on the right of (2.10) is non–negative since $\lambda_\nu^{i+1} \in H_-^{-1/2}(\Gamma_c)$ in virtue of the second inequality in $(\mathcal{P})_{i+1}$. Thus $\mathbf{u}^{i+1} \in \mathbb{K}$ solves the following implicit variational inequality:

$$
\left.
\begin{aligned}
& a(\mathbf{u}^{i+1}, \mathbf{w} - \mathbf{u}^{i+1}) - \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{w}_T + \frac{\beta}{\alpha}\mathbf{u}_T^i + \frac{\gamma}{\alpha}\mathbf{u}_T^{i-1}\| \rangle \\
& \quad + \langle \mathcal{F}(\frac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\lambda_\nu^{i+1}, \|\mathbf{u}_T^{i+1} + \frac{\beta}{\alpha}\mathbf{u}_T^i + \frac{\gamma}{\alpha}\mathbf{u}_T^{i-1}\| \rangle \\
& \quad \geq L^{i+1}(\mathbf{w} - \mathbf{u}^{i+1}) \qquad \forall \mathbf{w} \in \mathbb{K} \,,
\end{aligned}
\right\} \tag{$\widetilde{\mathcal{Q}}$}_{i+1}
$$

which is the weak form of the state contact problem with Coulomb friction and the coefficient of friction $\mathcal{F}$ depending on the solution at the time $t = t_{i+1}$.

In particular, the friction conditions hidden in $(\widetilde{\mathcal{Q}})_{i+1}$ read for all $\mathbf{x} \in \Gamma_c$ as follows:

$$\left.\begin{array}{ll} \Delta^{i+1}\mathbf{u}_T(\mathbf{x}) = \mathbf{0} \;\Rightarrow & \|\boldsymbol{\tau}_T^{i+1}(\mathbf{x})\| \leq -\mathcal{F}(\mathbf{x},0)\tau_\nu^{i+1}(\mathbf{u}(\mathbf{x})) \\ \Delta^{i+1}\mathbf{u}_T(\mathbf{x}) \neq \mathbf{0} \;\Rightarrow & \boldsymbol{\tau}_T^{i+1}(\mathbf{x}) = \\ & \mathcal{F}(\mathbf{x}, \dfrac{\|\Delta^{i+1}\mathbf{u}_T\|}{\Delta t})\tau_\nu^{i+1}(\mathbf{u}(\mathbf{x}))\dfrac{\Delta^{i+1}\mathbf{u}_T}{\|\Delta^{i+1}\mathbf{u}_T\|}(\mathbf{x}) \end{array}\right\} \tag{2.11}$$

To make numerical relization of $(\widetilde{\mathcal{Q}})_{i+1}$ easier we replace the unknown function $\|\Delta^{i+1}\mathbf{u}_T\|$ in the argument of $\mathcal{F}$ by the known value $\|\Delta^i\mathbf{u}_T\|$ and denote $\mathcal{F}^i := \mathcal{F}(\|\Delta\mathbf{u}_T^i\|/\Delta t)$. The resulting problem involves the Coulomb friction law with the coefficient which does not depend on the solution on the current time level $t = t_{i+1}$.

The coefficients defining the finite diference (2.9) used in computations are $\alpha = 3/2$, $\beta = -2$ and $\gamma = 1/2$.

With such a choice of α, β and γ the final form of the problem we shall solve at each time level reads as follows:

$$\left.\begin{array}{l} \textit{Find } \mathbf{u}^{i+1} \in \mathbb{K} \textit{ such that} \quad \forall \mathbf{w} \in \mathbb{K} \\ a(\mathbf{u}^{i+1}, \mathbf{w} - \mathbf{u}^{i+1}) - \langle \mathcal{F}^i\lambda_\nu^{i+1}, \|\mathbf{w}_T - \frac{4}{3}\mathbf{u}_T^i + \frac{1}{3}\mathbf{u}_T^{i-1}\|\rangle \\ \quad + \langle \mathcal{F}^i\lambda_\nu^{i+1}, \|\mathbf{u}_T^{i+1} - \frac{4}{3}\mathbf{u}_T^i + \frac{1}{3}\mathbf{u}_T^{i-1}\|\rangle \geq L^{i+1}(\mathbf{w} - \mathbf{u}^{i+1})\,, \end{array}\right\} \tag{$(\mathcal{Q})_{i+1}$}$$

where $\lambda_\nu^{i+1} = \tau_\nu^{i+1}(\mathbf{u})_{|\Gamma_c}$.

3. Static contact problems with local Coulomb friction

In this section we shall describe how static contact problems $(\mathcal{Q})_{i+1}$ will be solved. To simplify our notation the index i denoting the time level will be skipped and we set $\mathbf{z} := \frac{4}{3}\mathbf{u}^i - \frac{1}{3}\mathbf{u}^{i-1}$. The problem we have to solve now reads as follows:

$$\left.\begin{array}{l} \textit{Find } \mathbf{u} \in \mathbb{K} \textit{ such that} \\ a(\mathbf{u}, \mathbf{w} - \mathbf{u}) + j(\lambda_\nu, \mathbf{w} - \mathbf{z}) - j(\lambda_\nu, \mathbf{u} - \mathbf{z}) \geq L(\mathbf{w} - \mathbf{u}) \quad \forall \mathbf{w} \in \mathbb{K}\ , \end{array}\right\} \tag{$\mathcal{Q}$}$$

where $\lambda_\nu = \tau_\nu(\mathbf{u})_{|\Gamma_c}$ and

$$j(\lambda_\nu, \mathbf{v}) = -\langle \mathcal{F}\lambda_\nu, \|\mathbf{v}_T\|\rangle\ . \tag{3.12}$$

Recall that due to our approach, the coefficient of friction $\mathcal{F}$ in $(\mathcal{Q})$ now does not depend on the solution $\mathbf{u}$ from actual timestep, but only on the

known solution from the previous ones. The unilateral constraint $\mathbf{u} \in \mathbb{K}$ can be released by means of Lagrange multipliers. We obtain the following equivalent formulation of $(\mathcal{Q})$:

$$\left.\begin{array}{l} \textit{Find } \mathbf{u} \in \mathbb{V} \textit{ and } \lambda_\nu \in H_-^{-1/2}(\Gamma_c) \textit{ such that} \\ a(\mathbf{u}, \mathbf{w}-\mathbf{u}) + j(\lambda_\nu, \mathbf{w}-\mathbf{z}) - j(\lambda_\nu, \mathbf{u}-\mathbf{z}) \geq \\ \qquad L(\mathbf{w}-\mathbf{u}) + \langle \lambda_\nu, w_\nu - u_\nu \rangle \quad \forall \mathbf{w} \in \mathbb{V} \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle \geq 0 \qquad \forall \mu_\nu \in H_-^{-1/2}(\Gamma_c) \end{array}\right\} \tag{3.13}$$

Problem (3.13) has an alternative formulation based on a *fixed–point* approach which turned out to be efficient from the numerical point of view. For given $g \in H_-^{-1/2}(\Gamma_c)$ defining the slip bound we consider the variational inequality:

$$\left.\begin{array}{l} \textit{Find } \mathbf{u} := \mathbf{u}(g) \in \mathbb{V} \textit{ and } \lambda_\nu := \lambda_\nu(g) \in H_-^{-1/2}(\Gamma_c) \textit{ such that} \\ a(\mathbf{u}, \mathbf{w}-\mathbf{u}) + j(g, \mathbf{w}-\mathbf{z}) - j(g, \mathbf{u}-\mathbf{z}) \geq \\ \qquad L(\mathbf{w}-\mathbf{u}) + \langle \lambda_\nu, w_\nu - u_\nu \rangle \qquad \forall \mathbf{w} \in \mathbb{V} \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle \geq 0 \qquad \forall \mu_\nu \in H_-^{-1/2}(\Gamma_c) \end{array}\right\} \tag{3.14}$$

This problem has a unique solution $(\mathbf{u}, \lambda_\nu)$ for any $g \in H_-^{-1/2}(\Gamma_c)$. The first component $\mathbf{u}(g)$ solves the contact problem with the Tresca model of friction in which the slip bound $-\mathcal{F}g$ is given a–priori and $\lambda_\nu = \tau_\nu(\mathbf{u}(g))_{|\Gamma_c}$ is the normal contact stress. This makes it possible to define a mapping $\Phi : H_-^{-1/2}(\Gamma_c) \mapsto H_-^{-1/2}(\Gamma_c)$ by

$$\Phi(g) = \lambda_\nu(g) \qquad \forall g \in H_-^{-1/2}(\Gamma_c)\,, \tag{3.15}$$

where $\lambda_\nu(g)$ is the second component of the solution to (3.14). If we compare (3.13) with (3.14) we see that $(\mathbf{u}, \lambda_\nu)$ solves (3.13) if and only if $\lambda_\nu \in H_-^{-1/2}(\Gamma_c)$ is a fixed–point of Φ, i.e.

$$\Phi(\lambda_\nu) = \lambda_\nu \tag{3.16}$$

A natural idea how to find λ_ν satisfying (3.15) is to use the method of *successive approximations*:

$$\left.\begin{array}{l} \lambda_\nu^{(0)} \in H_-^{-1/2}(\Gamma_c) \textit{ given;} \\ \textit{if } \lambda_\nu^{(k)} \in H_-^{-1/2}(\Gamma_c) \textit{ is known} \\ \textit{solve (3.14) with } g := \lambda_\nu^{(k)} \textit{ and} \\ \textit{set } \lambda_\nu^{(k+1)} := \lambda_\nu(\lambda_\nu^{(k)})\,,\ k := k+1 \\ \textit{repeat until stopping criterion .} \end{array}\right\} \tag{3.17}$$

It is worth noting that convergence of this method in the continuous setting of the static problem is not guaranteed since Φ *is not* a contractive mapping. The situation is rather different in the discrete case. After a suitable discretization of $\mathbb{V}$ and $H_-^{-1/2}(\Gamma_c)$ one can define a discrete version Φ_h of Φ whose fixed–points are solutions of discrete contact problems with Coulomb friction. It is known that Φ_h *is* already contractive provided that the coefficient of friction $\mathcal{F}$ is small enough but this property is mesh-dependent ([10]). In our computations we use the method of successive approximations (3.17). Since the efficiency of (3.17) depends mainly on how each iterative step can be realized we focus on this subject in the next section.

4. T–FETI for contact problems with given friction

The aim of this section is to present the efficient way of numerical realization of 3D contact problems with given friction which will be based on its mixed formulation combined with the T–FETI domain decomposition method. Recall that the solution to contact problems with given friction is equivalent to the following minimization problem:

$$\left.\begin{array}{l} \textit{Find } \mathbf{u} := \mathbf{u}(g) \in \mathbb{K} \textit{ such that} \\ J(\mathbf{u}) \leq J(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbb{K} \ , \end{array}\right\} \tag{4.18}$$

where

$$J(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + j(g, \mathbf{v} - \mathbf{z}) \ ,$$

with the same a, L as in Section 3, j defined by (3.12) and $\mathbf{z} \in \mathbb{V}$ given. This is a *constrained* optimization problem for the *non–differentiable* total potential energy functional J. In order to release the unilateral constraint $\mathbf{u} \in \mathbb{K}$ we introduced the Lagrange multipliers in (3.14). In what follows we shall suppose that the slip bound $g \in L^2_-(\Gamma_c)$, i.e. g is a non–positive square integrable function. To transform our problem into a smooth we write the non–differentiable term j as

$$j(g, \mathbf{v} - \mathbf{z}) = \sup_{\boldsymbol{\mu}_T \in \Lambda_T(g)} \int_{\Gamma_c} g \boldsymbol{\mu}_T \cdot (\mathbf{v}_T - \mathbf{z}_T) \, ds \ ,$$

where

$$\Lambda_T(g) = \{ \boldsymbol{\mu}_T \in (L^2(\Gamma_c))^2 \mid \|\boldsymbol{\mu}_T\| \leq -\mathcal{F} g \textit{ a.e. on } \Gamma_c \} \ . \tag{4.19}$$

Remark 4.1. *In* (4.19) *we used the fact that the tangential component* $\mathbf{v}_T$ *on* Γ_c *can be interpreted as a vector in* $\mathbb{R}^2$ *using the local orthogonal coordinate system* $(\boldsymbol{\nu}(\mathbf{x}), \mathbf{T}_1(\mathbf{x}), \mathbf{T}_2(\mathbf{x}))$, $\mathbf{x} \in \Gamma_c$.

The minimization problem (4.18) can be written in the following form:

$$\min_{\mathbf{v}\in\mathbb{K}} J(\mathbf{v}) = \min_{\mathbf{v}\in\mathbb{V}} \sup_{\mu_\nu \in H_-^{-1/2}(\Gamma_c)} \sup_{\boldsymbol{\mu}_T \in \Lambda_T(g)} \mathcal{L}(\mathbf{v}, \mu_\nu, \boldsymbol{\mu}_T) \ , \tag{4.20}$$

where

$$\mathcal{L}(\mathbf{v}, \mu_\nu, \boldsymbol{\mu}_T) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) - \langle \lambda_\nu, v_\nu \rangle - (\boldsymbol{\mu}_T, \mathbf{u}_T - \mathbf{z}_T)_{0,\Gamma_c}$$

is the Lagrangian and $(\ ,\ )_{0,\Gamma_c}$ stands for the scalar product in $(L^2(\Gamma_c))^2$.

Let us denote by $(\mathbf{u}, \lambda_\nu, \boldsymbol{\lambda}_T)$ a saddle point of $\mathcal{L}$ on $\mathbb{V} \times H_-^{-1/2}(\Gamma_c) \times \Lambda_T(g)$, i.e.

$$\mathcal{L}(\mathbf{u}, \mu_\nu, \boldsymbol{\mu}_T) \le \mathcal{L}(\mathbf{u}, \lambda_\nu, \boldsymbol{\lambda}_T) \le \mathcal{L}(\mathbf{v}, \lambda_\nu, \boldsymbol{\lambda}_T) \tag{4.21}$$

holds for $\forall (\mathbf{v}, \mu_\nu, \boldsymbol{\mu}_T) \in \mathbb{V} \times H_-^{-1/2}(\Gamma_c) \times \Lambda_T(g)$. This is equivalent to the folowing problem:

$$\left.\begin{array}{l} \textit{Find } (\mathbf{u}, \lambda_\nu, \boldsymbol{\lambda}_T) \in \mathbb{V} \times H_-^{-1/2}(\Gamma_c) \times \Lambda_T(g) \textit{ such that} \\ a(\mathbf{u}, \mathbf{w}) = L(\mathbf{w}) + \langle \lambda_\nu, w_\nu \rangle + (\boldsymbol{\lambda}_T, \mathbf{w}_T)_{0,\Gamma_c} \quad \forall \mathbf{w} \in \mathbb{V} \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle \ge 0 \qquad \forall \mu_\nu \in H_-^{-1/2}(\Gamma_c) \\ (\boldsymbol{\mu}_T - \boldsymbol{\lambda}_T, \mathbf{u}_T)_{0,\Gamma_c} \ge (\boldsymbol{\mu}_T - \boldsymbol{\lambda}_T, \mathbf{z}_T)_{0,\Gamma_c} \qquad \forall \boldsymbol{\mu}_T \in \Lambda_T(g) \ . \end{array}\right\} \tag{4.22}$$

It is well–known (see [11]) that (4.21) has a unique solution. Its first component $\mathbf{u}$ is the solution of (4.18), $\lambda_\nu = \tau_\nu(\mathbf{u})_{|\Gamma_c}$, $\boldsymbol{\lambda}_T = \boldsymbol{\tau}_T(\mathbf{u})_{|\Gamma_c}$ is the normal and tangential contact stress, respectively.

To increase the efficiency of numerical realization of (4.21) we shall use in the next section the so–called T–FETI domain decomposition method [2] whose basic idea on the continuous level will be now presented.

Let $\{\Omega_i\}_{i=1}^s$ be a decomposition of $\overline{\Omega}$ into s non–overlapping subdomains which is compatible with the partition of $\partial\Omega$ into Γ_u, Γ_p and Γ_c. Denote by $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ a common interface between Ω_i and Ω_j, i.e. the two–dimensional Lebesgue measure $meas_2\, \Gamma_{ij} > 0$. Finally let

$$\mathcal{I} = \{(i,j) \mid 1 \le i < j \le s\,,\ meas_2\, \Gamma_{ij} > 0\}$$

$$\mathcal{D} = \{l \mid 1 \le l \le s\,,\ meas_2\, \Gamma_u \cap \partial\Omega_l > 0\}$$

be the sets identifying the common interfaces and parts of $\partial\Omega_l$ contained in Γ_u. On $\{\Omega_i\}_{i=1}^s$ we introduce the space

$$\mathbb{W} = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\, \mathbf{v}_i := \mathbf{v}_{|\Omega_i} \in (H^1(\Omega_i))^3,\ i = 1, \ldots, s\} \ .$$

It is easy to see that

$$\mathbf{v} \in \mathbb{V} \quad \textit{iff} \quad \mathbf{v} \in \mathbb{W} \ \wedge \ [\mathbf{v}]_{ij} = \mathbf{0} \ \forall (i,j) \in \mathcal{I} \ \wedge \ \mathbf{v}_{l|_{\Gamma_{ul}}} = \mathbf{0} \ \forall l \in \mathcal{D} \,, \tag{4.23}$$

where $[\mathbf{v}]_{ij} := (\mathbf{v}_i - \mathbf{v}_j)_{|\Gamma_{ij}}$, $(i,j) \in \mathcal{I}$, $\Gamma_{ul} := \Gamma_u \cap \partial\Omega_l$, $l \in \mathcal{D}$. To release the conditions on (4.23) we introduce the following sets of Lagrange multipliers:

$$\mathbb{Y}_{ij} := (H^1(\Omega_i))^3{}_{|\Gamma_{ij}} \ , \ (i,j) \in \mathcal{I}, \qquad \mathbb{Z}_l := (H^1(\Omega_l))^3{}_{|\Gamma_{ul}} \ , \ l \in \mathcal{D} \ ,$$

whose duals are denoted by $\mathbb{Y}'_{ij}$, and $\mathbb{Z}'_l$, respectively. Then

$$\left.\begin{array}{llll} \mathbf{v} \in \mathbb{V} \text{ iff } \quad \mathbf{v} \in \mathbb{W} & \wedge & \langle \boldsymbol{\mu}_{ij}, [\mathbf{v}]_{ij} \rangle = 0 & \forall \boldsymbol{\mu}_{ij} \in \mathbb{Y}'_{ij} \quad \forall (i,j) \in \mathcal{I} \\ & \wedge & \langle \boldsymbol{\mu}_l, \mathbf{v}_l \rangle = 0 & \forall \boldsymbol{\mu}_l \in \mathbb{Z}'_l \quad \forall l \in \mathcal{D} \ , \end{array}\right\}$$

where $\langle \ , \ \rangle$ denotes the duality pairings between $\mathbb{Y}'_{ij}$ and $\mathbb{Y}_{ij}$, $\mathbb{Z}'_l$ and $\mathbb{Z}_l$. Further let $\mathbb{Y}' = \prod_{(i,j)\in\mathcal{I}} \mathbb{Y}'_{ij}$, $\mathbb{Z}' = \prod_{l\in\mathcal{D}} \mathbb{Z}'_l$ be the cartesian products of $\mathbb{Y}'_{ij}$ and $\mathbb{Z}'_l$, respectively and define:

$$\langle \boldsymbol{\mu}_{int}, [\mathbf{v}] \rangle := \sum_{(i,j)\in\mathcal{I}} \langle \boldsymbol{\mu}_{ij}, [\mathbf{v}]_{ij} \rangle \,, \qquad \boldsymbol{\mu}_{ij} \in \mathbb{Y}'_{ij}$$

$$\langle \boldsymbol{\mu}_{Dir}, \mathbf{v} \rangle := \sum_{l\in\mathcal{D}} \langle \boldsymbol{\mu}_l, \mathbf{v}_l \rangle \,, \qquad \boldsymbol{\mu}_l \in \mathbb{Z}'_l \ .$$

Finally let $\mathcal{H}^{1/2}(\Gamma_c) := \mathbb{W}_{|\Gamma_c}$ be the trace space on Γ_c of functions from $\mathbb{W}$, $\mathcal{H}^{-1/2}(\Gamma_c)$ the dual of $\mathcal{H}^{1/2}(\Gamma_c)$ and $\mathcal{H}^{-1/2}_-(\Gamma_c)$ the cone of non–positive elements of $\mathcal{H}^{-1/2}(\Gamma_c)$. The respective duality pairing will be denoted again by $\langle \ , \ \rangle$. The T–FETI domain decomposition formulation of (4.21) reads as follows:

$$\left.\begin{array}{l} \textit{Find } (\mathbf{u}, \lambda_\nu, \boldsymbol{\lambda}_T, \boldsymbol{\lambda}_{int}, \boldsymbol{\lambda}_{Dir}) \in \mathbb{W} \times \mathcal{H}^{-1/2}_-(\Gamma_c) \times \Lambda_T(g) \times \mathbb{Y}' \times \mathbb{Z}' \ \textit{s.t.} \\ a(\mathbf{u}, \mathbf{w}) = L(\mathbf{w}) + \langle \lambda_\nu, w_\nu \rangle + (\boldsymbol{\lambda}_T, \mathbf{w}_T)_{0,\Gamma_c} + \langle \boldsymbol{\lambda}_{int}, [\mathbf{w}] \rangle \\ \qquad\qquad + \langle \boldsymbol{\lambda}_{Dir}, \mathbf{w} \rangle \qquad \forall \mathbf{w} \in \mathbb{W} \,, \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle + (\boldsymbol{\mu}_T - \boldsymbol{\lambda}_T, \mathbf{u}_T)_{0,\Gamma_c} \geq (\boldsymbol{\mu}_T - \boldsymbol{\lambda}_T, \mathbf{z}_T)_{0,\Gamma_c} \\ \qquad\qquad \forall \mu_\nu \in \mathcal{H}^{-1/2}_-(\Gamma_c) \ \forall \boldsymbol{\mu}_T \in \Lambda_T(g) \\ \langle \boldsymbol{\mu}_{int}, [\mathbf{u}] \rangle + \langle \boldsymbol{\mu}_{Dir}, \mathbf{u} \rangle = 0 \qquad \forall (\boldsymbol{\mu}_{int}, \boldsymbol{\mu}_{Dir}) \in \mathbb{Y}' \times \mathbb{Z}' \ . \end{array}\right\} \tag{4.24}$$

The elimination of $\mathbf{u}$ from (4.24) leads to the new formulation in terms of the Lagrange multipliers λ_ν, $\boldsymbol{\lambda}_T$, $\boldsymbol{\lambda}_{int}$ and $\boldsymbol{\lambda}_{Dir}$. This approach will be used in the next section for numerical realization of (4.24).

5. Algebraic T-FETI and algorithm

Let us suppose that the domains Ω_i, $i = 1, \dots, s$ are *polygonal* and denote by $\mathcal{T}_i$ a partition of $\overline{\Omega}_i$ into tetrahedrons. Next we suppose that $\mathcal{T}_{i|_{\Gamma_{ij}}} = \mathcal{T}_{j|_{\Gamma_{ij}}}$ for any $(i,j) \in \mathcal{I}$, i.e. the nodes $\mathcal{T}_i$ and $\mathcal{T}_j$ on the common interface Γ_{ij} coincide. Then $\mathcal{T} = \{\mathcal{T}_i\}_{i=1}^s$ defines the partition of $\overline{\Omega}$. With any $\mathcal{T}_i$ we associate the finite dimensional space $\mathbb{X}_i$, where

$$\mathbb{X}_i = \{\mathbf{v} \in (C(\overline{\Omega}_i))^3 \mid \mathbf{v}_{|_T} \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_i\}$$

and denote

$$\mathbb{X} = \prod_{i=1}^{s} \mathbb{X}_i \ .$$

The space $\mathbb{X}$ consists of all piecewise linear functions on $\mathcal{T}$ which are discontinuous on Γ_{ij}, $(i,j) \in \mathcal{I}$.

The spaces of the Lagrange multipliers used in (4.24) will be approximated by Dirac functions concentrated at some specific nodes of $\mathcal{T}$. To this end denote by

$$\left.\begin{array}{lll} \{\mathbf{x}_q^{ij}\}_{q=1}^{d_{ij}} & \dots \text{ nodes of } \mathcal{T}_i \text{ on } \overline{\Gamma}_{ij} \setminus \overline{\Gamma}_u \,, & (i,j) \in \mathcal{I} \\ \{\mathbf{y}_q^{l}\}_{q=1}^{d_{l}} & \dots \text{ nodes of } \mathcal{T}_l \text{ on } \overline{\Gamma}_{ul} \,, & l \in \mathcal{D} \\ \{\mathbf{z}_q^{k}\}_{q=1}^{d_{k}} & \dots \text{ nodes of } \mathcal{T}_k \text{ on } \overline{\Gamma}_{ck} \setminus \overline{\Gamma}_u \,, & k \in \mathcal{C} \ , \end{array}\right\} \tag{5.25}$$

where $\Gamma_{ck} := \Gamma_c \cap \partial\Omega_k$, d_{ij}, d_l, d_k denote appropriate dimensions, and

$$\mathcal{C} = \{k \mid 1 \le k \le s \,,\ meas_2\, \Gamma_{ck} > 0\} \ .$$

Dirac functions at $\{\mathbf{x}_q^{ij}\}_{q=1}^{d_{ij}}$, $\{\mathbf{y}_q^{l}\}_{q=1}^{d_l}$ and $\{\mathbf{z}_q^{k}\}_{q=1}^{d_k}$ will ensure the continuity across Γ_{ij}, the zero boundary condition on Γ_u and the satisfaction of the non–penetration and friction conditions on Γ_c, respectively. Let us concentrate now on the Dirac functions at $\{\mathbf{z}_q^{k}\}_{q=1}^{d_k}$, $k \in \mathcal{C}$. To simplify our notation denote by $\mathbf{w}$ one of theese nodes. Then the Lagrange multipliers acting at $\mathbf{w}$ are of the form $\boldsymbol{\nu}(\mathbf{w})\delta_{\mathbf{w}}$, $\mathbf{T}_1(\mathbf{w})\delta_{\mathbf{w}}$ and $\mathbf{T}_2(\mathbf{w})\delta_{\mathbf{w}}$, where $\mathbf{T}_1$ and $\mathbf{T}_2$ are the same as in Remark 4.1. If $\mathbf{v} \in \mathbb{X}$ then

$$\begin{aligned} [\boldsymbol{\nu}(\mathbf{w})\delta_{\mathbf{w}}, \mathbf{v}] &:= [\delta_{\mathbf{w}}, \mathbf{v} \cdot \boldsymbol{\nu}] = v_\nu(\mathbf{w}) \\ [\mathbf{T}_1(\mathbf{w})\delta_{\mathbf{w}} + \mathbf{T}_2(\mathbf{w})\delta_{\mathbf{w}}, \mathbf{v}] &:= [\delta_{\mathbf{w}}, \mathbf{v}_T] = \mathbf{v}_T(\mathbf{w}) \ . \end{aligned}$$

The spaces $\mathcal{H}_-^{-1/2}(\Gamma_c)$ and $\Lambda_T(g)$ in (4.24) are now discretized by

$$\mathbb{\Lambda}_\nu = \{\mu_\nu |\, \mu_\nu = \sum_{k\in\mathcal{C}} \sum_{q=1}^{d_k} \mu_q^k \boldsymbol{\nu}(\mathbf{z}_q^k)\delta_{\mathbf{z}_q^k}\,,\ \mu_q^k \le 0\}$$

$$\mathbb{\Lambda}_T(g) = \{\boldsymbol{\mu}_T |\, \boldsymbol{\mu}_T = \sum_{k\in\mathcal{C}} \sum_{q=1}^{d_k} (\mu_{q1}^k \mathbf{T}_1(\mathbf{z}_q^k) + \mu_{q2}^k \mathbf{T}_2(\mathbf{z}_q^k))\delta_{\mathbf{z}_q^k}\,,$$
$$||(\mu_{q1}^k, \mu_{q2}^k)||_2 \le (-\mathcal{F}g)(\mathbf{z}_q^k) \qquad \forall \mathbf{z}_q^k\}\ ,$$

where $||\ ||_2$ stands for the Euclidean norm of vectors. If $\mu_\nu \in \mathbb{\Lambda}_\nu$, $\boldsymbol{\mu}_T \in \mathbb{\Lambda}_T(g)$ and $\mathbf{v} \in \mathbb{X}$ are given then

$$[\mu_\nu, \mathbf{v}] := \sum_{k\in\mathcal{C}} \sum_{q=1}^{d_k} \mu_q^k v_\nu(\mathbf{z}_q^k)$$
$$[\boldsymbol{\mu}_T, \mathbf{v}_T] := \sum_{k\in\mathcal{C}} \sum_{q=1}^{d_k} \boldsymbol{\mu}_q^k \cdot \mathbf{v}_T(\mathbf{z}_q^k)\,, \quad \boldsymbol{\mu}_q^k = (\mu_{q1}^k, \mu_{q2}^k) \in \mathbb{R}^2\ .$$

Denote by $\mathbf{e}_i$, $i = 1, 2, 3$ the canonical basis of $\mathbb{R}^3$. The spaces $\mathbb{Y}'$ and $\mathbb{Z}'$ will be now discretized as follows :

$$\mathbb{\Lambda}_{int} = \{\boldsymbol{\mu}_{int} \mid \boldsymbol{\mu}_{int} = \sum_{(i,j)\in\mathcal{I}} \sum_{q=1}^{d_{ij}} \sum_{p=1}^{3} \mu_{qp}^{ij} \mathbf{e}_p (\delta_{\mathbf{x}_q^{ij}} - \delta_{\mathbf{x}_q^{ji}})\,, \quad \mu_{qp}^{ij} \in \mathbb{R}^1\}$$
$$\mathbb{\Lambda}_{Dir} = \{\boldsymbol{\mu}_{Dir} \mid \boldsymbol{\mu}_{Dir} = \sum_{l\in\mathcal{D}} \sum_{q=1}^{d_l} \sum_{p=1}^{3} \mu_{qp}^{l} \mathbf{e}_p \delta_{\mathbf{y}_q^l}\,, \quad \mu_{qp}^{l} \in \mathbb{R}^1\}\ ,$$

respectively. If $\boldsymbol{\mu}_{int} \in \mathbb{\Lambda}_{int}$ and $\mathbf{v} \in \mathbb{X}$ then

$$[\boldsymbol{\mu}_{int}, \mathbf{v}] := \sum_{(i,j)\in\mathcal{I}} \sum_{q=1}^{d_{ij}} \sum_{p=1}^{3} \mu_{qp}^{ij} [v_p(\mathbf{x}_q^{ij})]$$

where $[v_p(\mathbf{x}_q^{ij})]$ denotes the jump of the p-th component of $\mathbf{v}$ at $\mathbf{x}_q^{ij}$. Similarily

$$[\boldsymbol{\mu}_{Dir}, \mathbf{v}] := \sum_{l\in\mathcal{D}} \sum_{q=1}^{d_l} \sum_{p=1}^{3} \mu_{qp}^{l} v_p(\mathbf{y}_q^l)\ .$$

In order to obtain the algebraic formulation of the T–FETI domain decomposition method we shall identify the used sets with the respective Euclidean spaces or their subsets. Thus $\mathbb{X} \sim \mathbb{R}^p$, $\mathbb{\Lambda}_\nu \sim \mathbb{R}^{m_\nu}_-$, $\mathbb{\Lambda}_{int} \sim \mathbb{R}^{m_{int}}$, $\mathbb{\Lambda}_{Dir} \sim \mathbb{R}^{m_{Dir}}$ and $\mathbb{\Lambda}_T(g)$ with $\mathbf{\Lambda}_T(\vec{\mathbf{g}})$, where

$$\mathbf{\Lambda}_T(\vec{\mathbf{g}}) = \{(\vec{\boldsymbol{\mu}}_a^\top, \vec{\boldsymbol{\mu}}_b^\top)^\top \in \mathbb{R}^{2m_\nu} \mid \quad ||(\mu_{ai}, \mu_{bi})||_2 \leq -\mathcal{F}_i g_i \quad \forall i = 1, \ldots, m_\nu\}$$

and $\vec{\mathbf{g}} = \{g_i\}_{i=1}^{m_\nu}$. The duality pairings introduced above can be expressed in the matrix form as follows:

$$\begin{aligned} [\mu_\nu, \mathbf{v}] &= (\vec{\boldsymbol{\mu}}_\nu, \mathbf{N}\vec{\mathbf{v}}) & [\boldsymbol{\mu}_T, \mathbf{v}] &= (\vec{\boldsymbol{\mu}}_T, \mathbf{T}\vec{\mathbf{v}}) \\ [\boldsymbol{\mu}_{int}, \mathbf{v}] &= (\vec{\boldsymbol{\mu}}_{int}, \mathbf{B}_{int}\vec{\mathbf{v}}) & [\boldsymbol{\mu}_{Dir}, \mathbf{v}] &= (\vec{\boldsymbol{\mu}}_{Dir}, \mathbf{B}_{Dir}\vec{\mathbf{v}}) \ , \end{aligned}$$

where $\vec{\mathbf{v}}$ is the vector of all nodal values of $\mathbf{v} \in \mathbb{X}$ and $\mathbf{N} \in \mathbb{R}^{m_\nu \times p}$, $\mathbf{T} \in \mathbb{R}^{2m_\nu \times p}$, $\mathbf{B}_{int} \in \mathbb{R}^{m_{int} \times p}$, and $\mathbf{B}_{Dir} \in \mathbb{R}^{m_{Dir} \times p}$ are the matrices representing the corresponding linear mappings.

The algebraic form of (4.24) reads as follows:

$$\left.\begin{array}{l} \textit{Find } (\vec{\mathbf{u}}, \vec{\boldsymbol{\lambda}}) \in \mathbb{R}^p \times \mathbf{\Lambda}(\vec{\mathbf{g}}) \textit{ such that} \\ \mathbf{K}\vec{\mathbf{u}} = \vec{\mathbf{f}} + \mathbf{B}^\top \vec{\boldsymbol{\lambda}} \\ (\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\lambda}})^\top \mathbf{B}\vec{\mathbf{u}} \geq (\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\lambda}})^\top \mathbf{B}\vec{\mathbf{z}} \qquad \forall \vec{\boldsymbol{\mu}} \in \mathbf{\Lambda}(\vec{\mathbf{g}}) \ , \end{array}\right\} \qquad (\mathcal{M}(\vec{\mathbf{g}}))$$

where $\mathbf{K} \in \mathbb{R}^{p \times p}$ is the stiffness matrix, $\vec{\mathbf{f}} \in \mathbb{R}^p$ is the load vector,

$$\mathbf{\Lambda}(\vec{\mathbf{g}}) = \mathbb{R}^{m_\nu}_- \times \mathbf{\Lambda}_T(\vec{\mathbf{g}}) \times \mathbb{R}^{m_{int}} \times \mathbb{R}^{m_{Dir}},$$

and

$$\vec{\boldsymbol{\mu}} = \begin{pmatrix} \vec{\boldsymbol{\mu}}_n \\ \vec{\boldsymbol{\mu}}_T \\ \vec{\boldsymbol{\mu}}_{int} \\ \vec{\boldsymbol{\mu}}_{Dir} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{N} \\ \mathbf{T} \\ \mathbf{B}_{int} \\ \mathbf{B}_{Dir} \end{pmatrix}, \quad \vec{\mathbf{z}} = \begin{pmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{z}}_T \\ \vec{\mathbf{0}} \\ \vec{\mathbf{0}} \end{pmatrix}.$$

Eliminating the linearly dependent rows from $\mathbf{B}$ one obtains the matrix of full row rank which will be denoted by the same symbol. From this we obtain the existence and uniqueness of the solution to $(\mathcal{M}(\vec{\mathbf{g}}))$.

From the first equation in $(\mathcal{M}(\vec{\mathbf{g}}))$ one can express $\vec{\mathbf{u}}$. This equation is satisfied iff

$$\vec{\mathbf{f}} + \mathbf{B}^\top \vec{\boldsymbol{\lambda}} \in Im\,\mathbf{K} \tag{5.26}$$

and

$$\vec{\mathbf{u}} = \mathbf{K}^{\dagger}(\vec{\mathbf{f}} + \mathbf{B}^{\top}\vec{\boldsymbol{\lambda}}) + \mathbf{R}\vec{\boldsymbol{\alpha}} \tag{5.27}$$

for an appropriate $\vec{\boldsymbol{\alpha}} \in \mathbb{R}^{\dim Ker\mathbf{K}}$, where $\mathbf{K}^{\dagger} \in \mathbb{R}^{p\times p}$ is a generalized inverse of $\mathbf{K}$ and $\mathbf{R} \in \mathbb{R}^{p\times \dim Ker\mathbf{K}}$ is a matrix whose columns span $Ker\mathbf{K}$. Let us note that the Moore-Penrose inverse is easily available in the T-FETI domain decomposition method [12]. Moreover $\vec{\boldsymbol{\alpha}}$ can be computed solely from $\vec{\boldsymbol{\lambda}}$ provided that $\vec{\boldsymbol{\lambda}}$ is known [13].

Since $Ker\mathbf{K}$ is the orthogonal complement of $Im\mathbf{K}$ in $\mathbb{R}^n$, one can write (5.26) equivalently as

$$\mathbf{R}^{\top}(\vec{\mathbf{f}} + \mathbf{B}^{\top}\vec{\boldsymbol{\lambda}}) = \vec{\mathbf{0}}. \tag{5.28}$$

Eliminating $\vec{\mathbf{u}}$ from $(\mathcal{M}(\vec{\mathbf{g}}))$ by using (5.27) and adding the constraint (5.28) to the definition of the feasible set we arrive at the following problem for $\vec{\boldsymbol{\lambda}}$:

$$\left.\begin{array}{ll} \textit{Find } \vec{\boldsymbol{\lambda}} \in \boldsymbol{\Lambda}^{\#}(\vec{\mathbf{g}}) \textit{ such that} & \\ \mathcal{S}(\vec{\boldsymbol{\lambda}}) \leq \mathcal{S}(\vec{\boldsymbol{\mu}}) & \forall \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}^{\#}(\vec{\mathbf{g}}) \ , \end{array}\right\} \tag{$\mathcal{D}(\vec{\mathbf{g}})$}$$

where

$$\mathcal{S}(\vec{\boldsymbol{\mu}}) = \frac{1}{2}\vec{\boldsymbol{\mu}}^{\top}\mathbf{B}\mathbf{K}^{\dagger}\mathbf{B}^{\top}\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\mu}}^{\top}\mathbf{B}(\vec{\mathbf{z}} - \mathbf{K}^{\dagger}\vec{\mathbf{f}}),$$
$$\boldsymbol{\Lambda}^{\#}(\vec{\mathbf{g}}) = \{\vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}(\vec{\mathbf{g}}) \mid \mathbf{R}^{\top}\mathbf{B}^{\top}\vec{\boldsymbol{\mu}} = -\mathbf{R}^{\top}\vec{\mathbf{f}}\}.$$

To simplify the next presentation we denote:

$$\mathbf{F} = \mathbf{B}\mathbf{K}^{\dagger}\mathbf{B}^{\top},\ \mathbf{G} = \mathbf{R}^{\top}\mathbf{B}^{\top},\ \vec{\mathbf{e}} = -\mathbf{R}^{\top}\vec{\mathbf{f}}.$$

Then the solution $\vec{\boldsymbol{\lambda}}$ to $(\mathcal{D}(\vec{\mathbf{g}}))$ satisfies (see (5.28))

$$\mathbf{G}\vec{\boldsymbol{\lambda}} = \vec{\mathbf{e}}.$$

Since $\vec{\boldsymbol{\lambda}}$ can be uniquely split into $\vec{\boldsymbol{\lambda}}_{Im} \in Im\,\mathbf{G}^{\top}$ and $\vec{\boldsymbol{\lambda}}_{Ker} \in Ker\mathbf{G}$ as

$$\vec{\boldsymbol{\lambda}} = \vec{\boldsymbol{\lambda}}_{Im} + \vec{\boldsymbol{\lambda}}_{Ker} \tag{5.29}$$

and $\vec{\boldsymbol{\lambda}}_{Im}$ is easily computable by

$$\vec{\boldsymbol{\lambda}}_{Im} = \mathbf{G}^{\top}(\mathbf{G}\mathbf{G}^{\top})^{-1}\vec{\mathbf{e}},$$

it remains to show how to get $\vec{\boldsymbol{\lambda}}_{Ker}$. Inserting (5.29) into $(\mathcal{D}(\vec{\mathbf{g}}))$ we obtain the new minimization problem for $\vec{\boldsymbol{\lambda}}_{Ker}$:

$$\left.\begin{array}{l} Find\ \vec{\boldsymbol{\lambda}}_{Ker} \in \boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}})\ such\ that \\ \mathcal{S}_{Ker}(\vec{\boldsymbol{\lambda}}_{Ker}) \leq \mathcal{S}_{Ker}(\vec{\boldsymbol{\mu}}) \qquad \forall \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}}), \end{array}\right\} \qquad (\mathcal{D}'(\vec{\mathbf{g}}))$$

where

$$\mathcal{S}_{Ker}(\vec{\boldsymbol{\mu}}) = \tfrac{1}{2}\vec{\boldsymbol{\mu}}^\top \mathbf{F}\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\mu}}^\top \vec{\mathbf{h}}, \quad \vec{\mathbf{h}} = \mathbf{B}(\vec{\mathbf{z}} - \mathbf{K}^\dagger \vec{\mathbf{f}}) - \mathbf{F}\vec{\boldsymbol{\lambda}}_{Im},$$
$$\boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}}) = \{\vec{\boldsymbol{\mu}} \in \mathbb{R}^{3m_\nu + m_{int} + m_{Dir}} :\ \vec{\boldsymbol{\mu}} + \vec{\boldsymbol{\lambda}}_{Im} \in \boldsymbol{\Lambda}(\vec{\mathbf{g}}), \mathbf{G}\vec{\boldsymbol{\mu}} = \vec{\mathbf{0}}\}.$$

Finally we introduce the orthogonal projectors $\mathbf{Q}$ and $\mathbf{P}$ on $Im\,\mathbf{G}^\top$ and $Ker\,\mathbf{G}$:

$$\mathbf{Q} = \mathbf{G}^\top(\mathbf{G}\mathbf{G}^\top)^{-1}\mathbf{G} \quad \text{and} \quad \mathbf{P} = \mathbf{I} - \mathbf{Q},$$

respectively. It is easy to verify that $(\mathcal{D}'(\vec{\mathbf{g}}))$ is equivalent to:

$$\left.\begin{array}{l} Find\ \vec{\boldsymbol{\lambda}}_{Ker} \in \boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}})\ such\ that \\ \mathcal{S}^{Proj}_{Ker}(\vec{\boldsymbol{\lambda}}_{Ker}) \leq \mathcal{S}^{Proj}_{Ker}(\vec{\boldsymbol{\mu}}) \qquad \forall \vec{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}}), \end{array}\right\} \qquad (\mathcal{D}''(\vec{\mathbf{g}}))$$

where

$$\mathcal{S}^{Proj}_{Ker}(\vec{\boldsymbol{\mu}}) = \frac{1}{2}\vec{\boldsymbol{\mu}}^\top(\mathbf{PFP} + \rho\mathbf{Q})\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\mu}}^\top \mathbf{P}\vec{\mathbf{h}}, \quad \rho > 0.$$

Remark 5.1. *It can be shown that the eigenvalues of the Hessian* $\mathbf{PFP} + \rho\mathbf{Q}$ *of the quadratic form* $\mathcal{S}^{Proj}_{Ker}$ *belong to an interval in* $\mathbb{R}^1_+$ *whose bounds are independent of the mesh norms provided that the ratio* H/h *between the domain decomposition norm* H *and the finite element norm* h *is bounded (see [3, 7]).*

In the rest of this section we present the SMALSE algorithm of Dostál [4] (in a sense optimal) for solving $(\mathcal{D}''(\vec{\mathbf{g}}))$, which is based on the augmented Lagrangian method. To this end we introduce the Lagrange multiplier vector $\vec{\boldsymbol{\beta}} \in \mathbb{R}^{\dim Ker\mathbf{K}}$ releasing the equality constraint in $\boldsymbol{\Lambda}^{\#}_{Ker}(\vec{\mathbf{g}})$ and the augmented Lagrangian to the problem $(\mathcal{D}''(\vec{\mathbf{g}}))$ by:

$$L_\rho(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\beta}}) = \frac{1}{2}\vec{\boldsymbol{\mu}}^\top(\mathbf{PFP} + \rho\mathbf{Q})\vec{\boldsymbol{\mu}} - \vec{\boldsymbol{\mu}}^\top \mathbf{Ph} + \vec{\boldsymbol{\beta}}^\top \mathbf{G}\vec{\boldsymbol{\mu}}.$$

The algorithm generates two sequences $\{\vec{\boldsymbol{\mu}}^{(k)}\}$ and $\{\vec{\boldsymbol{\beta}}^{(k)}\}$ which approximate $\vec{\boldsymbol{\lambda}}_{Ker}$ and $\vec{\boldsymbol{\beta}}$, respectively. Each $\vec{\boldsymbol{\mu}}^{(k)}$ is computed by

$$\text{minimize } L_\rho(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\beta}}^{(k)}) \quad \text{subject to } \vec{\boldsymbol{\mu}} + \vec{\boldsymbol{\lambda}}_{Im} \in \boldsymbol{\Lambda}(\vec{\mathbf{g}}) \qquad (5.30)$$

for $\vec{\boldsymbol{\beta}}^{(k)}$ being fixed. In order to recognize a sufficiently accurate approximation of the minima, we need a suitable optimality criterion. To this end we use the so–called *K-gradient* $\mathbf{g}^K = \mathbf{g}^K(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\beta}}^{(k)})$ represented by the vector of the KKT-optimality conditions to the problem (5.30) (see [4] for more details).

ALGORITHM 4.1. Given $\vec{\boldsymbol{\beta}}^{(0)} \in \mathbb{R}^{\dim Ker \mathbf{K}}$, $\epsilon > 0$, $\rho > 0$, $M_0 > 0$, $\eta > 0$, and $\beta > 1$. Set $k := 0$ and $\varepsilon_1 = \varepsilon \|\mathbf{Ph}\|$.

(Step 1.) Find $\vec{\boldsymbol{\mu}}^{(k)}$ such that $\vec{\boldsymbol{\mu}}^{(k)} + \vec{\boldsymbol{\lambda}}_{Im} \in \boldsymbol{\Lambda}(\vec{\mathbf{g}})$ and

$$\|\vec{\mathbf{g}}^K(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\beta}}^{(k)})\| \leq \min\{M_k \|\mathbf{G}\vec{\boldsymbol{\mu}}^{(k)}\|, \eta\}.$$

(Step 2.) If $\|\vec{\mathbf{g}}^K(\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\beta}}^{(k)})\| \leq \varepsilon_1$ and $\|\mathbf{G}\vec{\boldsymbol{\mu}}^{(k)}\| \leq \varepsilon_1 M_0 \|\vec{\boldsymbol{\mu}}^{(k)}\|$ then return $\vec{\boldsymbol{\lambda}}_{Ker} := \vec{\boldsymbol{\mu}}^{(k)}$, else go to Step 3.

(Step 3.) Compute $\vec{\boldsymbol{\beta}}^{(k+1)} = \vec{\boldsymbol{\beta}}^{(k)} + \rho \mathbf{G}\vec{\boldsymbol{\mu}}^{(k)}$.

(Step 4.) Update the precision control M_k as follows: if $k > 0$ and

$$L_\rho(\vec{\boldsymbol{\mu}}^{(k)}, \vec{\boldsymbol{\beta}}^{(k)}) < L_\rho(\vec{\boldsymbol{\mu}}^{(k-1)}, \vec{\boldsymbol{\beta}}^{(k-1)}) + \frac{\rho}{2}\|\mathbf{G}\vec{\boldsymbol{\mu}}^{(k)}\|^2$$

then $M_{k+1} = M_k/\beta$, else $M_{k+1} = M_k$.

(Step 5.) Set $k := k + 1$ and go to Step 1.

Step 1 can be performed by the *K-optimal* KPRGP algorithm of Kučera [4] for solving (5.30), i.e., the algorithm that guarantees convergence of the K-gradient $\vec{\mathbf{g}}^K$ to zero. As (5.30) is a minimization problem for the strictly convex quadratic function with the separable convex constraints, we apply the algorithm based on the ideas proposed of [15]. The analysis in [16] (originally developed in [5] for simple bound constraints) shows that its convergence rate can be expressed in terms of the spectral condition number of the Hessian of the objective function. This result is the key for proving that ALGORITHM 4.1 computes the solution in $\mathcal{O}(1)$ iterations provided that the eigenvalues of the Hessian $\mathbf{PFP} + \rho\mathbf{Q}$ belong to an interval in $\mathbb{R}^1_+$ whose bounds are independent of the size of the problem [4]. Under the condition of Remark 5.1 we conclude that ALGORITHM 4.1 is *scalable*, i.e., it finds the solution of $(\mathcal{D}''(\vec{\mathbf{g}}))$ in a number of iterations independent of the mesh norms.

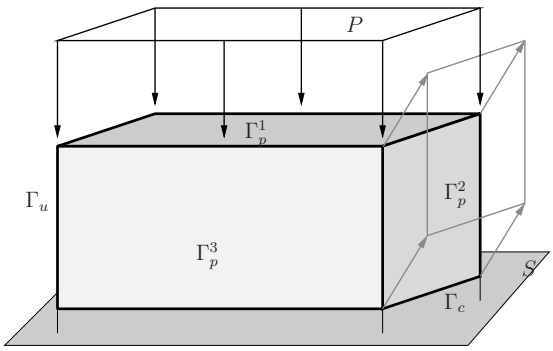

Figure 6.1: Geometry of a model example

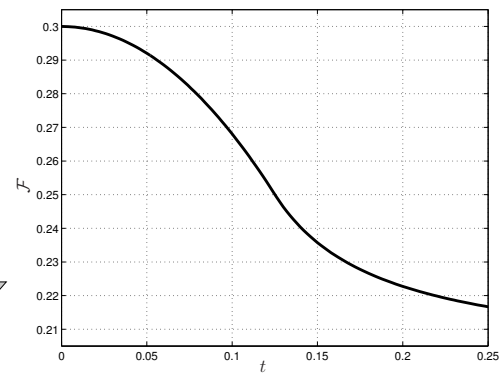

Figure 6.2: Coefficient of friction $\mathcal{F}(t)$

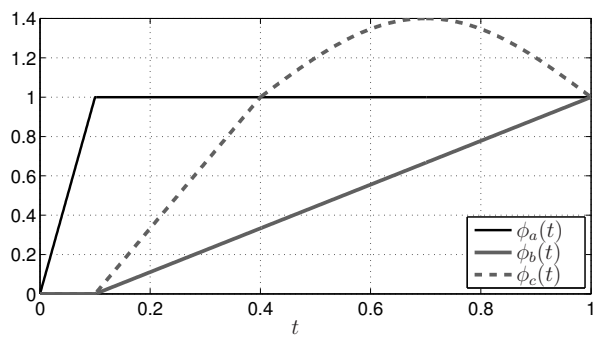

Figure 6.3: Functions describing the loading history

6. Numerical examples

We shall consider an elastic body represented by a brick $\Omega = (0,3) \times (0,1) \times (0,1)$ (in *meters*) made of a homogenous and isotropic material which is characterized by Young's modulus $E = 211.9e9[Pa]$ and Poisson's ratio $\sigma = 0.277$. The partition of $\partial\Omega$ into Γ_u, $\Gamma_p = \Gamma_p^1 \cup \Gamma_p^2 \cup \Gamma_p^3$ and Γ_c is depicted on Fig. 6.1. The rigid foundation is represented by the halfspace $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_3 \geq -0.1\}$, i.e. the initial gap between the body and S is 0.1 [m]. The part Γ_p is subject to surface tractions of density P, where $P(t) = (0,0,30)\phi_a(t)$ on Γ_p^1, $P(t) = (9,0,15)\phi_x(t)$ on Γ_p^2 and $P(t) = (0,0,0)$ on Γ_p^3. The functions $\phi_a, \phi_x : [0,1] \mapsto \mathbb{R}^1$ characterize the history of the loading process. Since friction is a non–conservative phenomenon, the equilibrium state of Ω at the end of loading depends also on its history,

i.e. on a particular choice of ϕ_a and ϕ_x. We shall consider monotone loading $\phi_x := \phi_b$ and nonmonotone loading $\phi_x := \phi_c$ (see Fig. 6.3 for the graphs). Let us note that both loading processes start and end with the same loads. The function describing the coefficient of friction $\mathcal{F}(||\dot{\mathbf{u}}_T||)$ is shown on Fig. 6.2.

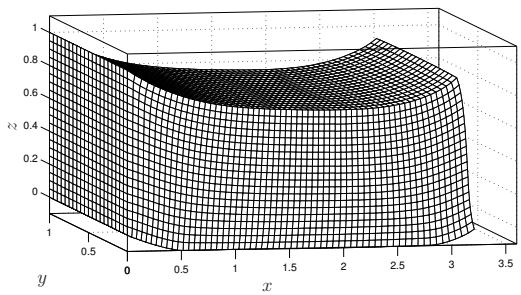

Figure 6.4: total deformation (ϕ_b)

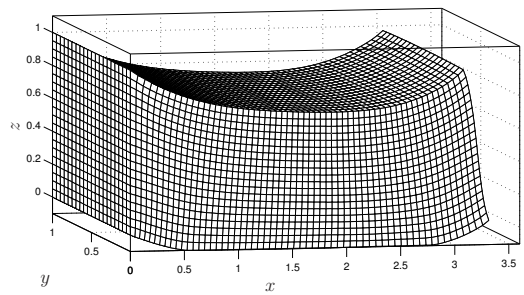

Figure 6.5: total deformation (ϕ_c)

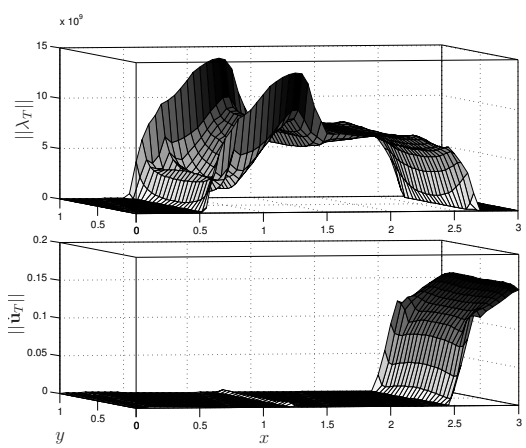

Figure 6.6: Distrib. of $||\boldsymbol{\lambda}_T||$ and $||\dot{\mathbf{u}}_T||$ (ϕ_b)

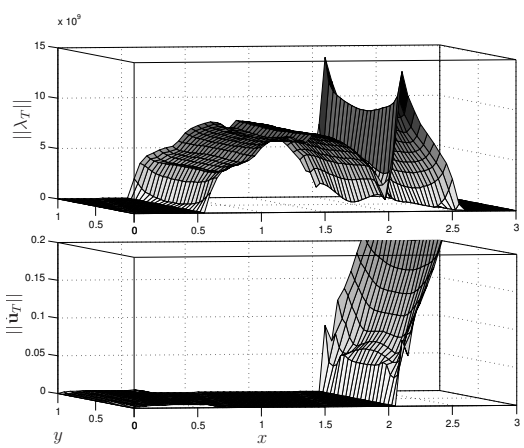

Figure 6.7: Distrib. of $||\boldsymbol{\lambda}_T||$ and $||\dot{\mathbf{u}}_T||$ (ϕ_c)

The domain Ω is divided into $3n \times n \times n$ cubic subdomains Ω_i for $n = 1, 2, 3, 4$ and 5. Each subdomain Ω_i is divided into $5 \times 5 \times 5$ cubes. Each cube is finally divided into 5 tetrahedra. Thus we obtain four different partitions $\mathcal{T}_h$ of $\overline{\Omega}$ with the total number of the primal variables $n_p = 1944$, 15552, 52488, 124416 and 243000. The time step was chosen $\Delta t = 0.05$ representing 50 time–steps. The initial state was $u_0 = 0$.

To find a solution to the corresponding contact problem with Coulomb friction at each time level we use the method of successive approximations

presented in the previous sections. Each iterative step which leads to a contact problem with given friction was realized in its dual form by using the modified version of the quadratic programming algorithm MPGRP (for the detailed description we refer to [16]). The stopping criterion for fixed–point iterations (Coulomb's loop) was chosen the same at all time levels: $\|\boldsymbol{\lambda}_\nu^{(k+1)} - \boldsymbol{\lambda}_\nu^{(k)}\| / \|\boldsymbol{\lambda}_\nu^{(k)}\| < 10^{-6}$, where $\boldsymbol{\lambda}_\nu^{(k)}$ is the k–th iteration of the normal contact stress and $\| \ \|$ stands for the Euclidean norm of a vector.

In Table 6.1 the following characteristics are summarized: $n_s = 3n \times n \times n$ denotes the number of subdomains; n_p and n_d is the total number of the primal and dual variables, respectively; it and n_m stand for the total number of the fixed–point iterations in Coulomb's loop and for the total number of multiplications by the dual matrix (at all time levels). The respective numbers in the last three columns of Table 6.1 pertains to ϕ_b and ϕ_c. The computations were performed by Matlab code on AMD Opteron 2210 HE, 8GB DDR2.

n	n_s	n_p	n_d	it	n_m	time [s]
1	3	1944	612	563/908	17798/18194	$2.2e2/2.1e2$
2	24	15552	5685	639/1101	23612/35279	$2.0e3/2.9e3$
3	81	52488	20136	743/1342	29763/33444	$9.7e3/1.2e4$
4	192	124416	48879	835/1627	29252/33950	$2.1e4/2.4e4$
5	375	243000	96828	913/1876	29741/39369	$4.7e4/6.1e4$

Table 6.1: the number of iterations and the solution times

Numerical results at time $t = 1$ are depicted in Figs. 6.4-6.7. The total deformation of Ω is shown in Figs. 6.4 and 6.5. As we have already mentioned, the final state of Ω depends also on the loading history, characterized by the functions ϕ_b, ϕ_c. This is clearly seen from Figs. 6.6 and 6.7 which show the distribution of the norm of the friction force $||\boldsymbol{\lambda}_T||$ and the norm of the tangential velocity $||\dot{\mathbf{u}}_T||$ on Γ_c.

7. Conclusions and comments

The present paper deals with the full discretization of 3D quasistatic contact problems with Coulomb friction and a solution dependent coefficient of friction. Such type of problems typically arises in geomechanics when modelling the movement of tectonic plates. The time discretization

is done by a three–step finite difference. The explicit form of the static contact problem with Coulomb friction solved at each time level is derived. Its numerical treatment is based on the method of successive approximations in which each iterative step is represented by a contact problem with Tresca friction. The finite element approximation of the Tresca problems is based on the T-FETI domain decomposition method [2] that results in minimizing a quadratic function subject to separable quadratic inequality and linear equality constraints. The total computational complexity of the overall method is determined by the scalable behavior of the augmented Lagrangian based algorithm [1, 3, 4] that we use for the minimization. The numerical experiments show that the relative efficiency (i.e., the number of matrix multiplications per the fixed-point iteration) is lower for finer discretizations. In addition, the benefit from applying the T-FETI consists in a simple identification of the kernel-spaces to the sub-body stiffness matrices and in a simple parallelization of computations.

Acknowledgement: *This research was supported by MSM 0021620839 and by the project 201/07/0294 of the Grant Agency of the Czech Republic. A part of this research was also done in the frame of the bilateral co-operation between Charles University, Prague and Aristotle University in Thessaloniki (the first and fourth author)*

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