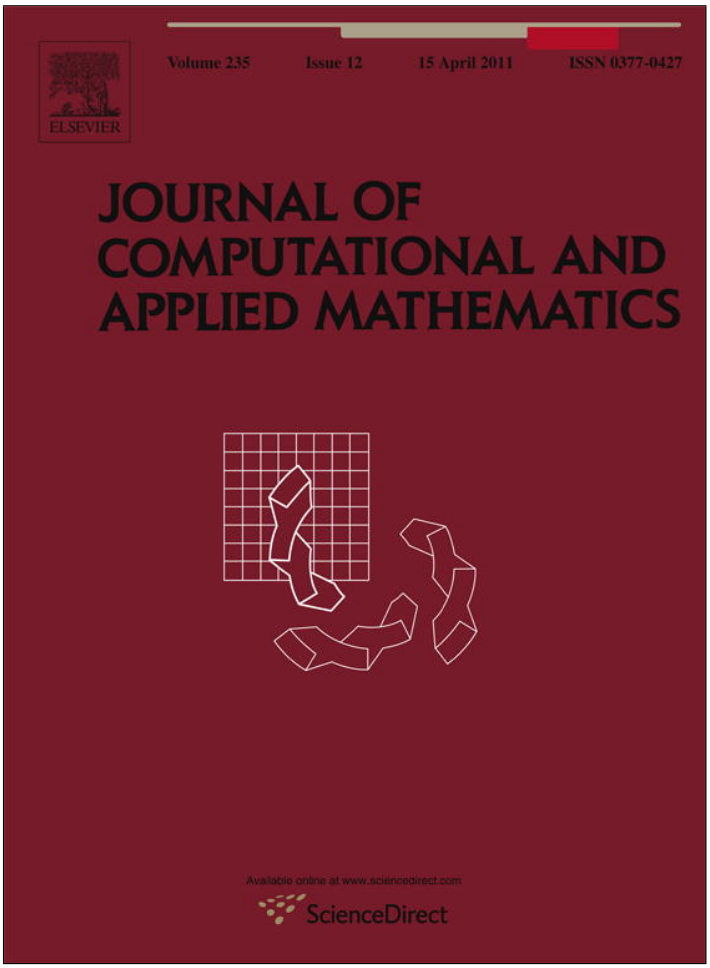

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Qualitative analysis of 3D elastostatic contact problems with orthotropic Coulomb friction and solution-dependent coefficients of friction

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ABSTRACT

This paper analyzes a discrete form of 3D contact problems with local orthotropic Coulomb friction and coefficients of friction which may depend on the solution itself. The analysis is based on the fixed-point reformulation of the original problem. Conditions guaranteeing the existence and uniqueness of discrete solutions are established. Finally, numerical results of a model example are presented.

1. Introduction

Contact mechanics is a special branch of solid mechanics analyzing the behavior of loaded deformable bodies which are in mutual contact. In addition to unilateral boundary conditions expressing non-penetration of the bodies in the structure, one has to also take into account the influence of friction on the contact zones. There are different models of friction, but the local Coulomb law is the most classical one. Although this model is seemingly simple, contrary is the case. The mathematical model involving static Coulomb friction leads to an implicit variational inequality, whose solution remained open for a long time. The existence analysis was done relatively not long ago. For the mathematical analysis of static, quasi-static and dynamic contact problems with Coulomb friction we refer to [1] and the references therein. In what follows we confine ourselves to static contact problems. Suppose first that the coefficient of friction $\mathscr{F}$ does not depend on the solution. Then a typical existence result says that a solution exists provided that $\mathscr{F}$ is sufficiently small (with additional technical assumptions on the regularity of data). As far as the structure of solutions is concerned, no general results are available at present unless a solution of this problem has some specific properties [2,3]. The situation is completely different for appropriate finite element discretizations of these problems. Using fixed-point arguments one can show that at least one solution exists for any $\mathscr{F}$ belonging to a large class of coefficients. Moreover, this solution is unique if $\mathscr{F}$ is small enough. Unfortunately, the bound $\mathscr{F}_{\max}$ on $\mathscr{F}$ ensuring uniqueness of the solution is mesh dependent. It is known (see [4]) that in the case of isotropic Coulomb friction, $\mathscr{F}_{\max}$ has to decay at least as $\sqrt{h}$, where h is the norm of a finite element partition. The same result has been obtained in [5] by using a penalty and regularization of the frictional term. The previous analysis has been extended to the isotropic Coulomb friction law in which the coefficient of friction $\mathscr{F}$ depends on the solution itself. It was shown that the uniqueness result depends not only on $\mathscr{F}_{\max}$ but also on the Lipschitz modulus L of $\mathscr{F}$. The goal of the present paper is to generalize these results to the case of orthotropic Coulomb friction in which both coefficients of friction

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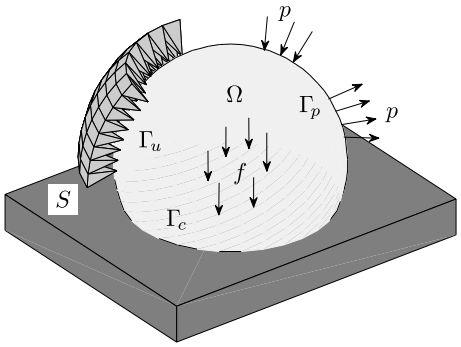

Fig. 1. Geometry of the problem.

in the directions of the principal axes of orthotropy depend on the magnitudes of the tangential components of contact displacements.

The paper is organized as follows: in Section 2, continuous setting of the problem is presented. A weak solution to our problem is defined in two different ways: (a) as a solution to an implicit variational inequality; (b) as a fixed point of an auxiliary mapping Ψ acting on the contact part of the boundary. The later is used for defining the discrete form of our problem. This form is based on an appropriate discretization of Ψ. Section 3 presents the existence and uniqueness analysis. We show that at least one discrete solution exists for any positive, bounded and continuous coefficients of friction. Assuming that the coefficients are Lipschitz continuous we prove that the discretization of Ψ is Lipschitz continuous as well. The estimate of its modulus of Lipschitz continuity will be derived in terms of $\mathscr{F}_{\max}$, L, the condition number of the friction coefficient matrix and the mesh norms of the respective finite element spaces used to build the discrete model. If $\mathscr{F}_{\max}$ and L are sufficiently small (expressed in terms of the mesh norms), then the modulus of Lipschitz continuity is less than one. Thus, as a by-product we obtain the mathematical justification of the method of successive approximations, one of the possible approaches for numerical realization of such problems. To illustrate its performance we present in Section 4 numerical results of a simple model example.

For other numerical methods for solving contact problems we refer to the following publications. The overview and the comparison of the most frequently used strategies can be found in [6]. To overcome the drawbacks of penalty and Lagrange multiplier techniques, augmented Lagrangian methods have been developed. The application of these methods in contact mechanics is described in [7]. The survey of algorithms of constrained optimization which are used in contact computational mechanics can be also found in [8]. Some algorithms are combined with multigrid or domain decomposition techniques in order to increase their performance for solving large scale problems of the real world; see, e.g., the primal–dual active set algorithm of Hűeber et al. [9], the non-smooth multiscale method of Krause [10], or the augmented Lagrangian based algorithm combined with the FETI method of Dostál et al. [11,12]. Our implementation of the method of successive approximations requires to solve a sequence of contact problems with Tresca friction that are represented by the minimization of strictly quadratic objective functions subject to ellipsoidal constraints. These minima are computed by the active set type algorithm of Kučera [13] that generalizes another one of Dostál and Schőberl originally developed for simple bound constraints. Note that this algorithm combined with the augmented Lagrangians [14] is the heart of the Matsol library [15] for solving 3D contact problems with friction. Results of numerical experiments presented in the paper illustrate the robustness of this algorithm for solving the orthotropic Coulomb friction law.

Throughout the paper we shall use the following notation: the Euclidean norm in $\mathbb{R}^n$ as well as the matrix norm in $\mathbb{R}^{n\times n}$ generated by the Euclidean vector norm are denoted by $\|.\|$, $u \cdot v$ stands for the scalar product of two vectors $u, v \in \mathbb{R}^n$. The symbol $W^{k,p}(G)$, $G \subset \mathbb{R}^n$, $k \geq 0$ integer, $p \in [1, +\infty]$, is used for the standard Sobolev space equipped with the norm $\|.\|_{k,p,G}$ ($W^{0,p}(G) = L^p(G)$). The analogous spaces of functions with values in $\mathbb{R}^m$ are denoted by $W^{k,p}(G;\mathbb{R}^m)$ (resp. $L^p(G;\mathbb{R}^m)$). If $p = 2$, we simply write $H^k(G)$ and $H^k(G;\mathbb{R}^m)$; $\|.\|_{k,G}$ and $(.,.)_{k,G}$ stands for the norm and the scalar product, respectively.

2. Setting of the problem

Let us consider a body made of a linear elastic material whose reference configuration is represented by a bounded domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary $\partial\Omega$. Let Γ_u, Γ_p and Γ_c be three disjoint, (relatively) open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_p \cup \overline{\Gamma}_c$ and $\mathrm{meas}_2(\Gamma_u)$, $\mathrm{meas}_2(\Gamma_c) > 0$. The body is fixed on Γ_u, surface tractions of density p act on Γ_p while a rigid foundation S unilaterally supports the body along Γ_c. For the sake of simplicity of our presentation we shall assume that S is a half-space and there is no gap between Γ_c and S, i.e. Γ_c is a part of a hyperplane (see Fig. 1). The effect of friction between Ω and S is described by the local orthotropic Coulomb friction law with coefficients of friction *depending* on the solution. In addition, volume forces of density f are applied to Ω. Our aim is to find an equilibrium state of the body.

By a solution to the pure elastostatic problem without contact (i.e. with $\Gamma_c = \emptyset$) we mean any displacement vector $u : \Omega \to \mathbb{R}^3$ satisfying *the equilibrium equations*, *linear Hooke's law* and the *kinematic* and *static boundary conditions* on Γ_u

and Γ_p, respectively:

$$\left.\begin{array}{ll} -\operatorname{div}\sigma(u) = f & \text{in } \Omega, \\ \sigma(u) = \mathcal{C}\varepsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_u, \\ \sigma(u)\nu = p & \text{on } \Gamma_p. \end{array}\right\} \tag{2.1}$$

Here $\sigma(u)$ is a stress tensor, $\varepsilon(u) = 1/2(\nabla u + \nabla^T u)$ is the linearized strain tensor associated with u and $\mathcal{C}$ is the fourth order elasticity tensor. Further, ν is the unit outward normal vector to $\partial\Omega$.

To formulate the contact and friction conditions, let $u_\nu := u \cdot \nu$, $\sigma_\nu(u) := (\sigma(u)\nu)\cdot\nu$ be the normal component of a displacement vector u and the stress vector $\sigma(u)\nu$ on Γ_c, respectively. Moreover, let t_1 and t_2 be principal axes of orthotropic friction on the tangent plane to Γ_c so that the triplet $\{\nu(x), t_1(x), t_2(x)\}$ forms a local orthonormal basis in $\mathbb{R}^3$ for any $x \in \Gamma_c$. By $u_t = (u_{t_1}, u_{t_2})$, $\sigma_t(u) = (\sigma_{t_1}(u), \sigma_{t_2}(u))$ we denote the tangential displacement and the tangential contact stress, respectively, with $u_{t_i} := u \cdot t_i$, $\sigma_{t_i} := (\sigma(u)\nu)\cdot t_i$, $1 \le i \le 2$. Finally, let $\mathscr{F}_1$ and $\mathscr{F}_2$ be coefficients of friction in the directions t_1 and t_2, respectively, and set

$$\mathscr{F} := \begin{pmatrix} \mathscr{F}_1 & 0 \\ 0 & \mathscr{F}_2 \end{pmatrix}.$$

In what follows we shall suppose that both $\mathscr{F}_1$ and $\mathscr{F}_2$ may depend on the magnitudes of u_{t_1} and u_{t_2} on Γ_c, i.e. $\mathscr{F}_i = \mathscr{F}_i(x, |u_{t_1}(x)|, |u_{t_2}(x)|)$, $x \in \Gamma_c$, $1 \le i \le 2$. The respective matrix $\mathscr{F}$ will be denoted by $\mathscr{F}(x, |u_{t_1}(x)|, |u_{t_2}(x)|)$ or for short $\mathscr{F}(|u_{t_1}|, |u_{t_2}|)$. *The non-penetration condition* and *the orthotropic Coulomb friction law* then read as follows:

$$\left.\begin{array}{l} u_\nu \le 0, \qquad \sigma_\nu(u) \le 0, \qquad u_\nu\sigma_\nu(u) = 0 \quad \text{on } \Gamma_c, \\ u_t(x) = 0 \Longrightarrow \|\mathscr{F}^{-1}(x, 0, 0)\sigma_t(u)(x)\| \le -\sigma_\nu(u)(x), \quad x \in \Gamma_c, \\ u_t(x) \ne 0 \Longrightarrow \mathscr{F}^{-1}(x, |u_{t_1}(x)|, |u_{t_2}(x)|)\sigma_t(u)(x) = \sigma_\nu(u)(x)\dfrac{\mathscr{F}(x, |u_{t_1}(x)|, |u_{t_2}(x)|)u_t(x)}{\|\mathscr{F}(x, |u_{t_1}(x)|, |u_{t_2}(x)|)u_t(x)\|}, \quad x \in \Gamma_c. \end{array}\right\} \tag{2.2}$$

The classical formulation of our problem is represented by (2.1) and (2.2). To give the weak formulation we introduce the following spaces and sets:

$$\begin{aligned} &V = \{v \in H^1(\Omega;\mathbb{R}^3) \mid v = 0 \text{ a.e. on } \Gamma_u\}, \qquad K = \{v \in V \mid v_\nu \le 0 \text{ a.e. on } \Gamma_c\}, \\ &W = \{v \in H^1(\Omega) \mid v = 0 \text{ a.e. on } \Gamma_u\}, \qquad X_\nu = \{v_{\nu|_{\Gamma_c}} \mid v \in V\}, \\ &X_{\nu+} = \{\varphi \in X_\nu \mid \varphi \ge 0 \text{ a.e. on } \Gamma_c\}, \qquad X_{t+} = \left\{(|v_{t_1|_{\Gamma_c}}|, |v_{t_2|_{\Gamma_c}}|) \mid v \in V\right\}, \\ &Y = W_{|_{\Gamma_c}} \end{aligned}$$

and endow X_ν with the norm:

$$\|\varphi\|_{X_\nu} := \inf_{\substack{v\in V \\ v_\nu|_{\Gamma_c}=\varphi}} \|v\|_{1,\Omega}.$$

By X_ν' we shall denote the (topological) dual of X_ν and $\langle .,. \rangle_\nu$ will be used for the corresponding duality pairing.

Furthermore, we shall assume that $f \in L^2(\Omega;\mathbb{R}^3)$, $p \in L^2(\Gamma_p;\mathbb{R}^3)$ and $\mathcal{C} = \{c_{ijkl}\}_{i,j,k,l=1}^3$ with $c_{ijkl} \in L^\infty(\Omega)$, $1 \le i, j, k, l \le 3$, satisfies the usual symmetry and ellipticity conditions:

$$\left.\begin{array}{l} c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in } \Omega, \ 1 \le i, j, k, l \le 3, \\ \exists c_0 > 0: \quad \mathcal{C}\xi : \xi \ge c_0(\xi : \xi) \text{ a.e. in } \Omega \text{ for every symmetric } \xi \in \mathbb{R}^{3\times 3}. \end{array}\right\} \tag{2.3}$$

We shall also suppose that the coefficients of friction $\mathscr{F}_1$ and $\mathscr{F}_2$ are continuous and bounded:

$$\left.\begin{array}{ll} \mathscr{F}_i \in C(\Gamma_c \times \mathbb{R}^2_+), & 1 \le i \le 2, \\ \mathscr{F}_{\min} \le \mathscr{F}_i(x,\xi) \le \mathscr{F}_{\max} & \forall x \in \Gamma_c \ \forall \xi \in \mathbb{R}^2_+, \ 1 \le i \le 2, \end{array}\right\} \tag{2.4}$$

where $0 < \mathscr{F}_{\min} \le \mathscr{F}_{\max}$ are given, and

$$\text{the mapping } x \mapsto (t_1(x), t_2(x)) \text{ belongs to } W^{1,\infty}(\Gamma_c;\mathbb{R}^6). \tag{2.5}$$

The weak formulation of (2.1) *and* (2.2) is given by the following *implicit* variational inequality:

$$\left.\begin{array}{l} \text{Find } u \in K \text{ such that} \\ a(u, v-u) - \langle \sigma_\nu(u), \|\mathscr{F}(|u_{t_1}|, |u_{t_2}|)v_t\|\rangle_\nu + \langle \sigma_\nu(u), \|\mathscr{F}(|u_{t_1}|, |u_{t_2}|)u_t\|\rangle_\nu \ge \ell(v-u) \ \forall v \in K, \end{array}\right\} \tag{$\mathscr{P}$}$$

where

$$a(u,v) := \int_\Omega \mathcal{C}\varepsilon(u) : \varepsilon(v)\,dx, \quad u, v \in V,$$
$$\ell(v) := \int_\Omega f \cdot v\,dx + \int_{\Gamma_p} p \cdot v\,ds, \quad v \in V.$$

Owing to (2.3) and Korn's inequality, a is a symmetric bilinear form which is V-elliptic and continuous on $V \times V$:

$$\exists \alpha > 0 : a(v,v) \geq \alpha \|v\|^2_{1,\Omega} \quad \forall v \in V, \tag{2.6}$$
$$\exists M > 0 : |a(u,v)| \leq M \|u\|_{1,\Omega} \|v\|_{1,\Omega} \quad \forall u, v \in V. \tag{2.7}$$

Remark 2.1. To make sense of the duality terms in $(\mathcal{P})$, one needs an additional smoothness of u and $\mathcal{F}$ (and of the mapping $x \mapsto (t_1(x), t_2(x)), x \in \Gamma_c$) ensuring that $\|\mathcal{F}(|u_{t_1}|, |u_{t_2}|)v_t\| \in X_\nu$ for any $v \in V$ (see [1]). To overcome this difficulty, we shall assume that $\sigma_\nu(u) \in L^2(\Gamma_c)$, in what follows. Then the duality pairing $\langle ., . \rangle_\nu$ can be replaced by the $L^2(\Gamma_c)$-scalar product and (2.4) is sufficient.

Below we introduce a fixed-point formulation of $(\mathcal{P})$, on which the finite element discretization will be based. To start with, we associate with any $(\varphi_1, \varphi_2) \in X_{t+}, g \in L^2_+(\Gamma_c)$ the following *auxiliary* problem:

$$\left.\begin{array}{l} \text{Find } u := u(\varphi_1, \varphi_2, g) \in K \text{ such that} \\ a(u, v-u) + j(\varphi_1, \varphi_2, g, v_t) - j(\varphi_1, \varphi_2, g, u_t) \geq \ell(v-u) \quad \forall v \in K, \end{array}\right\} \qquad (\mathcal{P}(\varphi_1, \varphi_2, g))$$

where

$$j(\varphi_1, \varphi_2, g, v_t) := (g, \|\mathcal{F}(\varphi_1, \varphi_2)v_t\|)_{0,\Gamma_c}, \quad (\varphi_1, \varphi_2) \in X_{t+},\ g \in L^2_+(\Gamma_c),\ v \in V.$$

Problem $(\mathcal{P}(\varphi_1, \varphi_2, g))$ is a weak formulation of a contact problem with orthotropic friction of *Tresca* type and the *fixed* matrix of friction coefficients $\mathcal{F}(\varphi_1, \varphi_2)$. The existence of a unique solution is guaranteed for any $(\varphi_1, \varphi_2) \in X_{t+}, g \in L^2_+(\Gamma_c)$, making use of its equivalence to a convex minimization problem (see [16, Chapter II]). This enables us to define the mapping $\Psi : X_{t+} \times L^2_+(\Gamma_c) \to X_{t+} \times X'_\nu$ by

$$\Psi(\varphi_1, \varphi_2, g) = (|u_{t_1}|, |u_{t_2}|, -\sigma_\nu(u)), \quad (\varphi_1, \varphi_2) \in X_{t+},\ g \in L^2_+(\Gamma_c),$$

where u solves $(\mathcal{P}(\varphi_1, \varphi_2, g))$ and $\sigma_\nu(u)$ is the corresponding normal contact stress. Comparing problems $(\mathcal{P})$ and $(\mathcal{P}(\varphi_1, \varphi_2, g))$, it is readily seen that if $(|u_{t_1}|, |u_{t_2}|, -\sigma_\nu(u))$ is a fixed point of Ψ in $X_{t+} \times L^2_+(\Gamma_c)$ then u is a solution to $(\mathcal{P})$.

Let $(\varphi_1, \varphi_2) \in X_{t+}$ and $g \in L^2_+(\Gamma_c)$ be fixed and Λ_ν be the cone of non-negative elements in X'_ν:

$$\Lambda_\nu = \{\mu \in X'_\nu \mid \langle \mu, \varphi \rangle_\nu \geq 0\ \forall \varphi \in X_{\nu+}\}.$$

To release the unilateral constraint $u \in K$, we introduce the following *mixed* formulation of $(\mathcal{P}(\varphi_1, \varphi_2, g))$:

$$\left.\begin{array}{l} \text{Find } (u, \lambda_\nu) := (u(\varphi_1, \varphi_2, g), \lambda_\nu(\varphi_1, \varphi_2, g)) \in V \times \Lambda_\nu \text{ such that} \\ a(u, v-u) + j(\varphi_1, \varphi_2, g, v_t) - j(\varphi_1, \varphi_2, g, u_t) \geq \ell(v-u) - \langle \lambda_\nu, v_\nu - u_\nu \rangle_\nu \quad \forall v \in V, \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle_\nu \leq 0 \quad \forall \mu_\nu \in \Lambda_\nu. \end{array}\right\} \qquad (\mathcal{M}(\varphi_1, \varphi_2, g))$$

It is known that $(\mathcal{M}(\varphi_1, \varphi_2, g))$ has a unique solution for any $(\varphi_1, \varphi_2) \in X_{t+}, g \in L^2_+(\Gamma_c)$. Moreover, u solves $(\mathcal{P}(\varphi_1, \varphi_2, g))$ and $\lambda_\nu = -\sigma_\nu(u)$, as follows from the Green formula [17]. This gives an equivalent expression for the mapping Ψ:

$$\Psi(\varphi_1, \varphi_2, g) = (|u_{t_1}|, |u_{t_2}|, \lambda_\nu) \quad \forall (\varphi_1, \varphi_2) \in X_{t+}\ \forall g \in L^2_+(\Gamma_c) \tag{2.8}$$

with (u, λ_ν) being the solution to $(\mathcal{M}(\varphi_1, \varphi_2, g))$.

3. Finite element discretization

This section deals with an approximation of problem $(\mathcal{P})$, which will be based on a fixed-point formulation for an appropriate discretization of the mapping Ψ. To this end we use (2.8) and a mixed finite element discretization of $(\mathcal{M}(\varphi_1, \varphi_2, g))$. We shall establish the existence as well as uniqueness of the solution to the resulting discrete problem. In addition, we shall investigate, how the uniqueness result depends on the size of the problem.

Let W^h, L^H be the following Lagrange finite element spaces corresponding to the partitions $\mathcal{T}^h_\Omega$ and $\mathcal{T}^H_{\Gamma_c}$ of $\overline{\Omega}$ and $\overline{\Gamma}_c$, respectively:

$$W^h = \{v^h \in C(\overline{\Omega}) \mid {v^h}_{|T} \in P_k(T)\ \forall T \in \mathcal{T}^h_\Omega \text{ and } v^h = 0 \text{ on } \Gamma_u\},$$
$$L^H = \{\mu^H \in L^2(\Gamma_c) \mid {\mu^H}_{|R} \in P_l(R)\ \forall R \in \mathcal{T}^H_{\Gamma_c}\}.$$

Here $k \geq 1$, $l \geq 0$ are integers and h, H stand for the norms of the partitions $\mathscr{T}_\Omega^h$ and $\mathscr{T}_{\Gamma_c}^H$, respectively. Only what we shall suppose at this moment is that $\mathscr{T}_\Omega^h$ is compatible with the decomposition of $\partial\Omega$ into Γ_u, Γ_p and Γ_c. In general, $\mathscr{T}_{\Gamma_c}^H$ is different from $\mathscr{T}_{\Omega|_{\overline{\Gamma}_c}}^h$, but the case when they equal each other is not excluded. Further, set

$$V^h = W^h \times W^h \times W^h, \qquad Y^h = W^h{}_{|_{\overline{\Gamma}_c}},$$
$$Y_+^h = \{\varphi^h \in Y^h \mid \varphi^h \geq 0 \text{ on } \Gamma_c\}, \qquad \Lambda_\nu^H = \{\mu^H \in L^H \mid \mu^H \geq 0 \text{ on } \Gamma_c\}.$$

Clearly, V^h and Λ_ν^H will serve as natural approximations of V and Λ_ν, respectively. In what follows, we shall suppose that the following condition is satisfied:

$$(\mu^H \in L^H \text{ and } (\mu^H, v_\nu^h)_{0,\Gamma_c} = 0 \ \forall v^h \in V^h) \Longrightarrow \mu^H = 0. \tag{3.1}$$

This makes it possible to endow the spaces L^H and $Y^h \times Y^h \times L^H$ with the following (mesh-dependent) norms:

$$\|\mu^H\|_{*,h} = \sup_{0 \neq v^h \in V^h} \frac{(\mu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}},$$
$$\|(\varphi_1^h, \varphi_2^h, \mu^H)\|_{Y^h \times Y^h \times L^H} = \|(\varphi_1^h, \varphi_2^h)\|_{0,\Gamma_c} + \|\mu^H\|_{*,h}.$$

Remark 3.1. Let us briefly mention two examples of the discretizations posited above.

(FE1) $\mathscr{T}_{\Gamma_c}^H = \mathscr{T}_{\Omega|_{\overline{\Gamma}_c}}^h$, $l = k$, $L^H = Y^h$.
Then condition (3.1) is always satisfied.

(FE2) $k = 1$, $l = 0$.
In this case, (3.1) is fulfilled provided that the ratio H/h is sufficiently large, i.e. the partition $\mathscr{T}_{\Gamma_c}^H$ is coarser than $\mathscr{T}_{\Omega|_{\overline{\Gamma}_c}}^h$ (see [18]).

For $(\varphi_1^h, \varphi_2^h, g^H) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H$ given, we introduce the following discrete form of problem $(\mathscr{M}(\varphi_1, \varphi_2, g))$:

$$\left.\begin{array}{l} \text{Find } (u^h, \lambda_\nu^H) := (u^h(\varphi_1^h, \varphi_2^h, g^H), \lambda_\nu^H(\varphi_1^h, \varphi_2^h, g^H)) \in V^h \times \Lambda_\nu^H \text{ such that} \\ a(u^h, v^h - u^h) + j(\varphi_1^h, \varphi_2^h, g^H, v_t^h) - j(\varphi_1^h, \varphi_2^h, g^H, u_t^h) \\ \quad \geq \ell(v^h - u^h) - (\lambda_\nu^H, v_\nu^h - u_\nu^h)_{0,\Gamma_c} \quad \forall v^h \in V^h, \\ (\mu_\nu^H - \lambda_\nu^H, u_\nu^h)_{0,\Gamma_c} \leq 0 \quad \forall \mu_\nu^H \in \Lambda_\nu^H. \end{array}\right\} \tag{$\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H)$}$$

Reformulating $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ as a saddle-point problem, condition (3.1) ensures that $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ has a unique solution (u^h, λ_ν^H) for any $(\varphi_1^h, \varphi_2^h, g^H) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H$ (see [16, Chapter VI]). Furthermore, its first component u^h solves:

$$\left.\begin{array}{l} \text{Find } u^h := u^h(\varphi_1^h, \varphi_2^h, g^H) \in K^{hH} \text{ such that} \\ a(u^h, v^h - u^h) + j(\varphi_1^h, \varphi_2^h, g^H, v_t^h) - j(\varphi_1^h, \varphi_2^h, g^H, u_t^h) \geq \ell(v^h - u^h) \quad \forall v^h \in K^{hH}, \end{array}\right\} \tag{$\mathscr{P}_{hH}(\varphi_1^h, \varphi_2^h, g^H)$}$$

where

$$K^{hH} := \{v^h \in V^h \mid (\mu^H, v_\nu^h)_{0,\Gamma_c} \leq 0 \ \forall \mu^H \in \Lambda_\nu^H\}.$$

Remark 3.2. Notice that K^{hH} is an external approximation of K, i.e. $K^{hH} \not\subset K$. On the other hand, Λ_ν^H is an internal approximation of Λ_ν.

To define a discretization of Ψ, let $r_h : H^1(\Gamma_c) \to Y^h$ be a linear interpolation operator preserving positivity:

$$(\varphi \in H^1(\Gamma_c) \text{ and } \varphi \geq 0 \text{ a.e. on } \Gamma_c) \Longrightarrow r_h\varphi \in Y_+^h \tag{3.2}$$

and possessing the following approximation property:

$$\exists c_r > 0 : \|\varphi - r_h\varphi\|_{0,\Gamma_c} \leq c_r h_{\Gamma_c} \|\varphi\|_{1,\Gamma_c} \quad \forall \varphi \in H^1(\Gamma_c) \cap Y, \tag{3.3}$$

where $h_{\Gamma_c} := \max_{F \in \mathscr{T}_{\Omega|_{\overline{\Gamma}_c}}^h} \operatorname{diam}(F)$. With such r_h at hand we introduce the mapping $\Psi_{hH} : Y_+^h \times Y_+^h \times \Lambda_\nu^H \to Y_+^h \times Y_+^h \times \Lambda_\nu^H$ by

$$\Psi_{hH}(\varphi_1^h, \varphi_2^h, g^H) = (r_h|u_{t_1}^h|, r_h|u_{t_2}^h|, \lambda_\nu^H),$$

where (u^h, λ_ν^H) solves $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$.

Definition 3.1. Any couple $(u^h, \lambda_\nu^H) \in V^h \times \Lambda_\nu^H$ is called a solution of the discrete contact problem with orthotropic Coulomb friction and solution-dependent coefficients of friction if $(r_h|u_{t_1}^h|, r_h|u_{t_2}^h|, \lambda_\nu^H)$ is a fixed point of Ψ_{hH}, i.e. (u^h, λ_ν^H) solves $(\mathscr{M}_{hH}(r_h|u_{t_1}^h|, r_h|u_{t_2}^h|, \lambda_\nu^H))$.

3.1. Existence result

The existence of a discrete solution will be done by using the fixed-point arguments. First we introduce two auxiliary results, the first one is a minor modification of Lemma 3.3 in [19]. Recall that $t_i(x) = (t_{i,j}(x))_{j=1}^3,\ 1 \le i \le 2$, are the principal axes of orthotropic friction and $\varphi_t = (\varphi_{t_1}, \varphi_{t_2})$ with $\varphi_{t_i} = \varphi \cdot t_i,\ 1 \le i \le 2$.

Lemma 3.1. *If $\varphi \in H^1(\Gamma_c)$ then $|\varphi| \in H^1(\Gamma_c)$ and*

$$\| \, |\varphi| \, \|_{1,\Gamma_c} \le \|\varphi\|_{1,\Gamma_c}.$$

Lemma 3.2. *Let (2.5) be satisfied. Then $\varphi_t \in H^1(\Gamma_c;\mathbb{R}^2)$ for any $\varphi \in H^1(\Gamma_c;\mathbb{R}^3)$ and there exists a constant $c_t > 0$ such that*

$$\|\varphi_t\|_{1,\Gamma_c} \le c_t \|\varphi\|_{1,\Gamma_c} \quad \forall \varphi \in H^1(\Gamma_c;\mathbb{R}^3).$$

Proof. Since Γ_c is supposed to be a flat part of $\partial\Omega$, we may assume without loss of generality that $\Gamma_c \subset \mathbb{R}^2 \times \{0\}$ (otherwise, one can introduce an appropriate orthonormal transformation of coordinates). The proof is then straightforward. □

With these results at our disposal we shall show by using the Brouwer fixed-point theorem that Ψ_{hH} has at least one fixed point in the set

$$\mathscr{C}(R_1, R_2) := \left\{(\varphi_1^h, \varphi_2^h, \mu^H) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H \mid \|(\varphi_1^h, \varphi_2^h)\|_{0,\Gamma_c} \le R_1 \text{ and } \|\mu^H\|_{*,h} \le R_2\right\}$$

for appropriate $R_1, R_2 > 0$.

Lemma 3.3. *Let $\mathscr{F}$ satisfy (2.4). Then there exist $R_1, R_2 > 0$ such that Ψ_{hH} maps $Y_+^h \times Y_+^h \times \Lambda_\nu^H$ into $\mathscr{C}(R_1, R_2)$.*

Proof. Let $(\varphi_1^h, \varphi_2^h, g^H) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H$ be given and (u^h, λ_ν^H) be the solution to $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$. Inserting $v^h := 0, 2u^h \in K^{hH}$ into $(\mathscr{P}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ we get

$$a(u^h, u^h) + j(\varphi_1^h, \varphi_2^h, g^H, u_t^h) = \ell(u^h), \tag{3.4}$$

which together with the non-negativeness of j imply that

$$\|u^h\|_{1,\Omega} \le \frac{\|\ell\|_{*,\Omega}}{\alpha}. \tag{3.5}$$

Here $\|.\|_{*,\Omega}$ stands for the norm in the dual to $H^1(\Omega;\mathbb{R}^3)$ and α is the constant from (2.6). Invoking (3.3), Lemmas 3.1 and 3.2,

$$\begin{aligned}
\|(r_h|u_{t_1}^h|, r_h|u_{t_2}^h|)\|_{0,\Gamma_c} &\le \|(r_h|u_{t_1}^h| - |u_{t_1}^h|, r_h|u_{t_2}^h| - |u_{t_2}^h|)\|_{0,\Gamma_c} + \|(|u_{t_1}^h|, |u_{t_2}^h|)\|_{0,\Gamma_c} \\
&\overset{(3.3)}{\le} c_r h_{\Gamma_c} \|(|u_{t_1}^h|, |u_{t_2}^h|)\|_{1,\Gamma_c} + \|u_t^h\|_{0,\Gamma_c} \le c_r h_{\Gamma_c} \|u_t^h\|_{1,\Gamma_c} + \|u_t^h\|_{0,\Gamma_c} \\
&\le c_r c_t h_{\Gamma_c} \|u^h\|_{1,\Gamma_c} + \|u^h\|_{0,\Gamma_c} \le (c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1)\|u^h\|_{0,\Gamma_c} \\
&\le c_{\mathrm{tr}}^{(2)} (c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1)\|u^h\|_{1,\Omega},
\end{aligned} \tag{3.6}$$

where $c_{\mathrm{tr}}^{(2)}$ is the norm of the trace mapping from $H^1(\Omega;\mathbb{R}^3)$ into $L^2(\partial\Omega;\mathbb{R}^3)$ and $c_{\mathrm{inv}}^{(1,0)}$ is the constant from the inverse inequality between the $H^1(\Gamma_c;\mathbb{R}^3)$ and $L^2(\Gamma_c;\mathbb{R}^3)$-norms for functions belonging to the finite-dimensional space $Y^h \times Y^h \times Y^h$:

$$\|\psi^h\|_{1,\Gamma_c} \le \frac{c_{\mathrm{inv}}^{(1,0)}}{h_{\Gamma_c}} \|\psi^h\|_{0,\Gamma_c} \quad \forall \psi^h \in Y^h \times Y^h \times Y^h. \tag{3.7}$$

In view of (3.5) and (3.6), the radius R_1 is of the form

$$R_1 := R_1(c_{\mathrm{inv}}^{(1,0)}, c_r, c_{\mathrm{tr}}^{(2)}, c_t, \alpha, \ell) := \frac{c_{\mathrm{tr}}^{(2)} (c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1)}{\alpha} \|\ell\|_{*,\Omega}.$$

Furthermore, introducing the subspace

$$V_0^h := \{v^h \in V^h \mid v_t^h = 0 \text{ on } \Gamma_c\},$$

one can see from $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ and (3.4) that

$$a(u^h, v^h) + j(\varphi_1^h, \varphi_2^h, g^H, v_t^h) \ge \ell(v^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c} \quad \forall v^h \in V^h.$$

Thus

$$a(u^h, v^h) = \ell(v^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c} \quad \forall v^h \in V_0^h,$$

from which, (2.7) and (3.5),

$$\frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}} = \frac{\ell(v^h) - a(u^h, v^h)}{\|v^h\|_{1,\Omega}} \le \left(1 + \frac{M}{\alpha}\right)\|\ell\|_{*,\Omega} \quad \forall v^h \in V_0^h. \tag{3.8}$$

To complete the proof, we may assume without loss of generality that $\Gamma_c \subset \mathbb{R}^2 \times \{0\}$ (otherwise, one can introduce an orthonormal transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $A(\Gamma_c) \subset \mathbb{R}^2 \times \{0\}$ and proceed with Av^h). Let

$$V_{00}^h := \{v^h = (v_1^h, v_2^h, v_3^h) \in V^h \mid v_1^h = v_2^h = 0 \text{ in } \Omega\} \subset V_0^h.$$

Then one has

$$\|\lambda_\nu^H\|_{*,h} = \sup_{0\neq v^h\in V^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}} \le \sup_{0\neq v^h\in V^h} \frac{(\lambda_\nu^H, v_3^h)_{0,\Gamma_c}}{\|v_3^h\|_{1,\Omega}} = \sup_{0\neq v^h\in V_{00}^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}} \le \sup_{0\neq v^h\in V_0^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}}.$$

From this and (3.8), we see that one can take

$$R_2 := R_2(M, \alpha, \ell) := \left(1 + \frac{M}{\alpha}\right)\|\ell\|_{*,\Omega}. \quad \square$$

Remark 3.3. Let us notice that at this moment the partitions $\mathcal{T}_\Omega^h$ and $\mathcal{T}_{\Gamma_c}^H$ are fixed and the constants c_r and $c_{\text{inv}}^{(1,0)}$ in (3.3) and (3.7), respectively, may depend on h. Later on we shall consider $\mathcal{T}_\Omega^h$ and $\mathcal{T}_{\Gamma_c}^H$ as elements of systems $\{\mathcal{T}_\Omega^h\}$, $\{\mathcal{T}_{\Gamma_c}^H\}$, $h, H \to 0+$, and we shall formulate conditions on these systems under which the constants do not depend on h.

Lemma 3.4. *The mapping Ψ_{hH} is continuous in $Y_+^h \times Y_+^h \times \Lambda_\nu^H$ provided that (2.4) is satisfied.*

Proof. Let $(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}), (\varphi_1^h, \varphi_2^h, g^H) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H$, $k \in \mathbb{N}$, be such that

$$(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}) \to (\varphi_1^h, \varphi_2^h, g^H) \quad \text{in } Y^h \times Y^h \times L^H,\ k \to +\infty,$$

and $(u^{h,k}, \lambda_\nu^{H,k})$ be the respective solutions to $(\mathscr{M}_{hH}(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}))$:

$$\left.\begin{array}{l} a(u^{h,k}, v^h - u^{h,k}) + j(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}, v_t^h) - j(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}, u_t^{h,k}) \\ \quad \ge \ell(v^h - u^{h,k}) - (\lambda_\nu^{H,k}, v_\nu^h - u_\nu^{h,k})_{0,\Gamma_c} \quad \forall v^h \in V^h, \\ (\mu_\nu^H - \lambda_\nu^{H,k}, u_\nu^{h,k})_{0,\Gamma_c} \le 0 \quad \forall \mu_\nu^H \in \Lambda_\nu^H. \end{array}\right\}$$

As we know, both sequences $\{u^{h,k}\}$ and $\{\lambda_\nu^{H,k}\}$ are bounded. Thus one can find $\{u^{h,k_l}\} \subset \{u^{h,k}\}$, $\{\lambda_\nu^{H,k_l}\} \subset \{\lambda_\nu^{H,k}\}$ and $u^h \in V^h$, $\lambda_\nu^H \in \Lambda_\nu^H$ such that

$$u^{h,k_l} \to u^h \quad \text{in } V^h, \qquad \lambda_\nu^{H,k_l} \to \lambda_\nu^H \quad \text{in } L^H,\ l \to +\infty.$$

Let $v^h \in V^h$ and $\mu_\nu^H \in \Lambda_\nu^H$ be arbitrarily chosen. Taking into account the equivalences of all norms in the finite-dimensional spaces involved, one can easily verify that

$$\begin{aligned} &a(u^{h,k_l}, v^h - u^{h,k_l}) - \ell(v^h - u^{h,k_l}) + (\lambda_\nu^{H,k_l}, v_\nu^h - u_\nu^{h,k_l})_{0,\Gamma_c} \overset{l\to+\infty}{\longrightarrow} a(u^h, v^h - u^h) - \ell(v^h - u^h) + (\lambda_\nu^H, v_\nu^h - u_\nu^h)_{0,\Gamma_c}, \\ &j(\varphi_1^{h,k_l}, \varphi_2^{h,k_l}, g^{H,k_l}, v_t^h) - j(\varphi_1^{h,k_l}, \varphi_2^{h,k_l}, g^{H,k_l}, u_t^{h,k_l}) \overset{l\to+\infty}{\longrightarrow} j(\varphi_1^h, \varphi_2^h, g^H, v_t^h) - j(\varphi_1^h, \varphi_2^h, g^H, u_t^h), \\ &(\mu_\nu^H - \lambda_\nu^{H,k_l}, u_\nu^{h,k_l})_{0,\Gamma_c} \overset{l\to+\infty}{\longrightarrow} (\mu_\nu^H - \lambda_\nu^H, u_\nu^h)_{0,\Gamma_c}, \end{aligned}$$

which shows that (u^h, λ_ν^H) solves $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$. Since this problem admits a unique solution, the original sequences $\{u^{h,k}\}$, $\{\lambda_\nu^{H,k}\}$ tend to u^h and λ_ν^H.

Furthermore, from the positivity preserving assumption (3.2) and the linearity of r_h it is readily seen that

$$|r_h(|u_{t_i}^{h,k}| - |u_{t_i}^h|)| \le r_h|u_{t_i}^{h,k} - u_{t_i}^h| \quad \text{on } \Gamma_c,\ 1 \le i \le 2,\ k \in \mathbb{N}.$$

Therefore, arguing as in (3.6) one gets

$$\begin{aligned} \|(r_h|u_{t_1}^{h,k}|, r_h|u_{t_2}^{h,k}|) - (r_h|u_{t_1}^h|, r_h|u_{t_2}^h|)\|_{0,\Gamma_c} &\le \|(r_h|u_{t_1}^{h,k} - u_{t_1}^h|, r_h|u_{t_2}^{h,k} - u_{t_2}^h|)\|_{0,\Gamma_c} \\ &\le c_{\text{tr}}^{(2)}(c_{\text{inv}}^{(1,0)} c_r c_t + 1)\|u^{h,k} - u^h\|_{1,\Omega}, \quad k \in \mathbb{N}, \end{aligned} \tag{3.9}$$

and the limit passage $k \to +\infty$ completes the proof. $\square$

We have arrived at the following existence result.

Theorem 3.1. *If (2.4) is fulfilled then the discrete problem given by Definition 3.1 has at least one solution.*

3.2. Uniqueness result

Applying the Banach fixed-point theorem, even uniqueness of the discrete solution can be ensured. Nevertheless, to establish the Lipschitz continuity of Ψ_{hH}, we shall need an additional assumption on $\mathscr{F}$, namely:

$$\exists L > 0: \ |\mathscr{F}_i(x,\xi) - \mathscr{F}_i(x,\bar{\xi})| \le L\|\xi - \bar{\xi}\| \quad \forall x \in \Gamma_c \ \forall \xi, \bar{\xi} \in \mathbb{R}^2_+, \ 1 \le i \le 2. \tag{3.10}$$

We start with a useful technical result.

Lemma 3.5. *If $\mathscr{F}$ satisfies (2.4) and (3.10) then it holds for any $u^h, \bar{u}^h \in V^h$ and any $(\varphi_1^h, \varphi_2^h), (\bar{\varphi}_1^h, \bar{\varphi}_2^h) \in Y_+^h \times Y_+^h$ that*

$$\begin{aligned}&\big|\|\mathscr{F}(\varphi_1^h,\varphi_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\varphi_1^h,\varphi_2^h)u_t^h\| - (\|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)u_t^h\|)\big| \\ &\quad \le L(2+\kappa(\mathscr{F}))\|(\varphi_1^h,\varphi_2^h) - (\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|\,\|u_t^h - \bar{u}_t^h\| \quad \text{on } \Gamma_c,\end{aligned} \tag{3.11}$$

where

$$\kappa(\mathscr{F}) := \sup_{\substack{x\in\Gamma_c \\ \xi\in\mathbb{R}^2_+}} \|\mathscr{F}(x,\xi)\|\,\|\mathscr{F}^{-1}(x,\xi)\| = \sup_{\substack{x\in\Gamma_c \\ \xi\in\mathbb{R}^2_+}} \frac{\max\{\mathscr{F}_1(x,\xi),\mathscr{F}_2(x,\xi)\}}{\min\{\mathscr{F}_1(x,\xi),\mathscr{F}_2(x,\xi)\}}.$$

Proof. For $x \in \Gamma_c$, $u^h, \bar{u}^h \in V^h$ and $(\varphi_1^h,\varphi_2^h), (\bar{\varphi}_1^h,\bar{\varphi}_2^h) \in Y_+^h \times Y_+^h$ given, set

$$\begin{aligned}&\boldsymbol{u} := u_t^h(x), \qquad \bar{\boldsymbol{u}} := \bar{u}_t^h(x), \\ &\boldsymbol{\phi} = (\phi_1,\phi_2) := (\varphi_1^h(x),\varphi_2^h(x)), \qquad \bar{\boldsymbol{\phi}} = (\bar{\phi}_1,\bar{\phi}_2) := (\bar{\varphi}_1^h(x),\bar{\varphi}_2^h(x))\end{aligned}$$

and define the function $h := G\circ F\circ H : \mathbb{R}\to\mathbb{R}$ with $H:\mathbb{R}\to\mathbb{R}^2$, $F:\mathbb{R}^2\to\mathbb{R}^2$, $G:\mathbb{R}^2\to\mathbb{R}$ introduced as follows:

$$H(r) = \bar{\boldsymbol{\phi}} + r(\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}), \quad r\in\mathbb{R},$$

$$F(\xi_1,\xi_2) := (F_1(\xi_1,\xi_2),F_2(\xi_1,\xi_2)) = \begin{cases} (\mathscr{F}_1(x,\xi_1,\xi_2),\mathscr{F}_2(x,\xi_1,\xi_2)) & \text{if } \xi_1,\xi_2 \ge 0, \\ (\mathscr{F}_1(x,\xi_1,0),\mathscr{F}_2(x,\xi_1,0)) & \text{if } \xi_1 \ge 0 > \xi_2, \\ (\mathscr{F}_1(x,0,\xi_2),\mathscr{F}_2(x,0,\xi_2)) & \text{if } \xi_2 \ge 0 > \xi_1, \\ (\mathscr{F}_1(x,0,0),\mathscr{F}_2(x,0,0)) & \text{if } 0 > \xi_1,\xi_2, \end{cases}$$

$$G(\xi_1,\xi_2) = \|\operatorname{Diag}\{\xi_1,\xi_2\}\bar{\boldsymbol{u}}\| - \|\operatorname{Diag}\{\xi_1,\xi_2\}\boldsymbol{u}\|, \quad (\xi_1,\xi_2)\in\mathbb{R}^2.$$

Obviously, h is Lipschitz continuous in $\mathbb{R}$ and the left-hand side of (3.11) at the point x equals $|h(1)-h(0)|$. From the Lebourg mean-value theorem it follows that there exists $\bar{r}\in(0,1)$ such that

$$h(1) - h(0) \in \partial h(\bar{r}),$$

where ∂h denotes the Clarke subdifferential of h (see [20]). So it suffices to estimate $|\theta|$ for any $\theta\in\partial h(r)$ and any $r\in(0,1)$ fixed.

Due to the continuous differentiability of H at r and G at $F(H(r))$, Chain Rule II for the Clarke subdifferential ∂h and the chain rule for $\partial(G\circ F)$ viewed as the generalized Jacobian imply that

$$\begin{aligned}&\partial h(r) \subset (\nabla H(r))^T \partial(G\circ F)(H(r)), \\ &\partial(G\circ F)(H(r)) = (\partial F(H(r)))^T \nabla G(F(H(r)))\end{aligned}$$

so that $\theta\in\partial h(r)$ is of the form

$$\theta = (\nabla H(r))^T \boldsymbol{Z}^T \nabla G(F(H(r)))$$

for some $\boldsymbol{Z} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix} \in \partial F(H(r))$.

Suppose first that $\boldsymbol{u}, \bar{\boldsymbol{u}} \ne \boldsymbol{0}$. If it is so then

$$\begin{aligned}&(\nabla H(r))^T \boldsymbol{Z}^T = \big((\phi_1-\bar{\phi}_1)z_{11} + (\phi_2-\bar{\phi}_2)z_{12}, (\phi_1-\bar{\phi}_1)z_{21} + (\phi_2-\bar{\phi}_2)z_{22}\big), \\ &(\zeta_1,\zeta_2)\nabla G(\xi_1,\xi_2) = \frac{\operatorname{Diag}\{\xi_1,\xi_2\}\bar{\boldsymbol{u}}\cdot\operatorname{Diag}\{\zeta_1,\zeta_2\}\bar{\boldsymbol{u}}}{\|\operatorname{Diag}\{\xi_1,\xi_2\}\bar{\boldsymbol{u}}\|} - \frac{\operatorname{Diag}\{\xi_1,\xi_2\}\boldsymbol{u}\cdot\operatorname{Diag}\{\zeta_1,\zeta_2\}\boldsymbol{u}}{\|\operatorname{Diag}\{\xi_1,\xi_2\}\boldsymbol{u}\|}\end{aligned}$$

and consequently,

$$\theta = \frac{\boldsymbol{F}\bar{\boldsymbol{u}}\cdot\boldsymbol{S}\bar{\boldsymbol{u}}}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\|} - \frac{\boldsymbol{F}\boldsymbol{u}\cdot\boldsymbol{S}\boldsymbol{u}}{\|\boldsymbol{F}\boldsymbol{u}\|}$$

with

$$\begin{aligned}&\boldsymbol{F} := \operatorname{Diag}\{F_1(\bar{\boldsymbol{\phi}} + r(\boldsymbol{\phi}-\bar{\boldsymbol{\phi}})), F_2(\bar{\boldsymbol{\phi}} + r(\boldsymbol{\phi}-\bar{\boldsymbol{\phi}}))\}, \\ &\boldsymbol{S} := \operatorname{Diag}\{(\phi_1-\bar{\phi}_1)z_{11} + (\phi_2-\bar{\phi}_2)z_{12}, (\phi_1-\bar{\phi}_1)z_{21} + (\phi_2-\bar{\phi}_2)z_{22}\}.\end{aligned}$$

Clearly,

$$|\theta| \le \left| \frac{\boldsymbol{F}\bar{\boldsymbol{u}} \cdot \boldsymbol{S}(\bar{\boldsymbol{u}} - \boldsymbol{u})}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\|} \right| + \left| \frac{\boldsymbol{F}\bar{\boldsymbol{u}} \cdot \boldsymbol{S}\boldsymbol{u}}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\|} - \frac{\boldsymbol{F}\bar{\boldsymbol{u}} \cdot \boldsymbol{S}\boldsymbol{u}}{\|\boldsymbol{F}\boldsymbol{u}\|} \right| + \left| \frac{\boldsymbol{F}(\bar{\boldsymbol{u}} - \boldsymbol{u}) \cdot \boldsymbol{S}\boldsymbol{u}}{\|\boldsymbol{F}\boldsymbol{u}\|} \right| =: s_1 + s_2 + s_3.$$

In virtue of the inequality $\|\boldsymbol{u}\| \le \|\boldsymbol{F}^{-1}\| \|\boldsymbol{F}\boldsymbol{u}\|$ and the fact that both $\boldsymbol{F}$ and $\boldsymbol{S}$ are diagonal matrices, one has

$$s_1 \le \frac{\|\boldsymbol{F}\bar{\boldsymbol{u}}\| \, \|\boldsymbol{S}(\bar{\boldsymbol{u}} - \boldsymbol{u})\|}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\|} \le \|\boldsymbol{S}\| \, \|\bar{\boldsymbol{u}} - \boldsymbol{u}\|,$$

$$s_2 = \left| \frac{(\boldsymbol{F}\bar{\boldsymbol{u}} \cdot \boldsymbol{S}\boldsymbol{u})(\|\boldsymbol{F}\boldsymbol{u}\| - \|\boldsymbol{F}\bar{\boldsymbol{u}}\|)}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\| \, \|\boldsymbol{F}\boldsymbol{u}\|} \right| \le \frac{\|\boldsymbol{F}\bar{\boldsymbol{u}}\| \, \|\boldsymbol{S}\| \, \|\boldsymbol{u}\| \, \|\boldsymbol{F}\boldsymbol{u} - \boldsymbol{F}\bar{\boldsymbol{u}}\| \, \|\boldsymbol{F}^{-1}\|}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\| \, \|\boldsymbol{u}\|} \le \kappa(\mathscr{F}) \|\boldsymbol{S}\| \, \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|,$$

$$s_3 = \left| \frac{\boldsymbol{S}(\bar{\boldsymbol{u}} - \boldsymbol{u}) \cdot \boldsymbol{F}\boldsymbol{u}}{\|\boldsymbol{F}\boldsymbol{u}\|} \right| \le \|\boldsymbol{S}\| \, \|\bar{\boldsymbol{u}} - \boldsymbol{u}\|.$$

Furthermore, let $\boldsymbol{z}_i$ denote the ith row vector of $\boldsymbol{Z}$. Then $\|\boldsymbol{z}_i\| \le L$ because $\boldsymbol{z}_i \in \partial F_i(H(r))$ and the Lipschitz modulus of F_i is less than or equal to L by (3.10). Thus,

$$\|\boldsymbol{S}\| = \max_{1 \le i \le 2} \{|(\phi_1 - \bar{\phi}_1) z_{i1} + (\phi_2 - \bar{\phi}_2) z_{i2}|\} \le \max_{1 \le i \le 2} \|\boldsymbol{z}_i\| \, \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}\| \le L \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}\|.$$

Combining the previous estimates we get:

$$|\theta| \le L(2 + \kappa(\mathscr{F})) \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}\| \, \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|. \tag{3.12}$$

To complete the assertion, let $\boldsymbol{u} = \boldsymbol{0} \ne \bar{\boldsymbol{u}}$. In this case,

$$|\theta| = \left| \frac{\boldsymbol{F}\bar{\boldsymbol{u}} \cdot \boldsymbol{S}\bar{\boldsymbol{u}}}{\|\boldsymbol{F}\bar{\boldsymbol{u}}\|} \right| \le \|\boldsymbol{S}\| \, \|\bar{\boldsymbol{u}} - \boldsymbol{0}\| \le L \|\boldsymbol{\phi} - \bar{\boldsymbol{\phi}}\| \, \|\bar{\boldsymbol{u}} - \boldsymbol{u}\|,$$

i.e. (3.12) holds as well and so it is for $\bar{\boldsymbol{u}} = \boldsymbol{0}$. □

Proposition 3.1. *Let (2.4) and (3.10) be satisfied. For any $R_1, R_2 > 0$, Ψ_{hH} is Lipschitz continuous in $\mathscr{C}(R_1, R_2)$:*

$$\begin{aligned} &\exists C_1, C_2 > 0 : \ \|\Psi_{hH}(\varphi_1^h, \varphi_2^h, g^H) - \Psi_{hH}(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{Y^h \times Y^h \times L^H} \\ &\le \max\left\{ \frac{\mathscr{F}_{\max}}{\sqrt{H}} C_1, \frac{L(2 + \kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_c} H}} C_2 R_2 \right\} \|(\varphi_1^h, \varphi_2^h, g^H) - (\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{Y^h \times Y^h \times L^H} \\ &\quad \forall (\varphi_1^h, \varphi_2^h, g^H), (\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H) \in \mathscr{C}(R_1, R_2). \end{aligned} \tag{3.13}$$

Proof. For $(\varphi_1^h, \varphi_2^h, g^H), (\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H) \in \mathscr{C}(R_1, R_2)$ denote by $(u^h, \lambda_\nu^H), (\bar{u}^h, \bar{\lambda}_\nu^H)$ the solutions to $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ and $(\mathscr{M}_{hH}(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H))$, respectively. Inserting $v^h := \bar{u}^h \in K^{hH}$ and $v^h := u^h \in K^{hH}$ into $(\mathscr{P}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ and $(\mathscr{P}_{hH}(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H))$, respectively, we have:

$$\begin{aligned} &a(u^h, \bar{u}^h - u^h) + j(\varphi_1^h, \varphi_2^h, g^H, \bar{u}_t^h) - j(\varphi_1^h, \varphi_2^h, g^H, u_t^h) \ge \ell(\bar{u}^h - u^h), \\ &a(\bar{u}^h, u^h - \bar{u}^h) + j(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H, u_t^h) - j(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H, \bar{u}_t^h) \ge \ell(u^h - \bar{u}^h). \end{aligned}$$

Summing both inequalities and using (2.6) we arrive at

$$\begin{aligned} \alpha \|u^h - \bar{u}^h\|_{1,\Omega}^2 &\le a(u^h - \bar{u}^h, u^h - \bar{u}^h) \\ &\le j(\varphi_1^h, \varphi_2^h, g^H, \bar{u}_t^h) - j(\varphi_1^h, \varphi_2^h, g^H, u_t^h) + j(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H, u_t^h) - j(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H, \bar{u}_t^h) \\ &= \big(g^H, \|\mathscr{F}(\varphi_1^h, \varphi_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\varphi_1^h, \varphi_2^h)u_t^h\|\big)_{0,\Gamma_c} - \big(\bar{g}^H, \|\mathscr{F}(\bar{\varphi}_1^h, \bar{\varphi}_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\bar{\varphi}_1^h, \bar{\varphi}_2^h)u_t^h\|\big)_{0,\Gamma_c} \\ &= \big(g^H - \bar{g}^H, \|\mathscr{F}(\varphi_1^h, \varphi_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\varphi_1^h, \varphi_2^h)u_t^h\|\big)_{0,\Gamma_c} \\ &\quad + \big(\bar{g}^H, \|\mathscr{F}(\varphi_1^h, \varphi_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\varphi_1^h, \varphi_2^h)u_t^h\| - (\|\mathscr{F}(\bar{\varphi}_1^h, \bar{\varphi}_2^h)\bar{u}_t^h\| - \|\mathscr{F}(\bar{\varphi}_1^h, \bar{\varphi}_2^h)u_t^h\|)\big)_{0,\Gamma_c} \\ &=: s_1 + s_2. \end{aligned} \tag{3.14}$$

The first term can be estimated as follows:

$$\begin{aligned} s_1 &\le \|g^H - \bar{g}^H\|_{0,\Gamma_c} \big\| \, \|\mathscr{F}(\varphi_1^h, \varphi_2^h)\bar{u}_t^h - \mathscr{F}(\varphi_1^h, \varphi_2^h)u_t^h\| \, \big\|_{0,\Gamma_c} = \|g^H - \bar{g}^H\|_{0,\Gamma_c} \|\mathscr{F}(\varphi_1^h, \varphi_2^h)(\bar{u}_t^h - u_t^h)\|_{0,\Gamma_c} \\ &\le \mathscr{F}_{\max} \|g^H - \bar{g}^H\|_{0,\Gamma_c} \|\bar{u}^h - u^h\|_{0,\Gamma_c} \le \frac{\mathscr{F}_{\max}}{\sqrt{H}} c_{\text{inv}}^{(0,-1/2)} c_{\text{tr}}^{(2)} \|g^H - \bar{g}^H\|_{*,h} \|\bar{u}^h - u^h\|_{1,\Omega}, \end{aligned} \tag{3.15}$$

where $c_{\mathrm{tr}}^{(2)}$ is the norm of the trace mapping from $H^1(\Omega;\mathbb{R}^3)$ into $L^2(\partial\Omega;\mathbb{R}^3)$ and $c_{\mathrm{inv}}^{(0,-1/2)}$ is the constant from the equivalence of the corresponding norms in the finite-dimensional space L^H:

$$\|\mu^H\|_{0,\Gamma_c} \le \frac{c_{\mathrm{inv}}^{(0,-1/2)}}{\sqrt{H}}\|\mu^H\|_{*,h} \quad \forall \mu^H \in L^H. \tag{3.16}$$

Further, from the previous lemma,

$$\begin{aligned} s_2 &\le L(2+\kappa(\mathscr{F}))\|\bar{g}^H\|_{0,\Gamma_c}\big\|\,\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|\,\|u_t^h-\bar{u}_t^h\|\big\|_{0,\Gamma_c}\\ &\le L(2+\kappa(\mathscr{F}))\|\bar{g}^H\|_{0,\Gamma_c}\|u^h-\bar{u}^h\|_{0,\infty,\Gamma_c}\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_c}. \end{aligned}$$

Due to the equivalence of norms in $Y^h\times Y^h\times Y^h$, namely:

$$\|\psi^h\|_{0,\infty,\Gamma_c} \le \frac{c_{\mathrm{inv}}^{(\infty)}}{\sqrt{h_{\Gamma_c}}}\|\psi^h\|_{0,4,\Gamma_c} \quad \forall \psi^h \in Y^h\times Y^h\times Y^h \tag{3.17}$$

with an appropriate $c_{\mathrm{inv}}^{(\infty)} > 0$, and the continuity of the trace mapping from $H^1(\Omega;\mathbb{R}^3)$ into $L^4(\partial\Omega;\mathbb{R}^3)$, whose norm is denoted by $c_{\mathrm{tr}}^{(4)}$, one obtains:

$$\|u^h-\bar{u}^h\|_{0,\infty,\Gamma_c} \le \frac{c_{\mathrm{inv}}^{(\infty)}c_{\mathrm{tr}}^{(4)}}{\sqrt{h_{\Gamma_c}}}\|u^h-\bar{u}^h\|_{1,\Omega}.$$

Using (3.16) once again, we get:

$$\|\bar{g}^H\|_{0,\Gamma_c} \le \frac{c_{\mathrm{inv}}^{(0,-1/2)}}{\sqrt{H}}\|\bar{g}^H\|_{*,h} \le \frac{c_{\mathrm{inv}}^{(0,-1/2)}}{\sqrt{H}}R_2,$$

making use of the definition of $\mathscr{C}(R_1,R_2)$. Therefore

$$s_2 \le \frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_c}H}}c_{\mathrm{inv}}^{(0,-1/2)}c_{\mathrm{inv}}^{(\infty)}c_{\mathrm{tr}}^{(4)}R_2\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_c}\|u^h-\bar{u}^h\|_{1,\Omega}. \tag{3.18}$$

The inequality (3.14) together with (3.15) and (3.18) implies that

$$\begin{aligned} \|u^h-\bar{u}^h\|_{1,\Omega} &\le \frac{\mathscr{F}_{\max}}{\sqrt{H}}\tilde{C}_1\|g^H-\bar{g}^H\|_{*,h}+\frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_c}H}}\tilde{C}_2R_2\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_c}\\ &\le \max\left\{\frac{\mathscr{F}_{\max}}{\sqrt{H}}\tilde{C}_1,\frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_c}H}}\tilde{C}_2R_2\right\}\|(\varphi_1^h,\varphi_2^h,g^H)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H)\|_{Y^h\times Y^h\times L^H} \end{aligned}$$

with

$$\begin{aligned} \tilde{C}_1 &:= \tilde{C}_1(c_{\mathrm{inv}}^{(0,-1/2)},c_{\mathrm{tr}}^{(2)},\alpha) := \frac{c_{\mathrm{inv}}^{(0,-1/2)}c_{\mathrm{tr}}^{(2)}}{\alpha},\\ \tilde{C}_2 &:= \tilde{C}_2(c_{\mathrm{inv}}^{(0,-1/2)},c_{\mathrm{inv}}^{(\infty)},c_{\mathrm{tr}}^{(4)},\alpha) := \frac{c_{\mathrm{inv}}^{(0,-1/2)}c_{\mathrm{inv}}^{(\infty)}c_{\mathrm{tr}}^{(4)}}{\alpha}. \end{aligned}$$

Following the steps in (3.9) one can see that

$$\|(r_h|u_{t_1}^h|,r_h|u_{t_2}^h|)-(r_h|\bar{u}_{t_1}^h|,r_h|\bar{u}_{t_2}^h|)\|_{0,\Gamma_c} \le c_{\mathrm{tr}}^{(2)}(c_{\mathrm{inv}}^{(1,0)}c_rc_t+1)\|u^h-\bar{u}^h\|_{1,\Omega}.$$

Finally, the Lagrange multipliers are treated similarly as in the proof of Lemma 3.3. The relations

$$\begin{aligned} a(u^h,v^h) &= \ell(v^h)-(\lambda_\nu^H,v_\nu^h)_{0,\Gamma_c} \quad \forall v^h\in V_0^h,\\ a(\bar{u}^h,v^h) &= \ell(v^h)-(\bar{\lambda}_\nu^H,v_\nu^h)_{0,\Gamma_c} \quad \forall v^h\in V_0^h \end{aligned}$$

give

$$\begin{aligned} &(\lambda_\nu^H-\bar{\lambda}_\nu^H,v_\nu^h)_{0,\Gamma_c} = a(\bar{u}^h-u^h,v^h) \quad \forall v^h\in V_0^h,\\ &\|\lambda_\nu^H-\bar{\lambda}_\nu^H\|_{*,h} = \sup_{0\ne v^h\in V^h}\frac{(\lambda_\nu^H-\bar{\lambda}_\nu^H,v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}} \le \sup_{0\ne v^h\in V_0^h}\frac{(\lambda_\nu^H-\bar{\lambda}_\nu^H,v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}}\\ &\qquad = \sup_{0\ne v^h\in V_0^h}\frac{a(\bar{u}^h-u^h,v^h)}{\|v^h\|_{1,\Omega}} \le M\|u^h-\bar{u}^h\|_{1,\Omega}. \end{aligned}$$

Thus, setting

$$C_1 := C_1(c_{\mathrm{inv}}^{(0,-1/2)}, c_{\mathrm{inv}}^{(1,0)}, c_r, c_{\mathrm{tr}}^{(2)}, c_t, M, \alpha) := (c_{\mathrm{tr}}^{(2)}(c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1) + M)\tilde{C}_1,$$
$$C_2 := C_2(c_{\mathrm{inv}}^{(0,-1/2)}, c_{\mathrm{inv}}^{(1,0)}, c_{\mathrm{inv}}^{(\infty)}, c_r, c_{\mathrm{tr}}^{(2)}, c_{\mathrm{tr}}^{(4)}, c_t, M, \alpha) := (c_{\mathrm{tr}}^{(2)}(c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1) + M)\tilde{C}_2,$$

we have:

$$\begin{aligned}
&\|\Psi_{hH}(\varphi_1^h, \varphi_2^h, g^H) - \Psi_{hH}(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{Y^h \times Y^h \times L^H} \\
&\quad = \|(r_h|u_{t_1}^h|, r_h|u_{t_2}^h|) - (r_h|\bar{u}_{t_1}^h|, r_h|\bar{u}_{t_2}^h|)\|_{0,\Gamma_c} + \|\lambda_\nu^H - \bar{\lambda}_\nu^H\|_{*,h} \\
&\quad \le (c_{\mathrm{tr}}^{(2)}(c_{\mathrm{inv}}^{(1,0)} c_r c_t + 1) + M)\|u^h - \bar{u}^h\|_{1,\Omega} \\
&\quad \le \max\left\{\frac{\mathscr{F}_{\max}}{\sqrt{H}} C_1, \frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_c} H}} C_2 R_2\right\} \|(\varphi_1^h, \varphi_2^h, g^H) - (\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{Y^h \times Y^h \times L^H}. \quad \square
\end{aligned}$$

Choosing R_1 and R_2 from Lemma 3.3, we obtain the following uniqueness result.

Theorem 3.2. *Let (2.4) and (3.10) be satisfied and $\mathscr{F}_{\max}$ and L be sufficiently small. Then the solution of our problem in the sense of Definition 3.1 is unique. In addition, it is the limit of the sequence generated by* the method of successive approximations*:*

$$\left.\begin{array}{l} \text{Let } (\varphi_1^{h,0}, \varphi_2^{h,0}, g^{H,0}) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H \text{ be given;} \\ \text{for } k = 0, 1, \ldots \text{ set} \\ \quad (\varphi_1^{h,k+1}, \varphi_2^{h,k+1}, g^{H,k+1}) := \Psi_{hH}(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}); \end{array}\right\}$$

for any choice of $(\varphi_1^{h,0}, \varphi_2^{h,0}, g^{H,0}) \in Y_+^h \times Y_+^h \times \Lambda_\nu^H$.

Proof. Consider R_1 and R_2 given by Lemma 3.3. In view of (3.13), Ψ_{hH} is contractive in $\mathscr{C}(R_1, R_2)$ for $\mathscr{F}_{\max}$ and L sufficiently small. The assertion now follows from the Banach fixed-point theorem. $\square$

So far, we have assumed that the partitions $\mathscr{T}_\Omega^h$ and $\mathscr{T}_{\Gamma_c}^H$ are fixed and the constants $c_{\mathrm{inv}}^{(0,-1/2)}$, $c_{\mathrm{inv}}^{(1,0)}$, $c_{\mathrm{inv}}^{(\infty)}$ and c_r may eventually depend on h and H. In what follows, we present sufficient conditions under which these constants do not depend on the mesh norms. To this end we shall consider systems of partitions $\{\mathscr{T}_\Omega^h\}$ and $\{\mathscr{T}_{\Gamma_c}^H\}$ for $h, H \to 0+$. We shall suppose that:

(i) $\{\mathscr{T}_{\Omega|_{\overline{\Gamma}_c}}^h\}$ and $\{\mathscr{T}_{\Gamma_c}^H\}$, $h, H \to 0+$, are regular systems of partitions of $\overline{\Gamma}_c$ which satisfy the so-called inverse assumption [21, (3.2.28)];

(ii) the Babuška–Brezzi condition is satisfied for (V^h, L^H):

$$\exists \beta > 0: \quad \sup_{0 \ne v^h \in V^h} \frac{(\mu^H, v_\nu^h)_{0,\Gamma_c}}{\|v^h\|_{1,\Omega}} \ge \beta \|\mu^H\|_{*,\Gamma_c} \quad \forall \mu^H \in L^H \ \forall h, H \to 0+,$$

where $\|.\|_{*,\Gamma_c}$ is the dual norm in X_ν' (recall that the duality pairing between X_ν and X_ν' is realized by the $L^2(\Gamma_c)$-scalar product in our case):

$$\|\mu^H\|_{*,\Gamma_c} = \sup_{0 \ne \varphi \in X_\nu} \frac{(\mu^H, \varphi)_{0,\Gamma_c}}{\|\varphi\|_{X_\nu}}, \quad \mu^H \in L^H \ \forall H \to 0+;$$

(iii) the interpolation operator r_h is such that c_r in (3.3) does not depend on h_{Γ_c}.

From (ii) it is readily seen that

$$\beta \|\mu^H\|_{*,\Gamma_c} \le \|\mu^H\|_{*,h} \le \|\mu^H\|_{*,\Gamma_c} \quad \forall \mu^H \in L^H \ \forall h, H \to 0+,$$

which means that the mesh-dependent norm $\|.\|_{*,h}$ can be replaced by the dual norm $\|.\|_{*,\Gamma_c}$ in all the previous estimates. In addition, taking (i) into account, the constants from the inverse inequalities (3.7), (3.16) and (3.17) are independent of h_{Γ_c}, H (see [21]). For this reason, neither R_1, R_2 from Lemma 3.3, nor C_1, C_2 from Proposition 3.1 depend on h_{Γ_c}, H.

Remark 3.4. Let (i)–(iii) hold and $\kappa(\mathscr{F})$ be bounded. To guarantee the uniqueness of the discrete solutions for $h, H \to 0+$, the parameters $\mathscr{F}_{\max}$ and L have to decay at least as fast as $\sqrt{H}$ and $\sqrt{h_{\Gamma_c} H}$, respectively.

Notice that if $\mathscr{F}_1$ coincides with $\mathscr{F}_2$, i.e. $\kappa(\mathscr{F}) = 1$, orthotropic friction reduces to isotropic one. The latter model has been studied already in [22], where a stronger condition on the decay of L was derived, namely $L \sim h_{\Gamma_c}\sqrt{H}$. On the other hand if $\mathscr{F}$ does not depend on u, i.e. $L = 0$, the classical result from [4] is recovered.

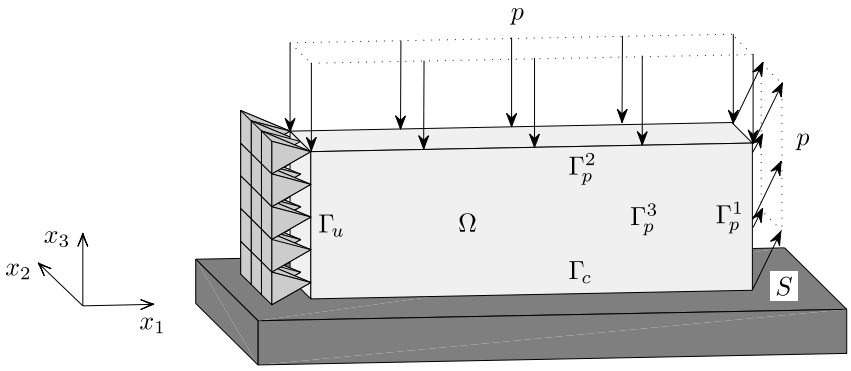

Fig. 2. Geometry.

Let us briefly comment on the satisfaction of the Babuška–Brezzi condition in (ii). It is shown in [23] that it is satisfied for (FE1) if $k = l = 1$. In the case of (FE2), (ii) is satisfied provided that the ratio H/h is sufficiently large and the auxiliary linear elasticity problem:

$$\left.\begin{array}{l}\text{Find } w_\mu \in V \text{ such that}\\ a(w_\mu, v) = \langle \mu, v_\nu\rangle_\nu \quad \forall v \in V\end{array}\right\}$$

is regular in the following sense: there exists $\varepsilon > 0$ such that for every $\mu \in X_\nu' \cap H^{-1/2+\varepsilon}(\Gamma_c)$, the solution $w_\mu \in V$ belongs to $H^{1+\varepsilon}(\Omega;\mathbb{R}^3)$ and

$$\|w_\mu\|_{1+\varepsilon,\Omega} \le c(\varepsilon)\|\mu\|_{-1/2+\varepsilon,\Gamma_c}$$

holds with a constant $c(\varepsilon)$ depending solely on ε (see [18]).

Finally, let us refer to an example of the interpolation operator r_h satisfying (3.2) and (3.3) with the constant c_r independent of h_{Γ_c}. To this end, let Γ_c be polygonal and $\overline{\Gamma}_c \cap \overline{\Gamma}_u$ be either empty or a union of non-degenerate segments, i.e. containing no isolated points. Moreover, let $\{\mathcal{T}^h_{\Omega|_{\overline{\Gamma}_c}}\}$, $h \to 0+$, be a regular system of triangulations of $\overline{\Gamma}_c$ such that any two triangles from $\mathcal{T}^h_{\Omega|_{\overline{\Gamma}_c}}$ are either disjoint, or have a vertex or a whole side in common. If we still suppose that $\{\mathcal{T}^h_\Omega\}$ is compatible with the decomposition of $\partial\Omega$ into Γ_u, Γ_p and Γ_c then we can take the following Clément interpolation operator [24] (with $k = 1$)[1]:

Let $\{x^i\}_{i\in I_c}$ be the set of all contact nodes of $\mathcal{T}^h_\Omega$, i.e. the nodes of $\mathcal{T}^h_\Omega$ lying on $\overline{\Gamma}_c \setminus \overline{\Gamma}_u$, and $\{\varphi_i\}_{i\in I_c}$ be the corresponding Courant basis of Y^h. For each $i \in I_c$, denote the support of φ_i by Δ^i and define $\pi_i : L^2(\Delta^i) \to P_0(\Delta^i)$ by

$$(\pi_i\varphi)(x) = \frac{1}{\mathrm{meas}_2(\Delta^i)}\int_{\Delta^i}\varphi\,\mathrm{d}s, \quad x \in \Delta^i,\ \varphi \in L^2(\Delta^i). \tag{3.19}$$

Then r_h is defined as follows:

$$r_h\varphi = \sum_{i\in I_c}(\pi_i\varphi)(x^i)\varphi_i, \quad \varphi \in L^2(\Gamma_c).$$

4. Numerical experiments

In our numerical experiments we shall consider an elastic, isotropic and homogeneous material characterized by Young's modulus $E = 21.19\text{e}10$ (Pa) and Poisson's ratio $\sigma = 0.277$ (steel). The initial configuration is represented by $\Omega = (0,3)\times(0,1)\times(0,1)$ (in (m)) with $\Gamma_u = \{0\}\times(0,1)\times(0,1)$, $\Gamma_c = (0,3)\times(0,1)\times\{0\}$, and $\Gamma_p = \Gamma_p^1 \cup \Gamma_p^2 \cup \Gamma_p^3$, where $\Gamma_p^1 = \{3\}\times(0,1)\times(0,1)$, $\Gamma_p^2 = (0,3)\times(0,1)\times\{1\}$ and $\Gamma_p^3 = (0,3)\times\{0,1\}\times(0,1)$. The density of surface tractions is described as follows:

$$p = (p_x^1, 0, p_z^1) \quad \text{on } \Gamma_p^1, \qquad p = (0,0,p_z^2) \quad \text{on } \Gamma_p^2, \qquad p = (0,0,0) \quad \text{on } \Gamma_p^3,$$

where $p_x^1 = 1\text{e}7$ (Pa), $p_z^1 = 2\text{e}7$ (Pa) and $p_z^2 = -3\text{e}7$ (Pa) (see Fig. 2).

The volume forces are neglected. The diagonal matrix $\mathcal{F}$ representing the coefficients of friction is independent of the spatial variable, $\mathcal{F}(x,\xi) := \mathcal{F}(\xi)$, $\xi = (\xi_1,\xi_2)$. We consider the following form of its diagonal elements $\mathcal{F}_1$, $\mathcal{F}_2$:

$$\mathcal{F}_1(\xi) = \phi_{\mathrm{par}_1}(\xi_1) \quad \text{and} \quad \mathcal{F}_2(\xi) = \phi_{\mathrm{par}_2}(\xi_2),$$

[1] In fact, the approximation property (3.3) is shown in [24] assuming that either $\overline{\Gamma}_c \cap \overline{\Gamma}_u = \emptyset$ or the whole relative boundary of Γ_c belongs to $\overline{\Gamma}_u$. However, the same argumentation is valid also for the case considered here.

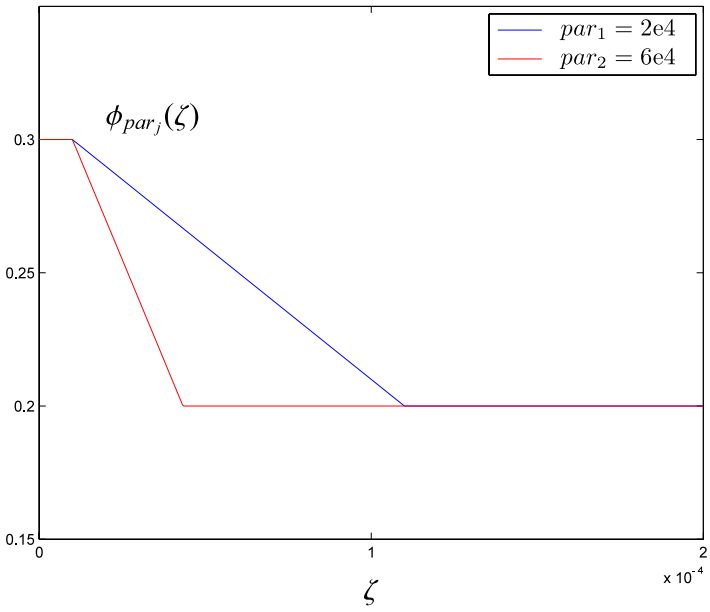

Fig. 3. Coefficients of friction.

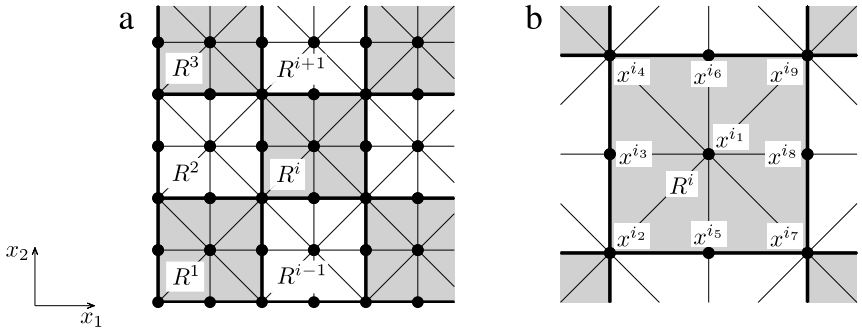

Fig. 4. (a) Partition $\mathscr{T}^H_{\Gamma_c}$; (b) Numbering of nodes.

where

$$\phi_{\mathrm{par}_j}(\zeta) = \begin{cases} 0.3 & \text{if } \zeta \le 10^{-5}; \\ 0.3 - \dfrac{0.1\mathrm{par}_j}{2}(\zeta - 10^{-5}) & \text{if } \zeta \in \left(10^{-5}, 10^{-5} + \dfrac{2}{\mathrm{par}_j}\right); \\ 0.2 & \text{if } \zeta \ge 10^{-5} + \dfrac{2}{\mathrm{par}_j} \end{cases}$$

for $j = 1, 2$, i.e., each coefficient depends only on one component of the tangential displacement. We will consider two different values of par_j, namely 2e4 and 6e4 (see Fig. 3). Finally, the principal axes of orthotropic friction t_1 and t_2 in (2.2) are $t_1 = (1, 0, 0)$ and $t_2 = (0, 1, 0)$, respectively.

The partition $\mathscr{T}^h_\Omega$ is constructed in two steps: Firstly, $\overline{\Omega}$ is cut into $3n_{\mathrm{div}} \times n_{\mathrm{div}} \times n_{\mathrm{div}}$ cubes, n_{div} even. Secondly, each of these cubes is divided into five tetrahedra. With such $\mathscr{T}^h_\Omega$ we associate the dual partition $\mathscr{T}^H_{\Gamma_c}$ as shown in Fig. 4(a). The fine lines and the black dots represent the triangulation $\mathscr{T}^h_{\Omega|_{\overline{\Gamma}_c}}$ and its nodes, respectively, while the "chessboard" with the panes R^i comprised of eight triangles belonging to $\mathscr{T}^h_{\Omega|_{\overline{\Gamma}_c}}$ constitutes the dual partition $\mathscr{T}^H_{\Gamma_c}$. The finite element spaces V^h, L^H consist of piecewise linear (vector) functions on $\mathscr{T}^h_\Omega$ and piecewise constant functions over $\mathscr{T}^H_{\Gamma_c}$, respectively.

Our computations are based on the method of successive approximations mentioned in Theorem 3.2. To evaluate the mapping Ψ_{hH} at $(\varphi^h_1, \varphi^h_2, g^H) \in Y^h_+ \times Y^h_+ \times \Lambda^H_\nu$ one has to solve problem $(\mathscr{M}_{hH}(\varphi^h_1, \varphi^h_2, g^H))$. This is a non-smooth problem due to the presence of the non-differentiable frictional term j. To regularize it, we introduce another Lagrange multiplier.

Instead of $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ we shall use in our computations the following three-field formulation:

$$\left.\begin{array}{l}\text{Find } (u^h, \lambda_\nu^H, \lambda_t^H) \in V^h \times \Lambda_\nu^H \times \Lambda_t^H(\varphi_1^h, \varphi_2^h, g^H) \text{ such that}\\ a(u^h, v^h) = \ell(v^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_c} - (\lambda_t^H, v_t^h)_{0,\Gamma_c} \quad \forall v^h \in V^h,\\ (\mu_\nu^H - \lambda_\nu^H, u_\nu^h)_{0,\Gamma_c} + (\mu_t^H - \lambda_t^H, u_t^h)_{0,\Gamma_c} \le 0 \quad \forall (\mu_\nu^H, \mu_t^H) \in \Lambda_\nu^H \times \Lambda_t^H(\varphi_1^h, \varphi_2^h, g^H)\end{array}\right\} \tag{4.1}$$

with

$$\Lambda_t^H(\varphi_1^h, \varphi_2^h, g^H) = \left\{\mu^H \in \left(L^H\right)^2 \,\middle|\, \left\| \frac{1}{\mathrm{meas}_2(R^i)} \int_{R^i} \mathscr{F}^{-1}(\varphi_1^h, \varphi_2^h)\mu^H \,\mathrm{ds} \right\| \le g^H{}_{|R^i} \;\forall\, R^i \in \mathscr{T}_{\Gamma_c}^H \right\},$$

where $\mathscr{F}^{-1}(\varphi_1^h, \varphi_2^h)$ stands for the inverse of $\mathscr{F}(\varphi_1^h, \varphi_2^h)$. For the approximation of the integrals in the definition of $\Lambda_t^H(\varphi_1^h, \varphi_2^h, g^H)$ we use the quadrature formula which is exact for continuous, piecewise-linear functions over $\mathscr{T}_{\Omega|_{R^i}}^h$ (for numbering of nodes see Fig. 4(b)):

$$\int_{R^i} \mathscr{F}^{-1}(\varphi_1^h, \varphi_2^h)\mu^H \,\mathrm{ds} \approx \frac{\mathrm{meas}_2(R^i)}{12}\left(4\mathscr{F}^{-1}(\varphi_1^h(x^{i_1}), \varphi_2^h(x^{i_1})) + \sum_{j=2}^{9} \mathscr{F}^{-1}\left(\varphi_1^h(x^{i_j}), \varphi_2^h(x^{i_j})\right)\right)\mu^H(x^{i_1}) \tag{4.2}$$

for any $R^i \in \mathscr{T}_{\Gamma_c}^H$.

Denote $n = \dim W^h$, $m = \dim L^H$ and $p =$ the number of the contact nodes of $\mathscr{T}_\Omega^h$. Then the algebraic counterpart of (4.1) reads as follows:

$$\left.\begin{array}{l}\text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^{3n} \times \mathbb{R}_+^m \times \boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{g}) \text{ such that}\\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{l} - \boldsymbol{N}^\top\boldsymbol{\lambda}_\nu - \boldsymbol{T}^\top\boldsymbol{\lambda}_t,\\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{N}\boldsymbol{u})_m + (\boldsymbol{\mu}_t - \boldsymbol{\lambda}_t, \boldsymbol{T}\boldsymbol{u})_{2m} \le 0 \quad \forall (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \mathbb{R}_+^m \times \boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{g}),\end{array}\right\} \tag{4.3}$$

where $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in \mathbb{R}_+^p, \boldsymbol{g} \in \mathbb{R}_+^m$ are given and

$$\boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{g}) = \{\boldsymbol{\mu}_t \in \mathbb{R}^{2m} |\, \|\boldsymbol{\mathscr{F}}_i^{-1}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)(\mu_{t,2i-1}, \mu_{t,2i})^\top\|^2 \le g_i^2,\ i = 1, \ldots, m\} \tag{4.4}$$

with $\boldsymbol{\mathscr{F}}_i^{-1}$ denoting a (2×2)-diagonal matrix whose elements can be computed using the quadrature formula (4.2) on R^i. Further $(\cdot, \cdot)_q$ stands for the inner product in $\mathbb{R}^q$, $\boldsymbol{A} \in \mathbb{R}^{3n \times 3n}$ is the symmetric, positive definite stiffness matrix, $\boldsymbol{l} \in \mathbb{R}^{3n}$ is the load vector, and $\boldsymbol{N} \in \mathbb{R}^{m \times 3n}$, $\boldsymbol{T} \in \mathbb{R}^{2m \times 3n}$ are the matrix representations of the linear mappings $v^h \mapsto v_\nu^h$, $v^h \mapsto v_t^h$, $v^h \in V^h$, coupling $\boldsymbol{u}$ with the dual variables $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_t$, respectively. Eliminating $\boldsymbol{u}$ from (4.3), we obtain the so-called *reciprocal variational formulation* of the problem:

$$(\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) := \mathrm{argmin}\, \mathscr{J}(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \quad \text{s.t. } (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \mathbb{R}_+^m \times \boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{g}), \tag{4.5}$$

where

$$\mathscr{J}(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) = \frac{1}{2}(\boldsymbol{\mu}_\nu^\top, \boldsymbol{\mu}_t^\top)\boldsymbol{S}(\boldsymbol{\mu}_\nu^\top, \boldsymbol{\mu}_t^\top)^\top - (\boldsymbol{\mu}_\nu^\top, \boldsymbol{\mu}_t^\top)\boldsymbol{h}$$

and $\boldsymbol{S} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^\top$, $\boldsymbol{h} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{l}$, $\boldsymbol{B} := (\boldsymbol{N}^\top, \boldsymbol{T}^\top)^\top$. We arrive at the following implementation of the method of successive approximations:

Algorithm 4.1. Let $\boldsymbol{\varphi}_1^{(0)}, \boldsymbol{\varphi}_2^{(0)} \in \mathbb{R}_+^p, \boldsymbol{g}^{(0)} \in \mathbb{R}_+^m$ and $\varepsilon > 0$ be given. Set $k := 0$.

(i) Solve $(\boldsymbol{\lambda}_\nu^{(k+1)}, \boldsymbol{\lambda}_t^{(k+1)}) := \mathrm{argmin}\, \mathscr{J}(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t)$ s.t. $(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \mathbb{R}_+^m \times \boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1^{(k)}, \boldsymbol{\varphi}_2^{(k)}, \boldsymbol{g}^{(k)})$.

(ii) Solve $\boldsymbol{A}\boldsymbol{u}^{(k+1)} = \boldsymbol{l} - \boldsymbol{N}^\top\boldsymbol{\lambda}_\nu^{(k+1)} - \boldsymbol{T}^\top\boldsymbol{\lambda}_t^{(k+1)}$.

(iii) Set $err^{(k)} := \|(\boldsymbol{\lambda}_\nu^{(k+1)}, \boldsymbol{\lambda}_t^{(k+1)}) - (\boldsymbol{\lambda}_\nu^{(k)}, \boldsymbol{\lambda}_t^{(k)})\| / \|(\boldsymbol{\lambda}_\nu^{(k+1)}, \boldsymbol{\lambda}_t^{(k+1)})\|$. If $err^{(k)} \le \varepsilon$, return $\boldsymbol{u} := \boldsymbol{u}^{(k+1)}$, $\boldsymbol{\lambda}_\nu := \boldsymbol{\lambda}_\nu^{(k+1)}$, $\boldsymbol{\lambda}_t := \boldsymbol{\lambda}_t^{(k+1)}$.

(iv) Set $k := k + 1$, assemble $\boldsymbol{u}_{t_1}^{(k)}, \boldsymbol{u}_{t_2}^{(k)} \in \mathbb{R}^p$ (the contact tangential displacements in the directions of t_1 and t_2), $\boldsymbol{\varphi}_1^{(k)} := |\boldsymbol{u}_{t_1}^{(k)}|$, $\boldsymbol{\varphi}_2^{(k)} := |\boldsymbol{u}_{t_2}^{(k)}|$, $\boldsymbol{g}^{(k)} := \boldsymbol{\lambda}_\nu^{(k)}$, where the absolute values are understood componentwisely, and go to step (i).

Let us mention that r_h is chosen to be the Lagrange interpolation operator for simplicity here. Nevertheless, it can be also seen as the Clément operator described at the end of the previous section when the integrals in (3.19) are approximated by an appropriate quadrature formula.

The total efficiency of our numerical approach depends on the algorithm used in step (i). As (4.5) is a strictly convex problem with the quadratic objective $\mathscr{J}$ subject to separable constraints (simple bounds and quadratic inequality constraints), we can solve it by the KPRGP-algorithm proposed and analyzed in [13,14]. Note that this algorithm is a direct generalization of the one in [25] for simple bound constraints. Its idea is based on combining conjugate gradient iterations with gradient projections in an active set strategy. Unlike the isotropic case investigated in [22] one has to compute

Table 1
$\mathscr{F}_1 = \phi_{6e4}$ and $\mathscr{F}_2 = \phi_{2e4}$.

n_{div}	$3n$	$3m$	*iter*		$n_{\boldsymbol{A}}$	
4	900	36	6	11	779	66
6	2646	81	8	18	1091	319
8	5832	144	8	24	1131	477
10	10890	225	8	20	1134	323
12	18252	324	8	25	1127	629
14	28350	441	9	24	1077	461
16	41616	576	9	29	1088	672

Table 2
$\mathscr{F}_1 = \phi_{2e4}$ and $\mathscr{F}_2 = \phi_{6e4}$.

n_{div}	$3n$	$3m$	*iter*		$n_{\boldsymbol{A}}$	
4	900	36	7	19	720	413
6	2646	81	8	21	849	366
8	5832	144	8	23	860	429
10	10890	225	8	27	911	479
12	18252	324	8	23	974	575
14	28350	441	8	18	977	293
16	41616	576	10	20	1044	332

projections onto the feasible set $\mathbb{R}^m_+ \times \boldsymbol{\Lambda}_t(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{g})$. Due to the separable structure of this set, each projection splits into independent projections onto $\mathbb{R}^1_+$ and onto ellipses in $\mathbb{R}^2$. The second case requires to solve non-linear equations (by the Newton method, e.g.). As the projected point on the ellipse is uniquely determined by its angle coordinate in the polar representation, the respective equation contains this coordinate as the only unknown [26]. Consequently, the increase of computational costs due to the Newton method is negligible.

Remark 4.1. To increase the efficiency of Algorithm 4.1, we initialize the KPRGP-algorithm in the kth iteration by the result of step (i) obtained in the previous iteration (and by the zero vectors, if $k = 0$). Moreover, we choose the terminating tolerance $\epsilon_\lambda := \epsilon_\lambda^{(k)}$ of the KPRGP-algorithm sufficiently accurate in order to achieve the terminating tolerance ε for the method of successive approximations. We use two strategies: (a) the fixed precision control $\epsilon_\lambda^{(k)} := (r_{tol} \times \varepsilon)\|\boldsymbol{h}\|$ with $0 < r_{tol} < 1$; (b) the adaptive precision control $\epsilon_\lambda^{(k)} := \min(r_{tol} \times err^{(k-1)}, c_{fact} \times \epsilon_\lambda^{(k-1)})\|\boldsymbol{h}\|$ with $0 < r_{tol} < 1$, $0 < c_{fact} < 1$, $err^{(-1)} = 1$ and $\epsilon_\lambda^{(-1)} = r_{tol}/c_{fact}$. While (a) makes it possible to obtain the solution in a small number of outer fixed-point iterations, (b) leads to a considerably more efficient procedure with a small number of matrix–vector multiplications. Note that the KPRGP-algorithm is terminated, if the reduced gradient [13,14] of the current (inner) iterate is less than or equal to $\epsilon_\lambda^{(k)}$.

The tables above show how our algorithm behaves for different meshes and different coefficients of friction. Table 1 summarizes experiments with $\mathscr{F}_1$ and $\mathscr{F}_2$ given by par1 = 6e4 and par2 = 2e4, i.e., $\mathscr{F}_1 = \phi_{6e4}$ and $\mathscr{F}_2 = \phi_{2e4}$, respectively. In Table 2 the role of $\mathscr{F}_1$ and $\mathscr{F}_2$ is interchanged, i.e., $\mathscr{F}_1 = \phi_{2e4}$ and $\mathscr{F}_2 = \phi_{6e4}$. Recall that $3n$, $3m$ stands for the total number of the primal and the dual variables, respectively. Further *iter* denotes the total number of the fixed-point iterations and $n_{\boldsymbol{A}}$ stands for the number of actions of $\boldsymbol{A}^{-1}$ (via the backward substitutions based on the pre-computed Cholesky factor). Since this step is the most expensive part of the KPRGP-algorithm, $n_{\boldsymbol{A}}$ expresses the total cost of computations. The first integer in the *iter* and $n_{\boldsymbol{A}}$ columns characterizes the fixed precision control (with $r_{tol} = 0.1$) while the second integer characterizes the adaptive one (with $r_{tol} = 0.1$ and $c_{fact} = 0.99$). The initial approximation and the terminating tolerance for the method of successive approximations were chosen to be $\boldsymbol{\varphi}_1^{(0)} = \boldsymbol{\varphi}_2^{(0)} = \boldsymbol{g}^{(0)} = \boldsymbol{0}$ and $\varepsilon = 1e-4$, respectively.

From the tables one can conclude that the total complexity as well as the behavior of Algorithm 4.1 depend on the way how the (inner) KPRGP-algorithm is terminated. If the inner terminating tolerance $\epsilon_\lambda^{(k)}$ is fixed and proportional to the final precision ε in all fixed-point iterations (strategy (a) of Remark 4.1) then the numbers *iter* and $n_{\boldsymbol{A}}$ are similar for all n_{div}. On the other hand, the inexact solving of the inner subproblems (strategy (b) of Remark 4.1) exhibits some oscillations in the values of *iter* and $n_{\boldsymbol{A}}$ with respect to n_{div}. In this case $\epsilon_\lambda^{(k)}$ is adaptive and proportional only to the current precision $err^{(k-1)}$ or, if the progress is not sufficient, to the improved inner tolerance $\epsilon_\lambda^{(k-1)}$ from the previous step. This strategy ensures that the KPRGP-algorithm performs as few steps as possible, but the number of outer fixed-point iterations increases. A heuristic explanation for this increase is simple. One can interpret several (usually three) outer iterations of the strategy (b) as one iteration of the strategy (a).

The results of our computations for $n_{\text{div}} = 16$ with $\mathscr{F}_1 = \phi_{6e4}$ and $\mathscr{F}_2 = \phi_{2e4}$ are seen in Fig. 5. The distribution of the normal contact stress ($\sigma_\nu(u) \approx -\lambda_\nu^H$) and the weighted norm of the tangential contact stress ($\|\mathscr{F}^{-1}(|u_{t_1}|, |u_{t_2}|)\sigma_t(u)\| \approx \|\mathscr{F}_h^{-1}\lambda_t^H\|$, where $\mathscr{F}_h^{-1} = \mathscr{F}^{-1}(|u_{t_1}^h|, |u_{t_2}^h|)$) are depicted in Fig. 5(a) and (b), respectively. All contact and friction phenomena appear on Γ_c in our model problem, i.e., the slipping and sticking contact zones as well as the zone of non-contact. Fig. 5(c) shows the deformed body while Fig. 5(d) enables us to check the satisfaction of the friction conditions (2.2). The lengths of

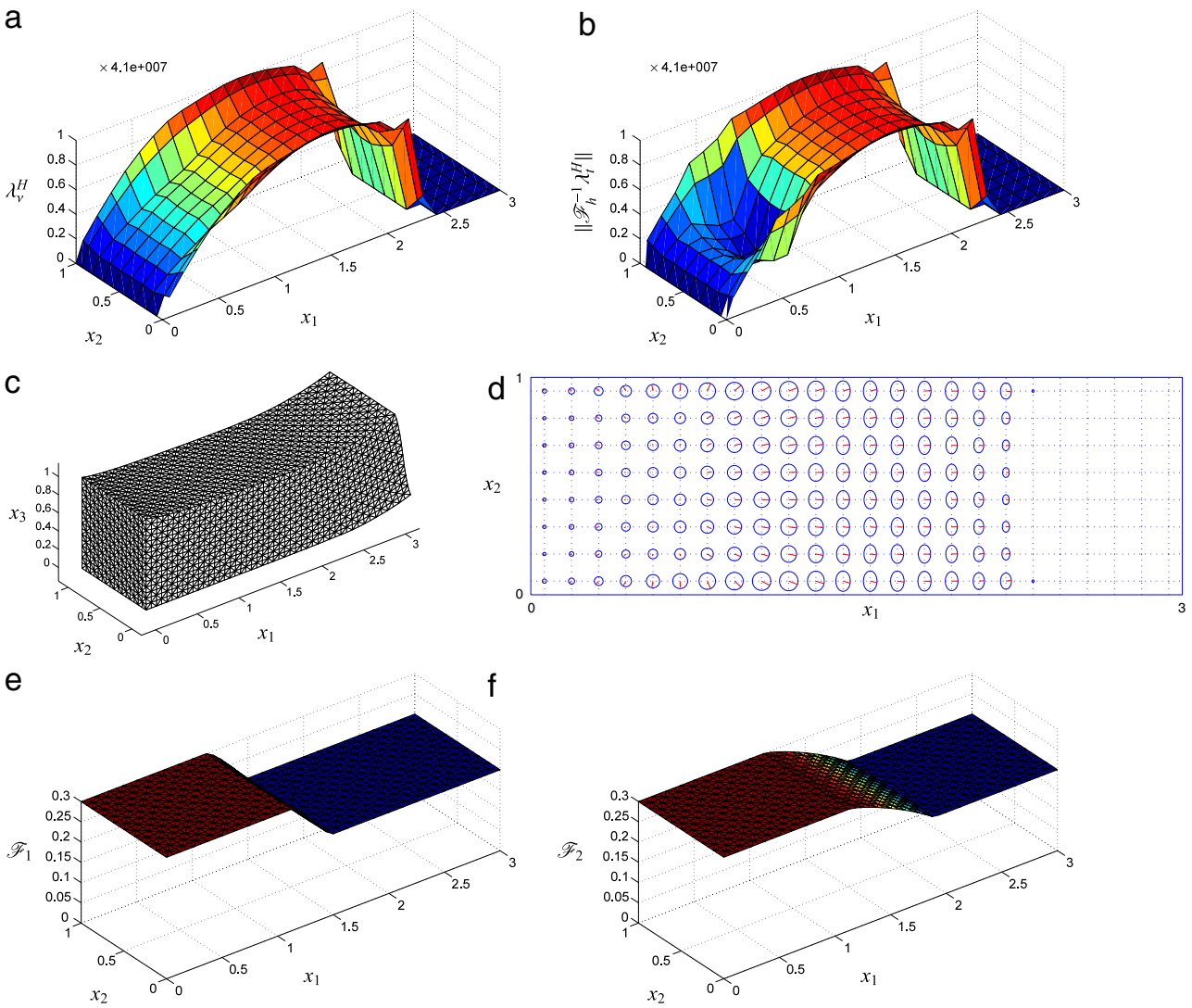

Fig. 5. (a) Normal contact stress; (b) weighted norm of the tangential contact stresses; (c) deformed body; (d) contact zone Γ_c; (e) distribution of $\mathscr{F}_1$; (f) distribution of $\mathscr{F}_2$.

the semi-axes of the ellipses in this figure are determined by the values of $\mathscr{F}_1$ and $\mathscr{F}_2$ at the solution. The small lines inside represent the tangential contact stress. Finally, Fig. 5(e) and (f) depict the distribution of $\mathscr{F}_1$ and $\mathscr{F}_2$ on Γ_c, respectively.

5. Conclusions and comments

The first, theoretical part is the main contribution of this paper. It is devoted to the existence and uniqueness analysis of solutions to discrete contact problems with orthotropic friction and coefficients of friction depending on the magnitude of the tangential contact displacements. Solutions are defined as fixed points of a mapping acting on the contact parts of the boundary. It was shown that at least one solution exists for the coefficients of friction represented by positive, bounded and continuous functions. If, in addition, these functions are Lipschitz continuous and sufficiently small together with the respective modulus of Lipschitz continuity then the solution is unique. The mesh-dependent bounds guaranteeing this property are derived. Such results are important not only from the theoretical but also from the practical point of view. As a consequence we obtain the justification of the method of successive approximations in which each iterative step is given by a contact problem with orthotropic Tresca friction. In the last section we used this approach for numerically solving of a model problem.

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