Numerical solution of contact problems with orthotropic Coulomb friction based on quadratic programming approach with the elliptic friction cone

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Abstract

This paper deals with a finite element discretization and numerical solution of contact problems for linearly elastic bodies with orthotropic Coulomb friction. We use the method of successive approximations for finding a solution to this problem. Each iterative step is represented by a contact problem with orthotropic Tresca friction. Numerical solution of this auxiliary problem is based on its dual formulation in terms of contact stresses. The minimization of the discrete dual energy functional is performed by the recently developed method of convex programming. The efficiency of computations is increased by using the T-FETI domain decomposition method.

Keywords: orthotropic friction, contact problems, domain decomposition methods

2008 MSC: 65K10, 65C20, 65N30, 65N55, 74M15

1. Introduction

Contact mechanics is a special branch of mechanics of solids which analyzes the behavior of a system of deformable loaded bodies in a mutual contact. Due to non-penetration and friction conditions, the problem we have to solve is highly non-linear. For materials obeying a linear Hook law for small deformations, an appropriate linearization of the non-penetration conditions (see [20, 14]) leads to a convex set of kinematically admissible displacements (geometrical nonlinearity). Another non-linearity comes from the presence of friction. The complexity of the resulting mathematical model depends on which model of friction is assumed.
friction is used. In the simplest case with an a-priori given slip bound (Tresca model), the mathematical model is represented by a variational inequality of the second kind (see [11, 14]). This model is however too simple since the non-penetration and friction phenomena are decoupled. For this reason more realistic models of friction have to be used and the Coulomb friction law is one of the most classical ones. The slip bound which is prescribed in Tresca model is now replaced by the product of a coefficient of friction $F$ and the magnitude of the normal contact force. Due to the coupling of unilateral and friction effects the resulting problem is now much more involved. It leads to the so-called implicit variational inequality (in terms of displacements) or to a quasivariational inequality (in terms of contact stresses) ([20, 14]). The mathematical analysis of this problem remained open for a longtime. The first results on the existence of solutions are in [21, 19]. So far we assumed the case of isotropic friction when friction works homogeneously in any direction. In many situations however this model does not reflect the reality since it does not take into account material and contact surface properties which might be anisotropic. In such a case one has to pass to a more general anisotropic friction law. This paper is devoted to a discretization and numerical solution of contact problems with orthotropic Coulomb friction characterized by two friction coefficients $F_1$ and $F_2$ in two mutually orthogonal directions. In this paper we do not use any approximation of the friction law such as a polyhedral friction law [31, 15] but we work with the original one, instead due to the applied minimization procedure [26, 5]. Analogously to the isotropic case, we use the fixed-point approach in which the solution to the original problem is defined as a fixed-point of an auxiliary mapping acting on the contact parts. To find fixed-points we use the method of successive approximations whose individual iterative steps are relatively simple: they are given by contact problems with orthotropic Tresca model of friction. Each iterative step represented by a variational inequality of the second kind for the displacements is then transformed into the dual form which is expressed in terms of the normal and tangential contact stresses. Its finite element discretization leads to a minimization of a quadratic function over a convex set given by simple bounds for the discrete normal contact stress and separable quadratic (elliptical) constraints for the discrete tangential (friction) forces. The theoretical analysis of discrete contact problems with orthotropic Coulomb law of friction is done in [16].

In the last decades, the number of numerical methods for the solution of frictional contact problems have been developed. These include non-smooth Newton methods and active set strategies [24, 18], monotone multigrid methods [23, 30], methods based on regularization [31, 29] or variants of domain decomposition methods [9, 27] (see the cited papers for other references). Here, we use the T-FETI variant of the FETI domain decomposition method [4] in which the bodies are decomposed into subdomains and the inter-connectivity of the computed solution as well as the Dirichlet boundary conditions are enforced by Lagrange multipliers. The usage of this method imposes additional equality constraints in the minimization problem. To compute the solution we apply the SMALSE-M algorithm [5] (semi-monotonic augmented Lagrangians for separable and equality constraints) that generalizes ideas of the SMALBE [3, 2] algorithm investigated for solving the frictionless contact problems. The algorithm updates the Lagrange multipliers for equality constraints
in the outer loop while the remaining constraints are treated by the inner loop with an effective precision control in which conjugate gradient iterations are combined with gradient projections. Unlike to the isotropic case the projections onto ellipses are needed that requires to solve non-linear equations, for instance by the Newton method. As the projected point on the ellipse is uniquely determined by its angle coordinate in the polar representation, the respective equation contains this coordinate as the only unknown so that the increase of computational costs due to the Newton method is negligible. The benefit from the usage of the T-FETI method consists in a simple parallelization of computations as well as in the fact that the resulting computations exhibit a scalable behavior (the number of iterations needed to get the solution with a given accuracy is independent of the size of the discrete problem [5, 7]). Let us mention that the scalability can be proved only for Tresca friction but one can observe it experimentally also for Coulomb friction. An industrial application for isotropic case solved by the above approach may be found in [7].

The paper is organized as follows: Section 2 deals with the continuous setting of the problem. First we present its classical formulation. The weak formulation is defined by means of the fixed-point approach having a contact problem with orthotropic Tresca model of friction as one iterative step in the method of successive approximations. Further we derive the mixed variational formulation of this step using the duality approach and also the formulation involving the T-FETI domain decomposition method. In Section 3 we present the finite element approximation of the problem. The displacements are approximated by linear functions on tetrahedrons. We derive the algebraic form of the discrete total potential energy functional. Using the algebraic Lagrange multipliers, the unilateral constraints are released and the frictional term is regularized. Eliminating the displacement field we finally arrive at the discrete dual formulation. This is done in Section 4. In Section 5 we present in more details the recently developed and mathematically justified method for the efficient minimization of quadratic functions subject to separable quadratic inequality and linear equality constraints [5]. Necessary modifications of this algorithm for elliptical constraints are described. Finally, in Section 6 numerical results of several model benchmarks are shown.

2. Setting of the problem

Let two elastic bodies be represented by two non-overlapping polyhedral domains \( \Omega^k \subset \mathbb{R}^3 \) with the boundaries \( \partial \Omega^k, k = 1, 2 \). Each boundary consists of three disjoint parts \( \Gamma^k_u, \Gamma^k_p, \) and \( \Gamma^k_c \) open in \( \partial \Omega^k \), so that \( \partial \Omega^k = \Gamma^k_u \cup \Gamma^k_p \cup \Gamma^k_c \). The zero displacements are prescribed on \( \Gamma^k_u \) while surface tractions of density \( p^k \) act on \( \Gamma^k_p \). On the contact interfaces given by \( \Gamma^1_c \) and \( \Gamma^2_c \) we consider contact conditions: the non-penetration of the bodies, the transmission of the contact stresses, and the effect of orthotropic Coulomb friction. Finally we suppose that each body \( \Omega^k \) is subject to volume forces of density \( f^k \).

We seek displacement fields \( u^k \) in \( \Omega^k \) satisfying the equilibrium equations and the Dirich-
\begin{align}
\text{let and Neumann boundary conditions:} \\
\begin{aligned}
\text{div } \sigma^k + f^k &= 0 \quad \text{in } \Omega^k, \\
u^k &= 0 \quad \text{on } \Gamma^k_{\text{in}}, \\
\sigma^k n^k - p^k &= 0 \quad \text{on } \Gamma^k_{\text{p}},
\end{aligned} \quad k = 1, 2, \tag{2.1}
\end{align}

where \( \sigma^k := \sigma(u^k) \) is the stress tensor in \( \Omega^k \) and \( n^k \) stands for the unit outward normal vector to \( \partial \Omega^k \), \( k = 1, 2 \). Stress tensors are related to linearized strain tensors \( e^k := \epsilon(u^k) = 1/2(\nabla u^k + \nabla^\top u^k) \) by Hooke’s law for linear isotropic materials:

\[
\sigma^k := \lambda^k \text{tr}(e^k) I + 2\mu^k e^k \quad \text{in } \Omega^k,
\]

where “tr” denotes the trace of matrices, \( I \in \mathbb{R}^{3 \times 3} \) is the identity matrix, and \( \lambda^k, \mu^k > 0 \) are the Lamè constants. Let us note that \( \lambda^k \) and \( \mu^k \) are given by the Young modulus \( E^k > 0 \) and the Poisson ratio \( \nu^k \in (0, 0.5) \) as \( \lambda^k := E^k \nu^k/(1 - (\nu^k)^2) \) and \( \mu^k := E^k/(2(1 + \nu^k)) \), respectively.

To formulate the contact conditions we introduce a predefined one-to-one transfer mapping \( \chi : \Gamma^1_c \mapsto \Gamma^2_c \) by means of which we define the initial distance between the contact surfaces at \( x \in \Gamma^1_c \) as \( d(x) := \|\chi(x) - x\| \) and the critical direction \( \nu(x) := (\chi(x) - x)/d(x) \) if \( d(x) \neq 0 \), or \( \nu(x) := n^1(x) \) if \( d(x) = 0 \). Here and in what follows, \( \| \cdot \| \) stands for the Euclidean norm of a vector. The non-penetration conditions read as follows:

\[
u_\nu - d \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu(u_\nu - d) = 0 \quad \text{on } \Gamma^1_c, \tag{2.2}
\]

where \( u_\nu(x) := (u^1(x) - u^2(\chi(x)))^\top \nu(x) \) is the relative contact displacement and \( \sigma_\nu(x) := \nu(x)^\top \sigma^1(x) \nu(x) \) is the contact stress at \( x \in \Gamma^1_c \) and both in the direction of \( \nu(x) \).

Let \( t_1 := t_1(x), t_2 := t_2(x) \) be two vectors orthogonal to \( \nu := \nu(x) \) and such that the triplet \( \{\nu, t_1, t_2\} \) is a local orthonormal basis in \( \mathbb{R}^3 \) with the origin at \( x \in \Gamma^1_c \), which is piecewise smooth on \( \Gamma^1_c \). By \( F_1 := F_1(x), F_2 := F_2(x) \) we denote given positive coefficients of friction in the directions \( t_1, t_2 \), respectively, and set

\[
F := F(x) = \begin{pmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{pmatrix} \in \mathbb{R}^{2 \times 2}.
\]

The orthotropic Coulomb friction law reads as follows:

\[
\begin{aligned}
u_i(x) = 0 & \implies \|F^{-1} \sigma_i(x)\| \leq \sigma_\nu(x) \\
u_i(x) \neq 0 & \implies F^{-1} \sigma_i(x) = \sigma_\nu(x) \frac{F \nu_i(x)}{\|F \nu_i(x)\|} \quad \text{for } x \in \Gamma^1_c, \tag{2.3}
\end{aligned}
\]

where \( u_i := (u_{i1}, u_{i2})^\top \) and \( \sigma_i := (\sigma_{i1}, \sigma_{i2})^\top \) are the relative tangential contact displacement and the tangential contact stress at \( x \in \Gamma^1_c \) with the components \( u_i(x) := (u^1(x) - u^2(\chi(x)))^\top t_i(x) \) and \( \sigma_i(x) := t_i(x)^\top \sigma^1(x) \nu(x), i = 1, 2 \), respectively.

Finally we require the transmission of the contact stresses:

\[
\sigma^i \nu = (\sigma^2 \circ \chi) \nu \quad \text{on } \Gamma^1_c, \tag{2.4}
\]
where \((\sigma^2 \circ \chi)(x) := \sigma^2(\chi(x))\) at \(x \in \Gamma^1_c\).

The classical formulation of the contact problem with orthotropic Coulomb friction consists in finding a pair \(u := (u^1, u^2)\) of the displacement fields \(u^k\) in \(\Omega^k\), \(k = 1, 2\), satisfying (2.1), (2.2), (2.3), and (2.4).

To give the weak formulation of this problem we introduce the following sets:

\[
\begin{align*}
\mathbb{V} &= \{ v := (v^1, v^2) \in (H^1(\Omega^1))^3 \times (H^1(\Omega^2))^3 \mid v^k = 0 \text{ on } \Gamma^k_u, \ k = 1, 2 \}, \\
\mathbb{K} &= \{ v \in \mathbb{V} \mid v_i - d \leq 0 \text{ on } \Gamma^1_c \}, \\
X_\nu &= \{ \varphi \in L^2(\Gamma^1_c) \mid \exists v \in \mathbb{V} : \varphi = v_\nu \text{ on } \Gamma^1_c \}, \\
X_{\nu+} &= \{ \varphi \in X_\nu \mid \varphi \geq 0 \text{ on } \Gamma^1_c \}, \\
X_{t+} &= \{ \varphi \in L^2(\Gamma^1_c) \mid \exists v \in \mathbb{V} : \varphi = \| \mathcal{F} v_\nu \| \text{ on } \Gamma^1_c \}.
\end{align*}
\] (2.5)

The symbol \(X'_\nu\) stands for the dual of \(X_\nu\) and \(X'_{\nu+}\) for the cone of all non-negative elements of \(X'_\nu\). The duality pairing between \(X_\nu\) and \(X_{\nu+}\) will be denoted by \(\langle \cdot, \cdot \rangle\), in what follows. We shall suppose that \(X_{t+} \subseteq X_\nu\) so that \((g, \| \mathcal{F} v_\nu \|)\) is well defined for any \(g \in X'_\nu\) and \(v \in \mathbb{V}\). Moreover we assume that \(f^k \in (L^2(\Omega^k))^3\), \(p^k \in (L^2(\Gamma^k_p))^3\), \(k = 1, 2\), \(d \in X_{\nu+}\). Finally \(\mathcal{F}_i\) will be sufficiently smooth and \(\mathcal{F}_{\min} \leq \mathcal{F}_i \leq \mathcal{F}_{\max}\) on \(\Gamma^1_c\), \(i = 1, 2\), where \(0 < \mathcal{F}_{\min} < \mathcal{F}_{\max}\) are given.

**Remark 2.1.** It is easy to see that \(H^{1/2}(\Gamma^1_c) \subseteq X_\nu \subseteq L^2(\Gamma^1_c)\), where \(H^{1/2}(\Gamma^1_c) := V_{1/2}\) is the trace space on \(\Gamma^1_c\) of functions belonging to \(V\) defined by

\[
V = \{ v \in H^1(\Omega^1) \mid v = 0 \text{ on } \Gamma^1_u \}.
\]

We start with the following auxiliary problem: for given \(g \in X'_{\nu+}\) find \(u := u(g) \in \mathbb{K}\) such that

\[
a(u, v - u) + \langle g, \| \mathcal{F} v_\nu \| - \| \mathcal{F} u_\nu \| \rangle \geq b(v - u) \quad \forall v \in \mathbb{K},
\] (2.6)

where

\[
a(u, v) = \sum_{k=1}^2 \int_{\Omega^k} \sigma(u^k) : \varepsilon(v^k) \, dx,
\]

\[
b(v) = \sum_{k=1}^2 \left( \int_{\Omega^k} (f^k)^\top a^k \, dx + \int_{\Gamma^k_p} (p^k)^\top v^k \, ds \right).
\]

It is easy to verify that (2.6) is the weak formulation of the contact problem with orthotropic friction and a given slip bound \(g\). The respective friction law is given by (2.3) with \(\sigma_\nu := -g\).

It is well-known that (2.6) is equivalent to the minimization problem:

\[
\text{Find } u := u(g) \in \mathbb{K} \text{ such that } \quad \left\{ \begin{array}{l}
J_g(u) \leq J_g(v) \quad \forall v \in \mathbb{K},
\end{array} \right. \quad (\mathcal{P}(g))
\]
where \( J_g(v) = \frac{1}{2}a(v, v) - b(v) + \langle g, \| Fv_1 \| \rangle \).

**Assumption:** Next we will assume that \((\mathcal{P}(g))\) has a unique solution (this is guaranteed, e.g., if \(\Gamma^1_u\) and \(\Gamma^2_u\) are non-empty sets) and that \(-\sigma_v(u(g)) \in X'_{\nu+}\) for every \(g \in X'_{\nu+}\).

On the basis of this assumption one can define the mapping \(\Psi : X'_{\nu+} \mapsto X'_{\nu+}\) by:

\[
\Psi : g \mapsto -\sigma_v(u(g)), \quad g \in X'_{\nu+},
\]

where \(\sigma_v(u(g))\) is the contact stress on \(\Gamma^1_c\) in the direction of \(\nu\) associated with the solution \(u(g)\) to \((\mathcal{P}(g))\).

By a weak solution to the contact problem with orthotropic Coulomb friction we call any \(u \in \mathbb{K}\) such that

\[
\Psi(-\sigma_v(u)) = -\sigma_v(u),
\]

i.e., \(-\sigma_v(u)\) is a fixed point of \(\Psi\) in \(X'_{\nu+}\).

Since the method of successive approximations will be the main tool for solving \((\mathcal{P})\), we will confine ourselves to the individual iterative step represented by (2.6) with \(g \in X'_{\nu+}\) given. For numerical solution of (2.6) we use its dual formulation involving the Lagrange multipliers on \(\Gamma^1_c\). For the sake of simplicity of our presentation we will suppose that \(g \in L^2(\Gamma^1_c)\) so that the duality pairing \(\langle \cdot, \cdot \rangle\) is represented by the \(L^2(\Gamma^1_c)\)-scalar product. The frictional term can be written as follows:

\[
\int_{\Gamma^1_c} g \| Fv_1 \| ds = \sup_{\mu_\nu \in (L^2(\Gamma^1_c))^2} \| \mu_\nu \| \leq g \text{ a.e. on } \Gamma^1_c \int_{\Gamma^1_c} \mu_\nu^T Fv_1 ds = \sup_{\mu_\nu \in X_\nu(g)} \int_{\Gamma^1_c} \mu_\nu^T v_1 ds,
\]

where

\[
X_\nu(g) = \{ \mu_\nu \in (L^2(\Gamma^1_c))^2 | \| F^{-1} \mu_\nu \| \leq g \text{ a.e. on } \Gamma^1_c \}.\]

Thus

\[
\min_{\mathbb{K}} J_g(v) = \min_{X'_{\nu+} \times X_\nu(g)} \sup_{\nu} L(v, \nu, \mu_\nu),
\]

where

\[
L(v, \nu, \mu_\nu) := \frac{1}{2}a(v, v) - b(v) + \langle \mu_\nu, v_\nu - d \rangle + (\mu_\nu, v_\nu)_{0, \Gamma^1_c}
\]

is the Lagrangian.

By the mixed formulation of \((\mathcal{P}(g))\) we call the problem of finding a saddle-point \((w, \lambda_\nu, \lambda_\nu)\) of \(L\) on \(\mathbb{V} \times X'_{\nu+} \times X_\nu(g)\):

\[
L(w, \nu, \mu_\nu) \leq L(w, \nu, \lambda_\nu) \leq L(v, \nu, \lambda_\nu) \quad \forall (v, \nu, \mu_\nu) \in \mathbb{V} \times X'_{\nu+} \times X_\nu(g),
\]

or, equivalently,

\[
\text{Find } (w, \nu, \lambda_\nu) \in \mathbb{V} \times X'_{\nu+} \times X_\nu(g) \text{ such that}
\]

\[
\begin{cases}
a(w, v) = b(v) - \langle \lambda_\nu, v_\nu \rangle - (\lambda_\nu, v_\nu)_{0, \Gamma^1_c} & \forall v \in \mathbb{V},

\langle \mu_\nu - \lambda_\nu, u_\nu - d \rangle + (\mu_\nu - \lambda_\nu, u_\nu)_{0, \Gamma^1_c} \leq 0 & \forall (\mu_\nu, \mu_\nu) \in X'_{\nu+} \times X_\nu(g).
\end{cases}
\]

\((\mathcal{M}(g))\)
It is well-known [10] that \((\mathcal{M}(g))\) has a unique solution \((w, \lambda_\nu, \lambda_i)\) and \(w = u, \lambda_\nu = -\sigma_\nu(u), \lambda_i = -\sigma_i(u)\), where \(u \in \mathbb{K}\) solves \((\mathcal{P}(g))\).

Instead of \(w\) we will write again \(u\). The relation between \((\mathcal{P}(g))\) and \((\mathcal{M}(g))\) enables us to give the equivalent definition of the mapping \(\Psi\) from (2.7), namely
\[
\Psi(g) = \lambda_\nu,
\]
where \((u, \lambda_\nu, \lambda_i)\) is the solution to \((\mathcal{M}(g))\).

To improve the efficiency of numerical solution of \((\mathcal{P}(g))\) we shall use in the next sections the so-called T-FETI domain decomposition method [4]. Its main idea in the continuous setting will be now introduced.

Let \(\{\Omega^k_i\}_{i=1}^{s_k}, k = 1, 2\), be a decomposition of \(\Omega^k\) into \(s_k\) subdomains \(\Omega^k_i\) with the Lipschitz boundaries \(\partial \Omega^k_i\) such that \(\Omega^k = \bigcup_{i=1}^{s_k} \Omega^k_i, \Omega^k_i \cap \Omega^k_j = \emptyset, i \neq j\). In addition we shall suppose that these decompositions are compatible with the partition of \(\partial \Omega^k\) into \(\Gamma^k_p, \Gamma^k_c, k = 1, 2\). We say that \(\Gamma^k_{ij}\) is a common interface between \(\Omega^k_i\) and \(\Omega^k_j, i \neq j\), iff \(\text{meas}_2 \Gamma^k_{ij} > 0\), where \(\Gamma^k_{ij} := \partial \Omega^k_i \cap \partial \Omega^k_j\) and \(\text{meas}_2\) stands for the two-dimensional Lebesgue measure of a set. To identify common interfaces we introduce the index sets:
\[
\mathcal{I}^k := \{(i, j)| \ 1 \leq i < j \leq s_k, \ \text{meas}_2(\Gamma^k_{ij}) > 0\}, \ k = 1, 2.
\]

Similarly, let
\[
\mathcal{D}^k := \{l| \ 1 \leq l \leq s_k, \ \text{meas}_2(\Gamma^k_{ul}) > 0\}, \ k = 1, 2,
\]
where \(\Gamma^k_{ul} := \partial \Omega^k_i \cap \partial \Omega^k_u\), be the index sets identifying those \(\Omega^k_i, 1 \leq l \leq s_k\), whose part of the boundary belongs to \(\Gamma^k_u\). Further, let
\[
\mathcal{W}^k = \{v^k \in (L^2(\Omega^k))^3| v^k_i := v^k|_{\Omega^k_i}, 1 \leq i \leq s_k\}, \ k = 1, 2,
\]
and
\[
\mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.
\]
It is readily seen that
\[
v = (v^1, v^2) \in \mathcal{V} \iff v \in \mathcal{W} \& [v^k]_{ij} = 0 \ \forall (i, j) \in \mathcal{I}^k \& \ v^k_{|_{\Gamma^k_{ul}}} = 0 \ \forall l \in \mathcal{D}^k, \ k = 1, 2,
\]
where
\[
[v^k]_{ij} := (v^k_i - v^k_j)|_{\Gamma^k_{ij}}, \ (i, j) \in \mathcal{I}^k
\]
is the jump of \(v^k_i\) and \(v^k_j\) across \(\Gamma^k_{ij}\). The conditions on \(\Gamma^k_{ij}\) and \(\Gamma^k_{ul}\) will be realized by means of Lagrange multipliers. To this end we introduce the following trace spaces:
\[
\mathcal{N}^k_{ij} := (H^1(\Omega^k))^3|_{\Gamma^k_{ij}} = (H^1(\Omega^k))^3|_{\Gamma^k_{ij}}, \ (i, j) \in \mathcal{I}^k,
\]
\[
\mathcal{Z}^k_l := (H^1(\Omega^k))^3|_{\Gamma^k_{ul}} \ l \in \mathcal{D}^k,
\]
for $k = 1, 2$. By $\mathcal{Y}^{-k}_{ij}$ and $\mathcal{Z}^{-k}_{ij}$ we denote the duals to $\mathcal{Y}^{k}_{ij}$ and $\mathcal{Z}^{k}_{ij}$, $(i, j) \in \mathcal{T}^{k}$, $l \in \mathcal{D}^{k}$, $k = 1, 2$, respectively, and for the duality pairing we use again the symbol $\langle \cdot, \cdot \rangle$. The space $\mathcal{V}$ can be equivalently characterized as follows:

$$v \in \mathcal{V} \iff v \in \mathcal{W} \quad \& \quad \langle \mu^{k}_{ij}, [v]_{ij} \rangle = 0 \ \forall \mu^{k}_{ij} \in \mathcal{Y}^{-k}_{ij} \ \forall (i, j) \in \mathcal{T}^{k}$$

$$\& \langle \mu^{k}_{l}, v^{k}_{l} \rangle = 0 \ \forall \mu^{k}_{l} \in \mathcal{Z}^{-k}_{l} \ \forall l \in \mathcal{D}^{k}, \quad k = 1, 2.$$

Let $\mu_{\text{int}}$, $\mu_{\text{Dir}}$ be the elements whose components are $\mu^{k}_{ij} \in \mathcal{Y}^{-k}_{ij}$, $(i, j) \in \mathcal{T}^{k}$ and $\mu^{k}_{l} \in \mathcal{Z}^{-k}_{l}$, $l \in \mathcal{D}^{k}$, respectively, for $k = 1, 2$ and define

$$\langle \mu_{\text{int}}, [v] \rangle := \sum_{k=1}^{2} \sum_{(i, j) \in \mathcal{T}^{k}} \langle \mu^{k}_{ij}, [v]^{k}_{ij} \rangle,$$

$$\langle \mu_{\text{Dir}}, v \rangle := \sum_{k=1}^{2} \sum_{l \in \mathcal{D}^{k}} \langle \mu^{k}_{l}, v^{k}_{l} \rangle.$$

The spaces of all such $\mu_{\text{int}}$ and $\mu_{\text{Dir}}$ will be denoted by $\mathcal{X}'_{\text{int}}$ and $\mathcal{X}'_{\text{Dir}}$, respectively.

To give the T-FETI domain decomposition formulation of $(\mathcal{M}(g))$ we introduce the sets $\tilde{X}_{\nu}$, $\tilde{X}_{\nu+}$ on $\Gamma_{c}^{1}$ in a similar way as in $(2.5)_{3,4}$ with the following minor change: instead of $\mathcal{V}$ we use the broken space $\mathcal{W}$. Again $\tilde{X}_{\nu}'$ denotes the dual of $\tilde{X}_{\nu}$, $\tilde{X}_{\nu+}'$ the cone of all non-negative elements of $\tilde{X}_{\nu}'$ and $\langle \cdot, \cdot \rangle$ stands for the duality pairing. The T-FETI domain decomposition formulation of $(\mathcal{M}(g))$ reads as follows:

$$\text{Find } (u, \lambda_{\nu}, \lambda_{t}, \lambda_{\text{int}}, \lambda_{\text{Dir}}) \in \mathcal{W} \times \tilde{X}_{\nu+}' \times \mathcal{X}_{t}(g) \times \mathcal{X}'_{\text{int}} \times \mathcal{X}'_{\text{Dir}} \text{ such that }$$

$$a(u, v) = b(v) - \langle \lambda_{\nu}, v_{\nu} \rangle - \langle \lambda_{t}, v_{t} \rangle_{0, \Gamma_{t}^{1}} + \langle \lambda_{\text{int}}, [v] \rangle + \langle \lambda_{\text{Dir}}, v \rangle \quad \forall v \in \mathcal{W},$$

$$\langle \mu_{\nu} - \mu_{\nu}, u_{\nu} - d \rangle + \langle \mu_{t} - \mu_{t}, u_{t} \rangle_{0, \Gamma_{t}^{1}} \leq 0 \quad \forall (\mu_{\nu}, \mu_{t}) \in \tilde{X}_{\nu+}' \times \mathcal{X}_{t}(g),$$

$$\langle \mu_{\text{int}}, [u] \rangle = 0 \quad \forall \mu_{\text{int}} \in \mathcal{X}'_{\text{int}},$$

$$\langle \mu_{\text{Dir}}, u \rangle = 0 \quad \forall \mu_{\text{Dir}} \in \mathcal{X}'_{\text{Dir}}.$$

Note that $\mathcal{X}_{t}(g)$ is still defined by $(2.8)$.

3. Discretization and domain decomposition

In this section we describe the finite element discretization of our problem combined with the T-FETI domain decomposition technique whose variant in the continuous setting was presented in the previous section. We start with the discretization of $(\mathcal{P}(g))$.

Let $\Omega^{k}$ be decomposed into $s_{k}$ non-overlapping polyhedral subdomains $\Omega^{k}_{i}$ so that $\overline{\Omega}^{k} = \bigcup_{i=1}^{s_{k}} \overline{\Omega}^{k}_{i}$, $k = 1, 2$. As in the previous section we denote by $\Gamma^{k}_{ij}$ a common (plane) interface between $\Omega^{k}_{i}$ and $\Omega^{k}_{j}$, $i \neq j$. All common interfaces will be again identified by the index sets $\mathcal{T}^{k}$. A regular partition of $\overline{\Omega}^{k}_{i}$ into hexahedrons will be denoted by $\mathcal{T}^{k}_{h}$. We will assume
that \( T_{ji}^{k} = T_{kj}^{l} \) for all \((i, j) \in I^k\), i.e., the nodes of \( T_{ji}^{k} \) and \( T_{kj}^{l} \) coincide on the common interface between \( \Omega_i^k \) and \( \Omega_j^l \). Moreover, \( T_{ji}^{k} \) will be compatible with \( \Gamma_{ij}^k \cap \partial \Omega_i^k \), \( \Gamma_{ij}^l \cap \partial \Omega_j^l \), and \( \Gamma_{ij}^k \cap \partial \Omega_j^l \), \( k = 1, 2 \). On \( T_{ji}^{k} \) we define the finite element space:

\[
\mathcal{V}_h = \mathcal{V}^{k}_{h} = \{ v^k_h \in (C(\overline{\Omega}))^3 : v^k_h \in (Q_1(T))^3 \text{ for all } T \in T_{ji}^{k} \},
\]

where \( Q_1(T) \) denotes the set of all trilinear polynomials on \( T \). Let \( n_{ki} := \dim \mathcal{V}^{k}_{h} \). Finally we introduce the product space:

\[
\mathcal{V}_h := \prod_{i=1}^{s_1} \mathcal{V}_{1,i}^{h} \times \prod_{i=1}^{s_2} \mathcal{V}_{2,i}^{h}, \quad \mathcal{V}_h = (v^1_h, v^2_h) \in \mathcal{V}_h.
\]

Let us observe that the Dirichlet boundary conditions are not included in the definition of \( \mathcal{V}_h \) and the non-zero jumps \([v^k_h]_{i,j}\) are allowed on the interfaces \( \Gamma_{ij}^k \), \((i, j) \in I^k \), \( k = 1, 2 \), for \( v_h \in \mathcal{V}_h \). Thus \( n := \dim \mathcal{V}_h = \sum_{i=1}^{s_1} \sum_{c=1}^{n_{ki}} n_{ki} \). As we shall see later, the definition of \( \mathcal{V}_h \) simplifies considerably the inversion of the stiffness matrix. In order to define the discretization \( \mathbb{K}_h \) of \( \mathbb{K} \) we introduce the index set

\[
I^c := \{ i | 1 \leq i \leq s_1, \text{meas}_2(\Gamma_{ci}^1) > 0, \Gamma_{ci}^1 := \partial \Omega_i^1 \cap \Gamma_c^1 \}.
\]

Then \( \mathcal{T}^c_i = \bigcup_{t \in I^c} \mathcal{T}^i_{ci} \). Let \( N_i := \{ \chi_{iq}^1 \}_{q=1}^{n_i}, i \in I^c \), be the set of all contact nodes of \( T_{ji}^{k} \), i.e., \( \chi_{iq}^1 \) is the vertex of a hexahedron \( T \in T_{ji}^{k} \) such that \( \chi_{iq}^1 \in \Gamma_{ci}^1 \setminus \Gamma_u^k \). The admissible set \( \mathbb{K}_h \) is defined as follows:

\[
\mathbb{K}_h := \mathcal{V}_h | v_{h,v}(\chi_{iq}^1) - d(\chi_{iq}^1) \leq 0 \forall q = 1, \ldots, m_i, \forall i \in I^c, \quad [v^k_h]_{i,j} = 0 \forall (i, j) \in I^k, \quad v^k_h = 0 \text{ on } \Gamma_u^k, \quad k = 1, 2,
\]

i.e., \( \mathbb{K}_h \) contains all continuous piecewise-trilinear vector functions in \( \Omega^1 \), \( \Omega^2 \) satisfying the homogeneous Dirichlet condition on \( \Gamma_u^1 \cup \Gamma_u^2 \) and the non-penetration condition at all contact nodes. Recall that \( v_{h,v}(\chi_{iq}^1) = (v^1_h(\chi_{iq}^1) - v^2_h(\chi(\chi_{iq}^1)) \Gamma \nu(\chi_{iq}^1)) \). The set \( \mathbb{K}_h \) is an external approximation of \( \mathbb{K} \), i.e., \( \mathbb{K}_h \not\subset \mathbb{K} \), in general.

In what follows we describe how to discretize the frictional term. With any \( \chi_{iq}^1 \in N_i \) we associate an element \( R_{iq} \) such that \( \Gamma_{ci}^1 = \bigcup_{q=1}^{m_i} \Gamma_{ci}^q \). Then

\[
\int_{\Gamma_{ci}^q} g \| \mathcal{F}v_{ht} \| \, ds = \sum_{i \in I^c} \sum_{q=1}^{m_i} \int_{R_{iq}} g \| \mathcal{F}v_{ht} \| \, ds.
\]

For the numerical evaluation of the integral over \( R_{iq} \) we use the cubature formula:

\[
\int_{R_{iq}} g \| \mathcal{F}v_{ht} \| \, ds \approx \text{meas}(R_{iq}) \bar{g}_{iq} \| \mathcal{F}v_{ht}(\chi_{iq}^1) \|,
\]

where

\[
\bar{g}_{iq} := \frac{1}{\text{meas}(R_{iq})} \int_{R_{iq}} g \, ds.
\]
where

\[ \int_{\Gamma^e} g \| \mathbf{Fv}_{ht} \| \, ds \approx \sum_{i \in I^e} \sum_{q=1}^{m_i} g_{iq} \| \mathbf{Fv}_{ht}(x_{iq}^1) \| =: j_h(v_h) \]

with \( g_{iq} := \int_{\Gamma_{iq}} g \, ds \).

The finite element approximation of (\( P(g) \)) reads as follows:

\[
\begin{align*}
\text{Find } u_h = u_h(g_h) \in \mathbb{K}_h & \text{ such that } \\
J_{gh}(u_h) & \leq J_{gh}(v_h) \quad \forall v_h \in \mathbb{K}_h, \\
\end{align*}
\]

(\( P_h(g_h) \))

where

\[ J_{gh}(v_h) := \frac{1}{2} a(v_h, v_h) - b(v_h) + j_h(v_h). \]

Problem (\( P_h(g_h) \)) can be written in the following algebraic form:

\[
\begin{align*}
\text{Find } \bar{u} \in \mathcal{K} & \text{ such that } \\
\mathcal{J}_g(\bar{u}) & \leq \mathcal{J}_g(\tilde{v}) \quad \forall \tilde{v} \in \mathcal{K}, \\
\end{align*}
\]

(\( \mathcal{P}(\tilde{g}) \))

where

\[ \mathcal{J}_g(\tilde{v}) := \frac{1}{2} \tilde{v}^T \mathbf{K} \tilde{v} - \tilde{v}^T \tilde{f} + \sum_{r=1}^{m} g_r \| \mathbf{F}_r \tilde{v}_r \|, \]

\[ \mathcal{K} := \{ \tilde{v} \in \mathbb{R}^n \mid \mathbf{N} \tilde{v} - \tilde{d} \leq 0, \mathbf{B}_e \tilde{v} = 0, \mathbf{B}_d \tilde{v} = 0 \}. \]

Here, \( \tilde{v} \in \mathbb{R}^n \) is a nodal displacement vector, \( \mathbf{K} \in \mathbb{R}^{n \times n} \) denotes the symmetric, positive semi-definite stiffness matrix, \( \tilde{f} \in \mathbb{R}^n \) is the load vector, and \( \mathbf{N} \in \mathbb{R}^{m \times n} \), \( m := \sum_{i \in I^e} m_i \), is the matrix whose the \( r \)-th row \( \mathbf{N}_r \) is such that \( \tilde{v} \mapsto \mathbf{N}_r \tilde{v}, \tilde{v} \in \mathbb{R}^n \) is the algebraic representation of the mapping \( v_h \mapsto v_{h,i} \chi(x_{iq}^1) \), \( v_h \in \mathcal{V}_h \) for an appropriate \( (i, q) \). Further \( \mathbf{B}_e \in \mathbb{R}^{m_e \times n} \) and \( \mathbf{B}_d \in \mathbb{R}^{m_d \times n} \) are the matrices representing the jumps across the inter-element boundaries and the Dirichlet boundary conditions, respectively. Finally, \( \mathbf{T}_1, \mathbf{T}_2 \in \mathbb{R}^{m \times n} \) are the matrices whose the \( r \)-th rows \( \mathbf{T}_r \) are such that \( \tilde{v} \mapsto \mathbf{T}_r \tilde{v}, \tilde{v} \in \mathbb{R}^n \) are the algebraic representations of the mappings \( v_h \mapsto v_{h,j} \chi(x_{iq}^1) := (v^1_{ij}(x_{iq}^1) - v^2_{ij}(x_{iq}^1)) \chi_j(x_{iq}^1) \), \( j = 1, 2 \) for an appropriate \( (i, q) \). We denote \( \tilde{v}_r = (\mathbf{T}_1 \tilde{v}, \mathbf{T}_2 \tilde{v})^T \in \mathbb{R}^2 \). This notation requires a one-to-one correspondence between the global indices \( r := r(i, q) \) and the local indices \( i, q \). Thus \( \mathbf{F}_r := \mathbf{F}(x_{iq}^1) \in \mathbb{R}^{2 \times 2} \) and \( \tilde{g}, \tilde{d} \in \mathbb{R}^m \) have the components \( g_r := g_{iq}, d_r := d(x_{iq}^1) \).

Remark 3.2. In general, the rows of the matrices \( \mathbf{N}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{B}_e, \mathbf{B}_d \) are linearly dependent that is not acceptable for the algorithm discussed below. Therefore redundant rows will be eliminated and the resulting full row-rank matrices will be denoted by the same symbols.

4. Dual problem

Formulation (\( \mathcal{P}(\tilde{g}) \)) is not suitable for direct computations as the constraints in \( \mathcal{K} \) can be hardly handled for large-scale problems. Moreover, the functional \( \mathcal{J}_{\tilde{g}} \) is non-differentiable.
due to the frictional term. To overcome these difficulties, we will use the dual formulation of \( (\mathcal{P}(\vec{g})) \).

Let us denote:

\[
\begin{align*}
X(\vec{g}) := X_{\nu^+} \times X_t(\vec{g}) \times \mathbb{R}^{m_c} \times \mathbb{R}^{m_d},
\end{align*}
\]

where

\[
\begin{align*}
X_{\nu^+} &= \{ \vec{\mu}_\nu \in \mathbb{R}^m | \vec{\mu}_\nu \geq 0 \}, \\
X_t(\vec{g}) &= \{ (\vec{\mu}_t_1, \vec{\mu}_t_2) \in \mathbb{R}^m \times \mathbb{R}^m | \| \mathbf{F}_r^{-1} \vec{\mu}_t^r \| \leq g_r, 1 \leq r \leq m \},
\end{align*}
\]

where \( \vec{\mu}_t^r := (\mu_{t_1r}, \mu_{t_2r})^T \in \mathbb{R}^2 \) and \( \mu_{t,r} \) is the \( r \)-th component of \( \vec{\mu}_{t_i} \in \mathbb{R}^m, i = 1, 2 \). In other words: \( \vec{\mu}_{t_i} \) collects the discrete tangential stress in the direction \( t_i, i = 1, 2 \), while \( \vec{\mu}_t^r \) is the complete tangential stress (in both directions) at the \( r \)-th contact node. The frictional term in \( (\mathcal{P}(\vec{g})) \) can be written as follows:

\[
\begin{align*}
\sum_{r=1}^{m} g_r \| \mathbf{F}_r \vec{v}_t^r \| = \sum_{r=1}^{m} \max_{\vec{\mu}_t^r \in \mathbb{R}^2 \| \vec{\mu}_t^r \| \leq g_r} (\vec{\mu}_t^r)^T \mathbf{F}_r \vec{v}_t^r = \max_{(\vec{\mu}_t_1, \vec{\mu}_t_2) \in X_t(\vec{g})} \sum_{r=1}^{m} (\vec{\mu}_t^r)^T \vec{v}_t^r. \quad (4.9)
\end{align*}
\]

Thus

\[
\min_{\vec{v} \in \mathcal{K}} J_g(\vec{v}) = \min_{\vec{\mu} \in X(\vec{g})} \max_{\vec{v} \in \mathcal{K}} \mathcal{L}(\vec{v}, \vec{\mu}),
\]

where

\[
\mathcal{L}(\vec{v}, \vec{\mu}) := \frac{1}{2} \vec{v}^\top K \vec{v} - \vec{v}^\top \vec{f} + \vec{\mu}^\top (B \vec{v} - \vec{c})
\]

is the Lagrangian and

\[
\begin{align*}
\vec{\mu} &:= (\vec{\mu}_\nu^T, \vec{\mu}_t_1, \vec{\mu}_t_2, \vec{\mu}_e^T, \vec{\mu}_d^T)^T \in X(\vec{g}), \\
B &:= (N^T, T_1^T, T_2^T, B_e^T, B_d^T)^T \in \mathbb{R}^{(3m + m_c + m_d) \times n}, \\
\vec{c} &:= (\vec{d}^T, \vec{0}^T, \vec{0}^T, \vec{0}^T)^T \in \mathbb{R}^{3m + m_c + m_d}.
\end{align*}
\]

Instead of \( (\mathcal{P}(\vec{g})) \) we shall use its saddle-point formulation:

\[
\begin{align*}
\text{Find } (\vec{u}, \vec{\lambda}) \in \mathbb{R}^n \times X(\vec{g}) \text{ such that } \\
\mathcal{L}(\vec{u}, \vec{\mu}) \leq \mathcal{L}(\vec{u}, \vec{\lambda}) \leq \mathcal{L}(\vec{v}, \vec{\lambda}) \quad \forall (\vec{v}, \vec{\mu}) \in \mathbb{R}^n \times X(\vec{g}),
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\text{Find } (\vec{u}, \vec{\lambda}) \in \mathbb{R}^n \times X(\vec{g}) \text{ such that } \\
K \vec{u} = \vec{f} - B^\top \vec{\lambda}, \\
(\vec{\mu} - \vec{\lambda})^T (B \vec{u} - \vec{c}) \leq 0 \quad \forall \vec{\mu} \in X(\vec{g}).
\end{align*}
\]

(\( \mathcal{M}(\vec{g}) \))
Remark 4.3. The set $X_t(\bar{g})$ is given by the system of the elliptical constraints associated with the contact nodes. If the frictional term in $\mathbf{J}_g$ was regularized by using the second expression in (4.9) (instead of the last one), we would arrive at $(\mathcal{M}(\bar{g}))$ with the following modification of $X_t(\bar{g})$ and $B$: $X_t(\bar{g}) = \{ (\bar{\mu}_t, \bar{\mu}_s) \in \mathbb{R}^m \times \mathbb{R}^m \mid \| \bar{\mu}_t \| \leq g_r, 1 \leq r \leq m \}$, while the blocks $T_1, T_2$ are replaced by $D_1 T_1, D_2 T_2$, where $D_i \in \mathbb{R}^{m \times m}$, $i = 1, 2$, is the diagonal matrix with the values of $\mathcal{F}_i$ at the contact nodes on its diagonal. This transform enables us to change the elliptical constraints to the circular ones. In the computational part of the paper we shall compare both strategies.

Now we eliminate the unknown $\bar{u}$. Let us note that $K$ is singular with a priori known kernel (rigid body modes). Therefore we use a generalized inverse $K^\dagger$ to $K$ satisfying $K = KK^\dagger K$ and a full rank matrix $R \in \mathbb{R}^{n \times l}$ whose columns span $\text{Ker} K$, $l = \dim \text{Ker} K$. The first equation in $(\mathcal{M}(\bar{g}))$ is solvable if $(\bar{f} - B^\top \bar{\lambda}) \in \text{Im} K$. As $\text{Ker} K$ is the orthogonal complement of $\text{Im} K$ in $\mathbb{R}^n$, this means $(\bar{f} - B^\top \bar{\lambda}) \perp \text{Ker} K$, i.e.,

$$R^\top (\bar{f} - B^\top \bar{\lambda}) = 0. \quad (4.11)$$

If (4.11) holds, then

$$\bar{u} = K^\dagger (\bar{f} - B^\top \bar{\lambda}) + R\bar{\alpha} \quad (4.12)$$

for an appropriate $\bar{\alpha} \in \mathbb{R}^l$. Substituting (4.12) into $(\mathcal{M}(\bar{g}))$ we arrive at the dual formulation of $(\mathcal{P}(\bar{g}))$:

Find $\bar{\lambda} \in \Lambda(\bar{g})$ such that

$$S(\bar{\lambda}) \leq S(\bar{\mu}) \quad \forall \bar{\mu} \in \Lambda(\bar{g}), \quad (\mathcal{D}(\bar{g}))$$

where

$$S(\bar{\mu}) := \frac{1}{2} \bar{\mu}^\top F \bar{\mu} - \bar{\mu}^\top \bar{h},$$

$$\Lambda(\bar{g}) := \{ \bar{\mu} \in X(\bar{g}) \mid G \bar{\mu} = \bar{c} \}$$

with $F := BK^\dagger B^\top$ being symmetric, positive semi-definite, $\bar{h} := BK^\dagger \bar{f} - \bar{c}$, $G := SR^\top B^\top$, $\bar{c} := SR^\top \bar{f}$, and $S$ being a nonsingular matrix that defines the orthonormalization of the rows of the full row-rank matrix $R^\top B^\top$.

It should be noted that $(\mathcal{D}(\bar{g}))$ is more convenient for numerical solution than $(\mathcal{P}(\bar{g}))$ as the functional $S$ is differentiable (quadratic) and the feasible set $\Lambda(\bar{g})$ is defined by the linear equality and separable quadratic inequality constraints. Recently developed algorithms [5, 25, 26, 6] for such type of problems will be shortly presented in the next section. Having $\bar{X}$ from $(\mathcal{D}(\bar{g}))$ at our disposal one can easily obtain $\bar{u}$ from (4.12) provided that $\bar{\alpha}$ is known. Next we describe how to get $\bar{\alpha}$ solely from $\bar{X}$.

Let us note that the last inequality in $(\mathcal{M}(\bar{g}))$ implies

$$(\bar{\mu}_v - \bar{\lambda}_v)^\top (N \bar{u} - \bar{d}) \leq 0 \quad \forall \bar{\mu}_v \in \mathcal{X}_{v+},$$

$$(\bar{\mu}_t_1 - \bar{\lambda}_t_1)^\top T_1 \bar{u} + (\bar{\mu}_t_2 - \bar{\lambda}_t_2)^\top T_2 \bar{u} \leq 0 \quad \forall (\bar{\mu}_t_1, \bar{\mu}_t_2)^\top \in \mathcal{X}_t(\bar{g}),$$
which is equivalent to
\[ N_r \vec{u} - d_r \leq 0, \quad \lambda_{\nu r} \geq 0, \quad \lambda_{\nu r}(N_r \vec{u} - d_r) = 0, \quad 1 \leq r \leq m, \] (4.13)
and
\[
\begin{align*}
\vec{u}_i^r &= \vec{0} & \Rightarrow & \quad \| \mathcal{F}_r^{-1} \vec{\lambda}_i^r \| \leq g_r \\
\vec{u}_i^r \neq \vec{0} & \Rightarrow & \quad \mathcal{F}_r^{-1} \vec{\lambda}_i^r = g_r \| \mathcal{F}_r \vec{u}_i^r \| 
\end{align*}
\] (4.14)
respectively, where \( \vec{u}_i^r = (T_{1r}, \vec{u}, T_{2r}, \vec{u})^T \in \mathbb{R}^2 \), \( \vec{\lambda}_i^r := (\lambda_{1r}, \lambda_{2r})^T \in \mathbb{R}^2 \). With the solution \( \vec{\lambda} \) of \((\mathcal{D}(\vec{g}))\) we associate two index sets:
\[ \mathcal{I}_\nu := \{ r | \lambda_{\nu r} > 0, 1 \leq r \leq m \}, \quad \mathcal{I}_t := \{ r | \| \mathcal{F}_r^{-1} \vec{\lambda}_i^r \| < g_r, 1 \leq r \leq m \}, \]
characterizing the nodes, where the strong contact and the strong stick, respectively, occurs. Let \( \mathbf{B}' \) and \( \vec{\epsilon}' \) be constructed from \( \mathbf{B} \) and \( \vec{\epsilon} \) by omitting those rows of \( \mathbf{N}, \mathbf{T}_1, \mathbf{T}_2 \) whose indices do not belong to \( \mathcal{I}_\nu \) and \( \mathcal{I}_t \), respectively, i.e.,
\[
\begin{align*}
\mathbf{B}' &:= (\mathbf{N}_{\mathcal{I}_\nu}, \mathbf{T}_{1\mathcal{I}_\nu}^T, \mathbf{T}_{2\mathcal{I}_\nu}^T, \mathbf{B}_e^T, \mathbf{B}_d^T)^T, \\
\vec{\epsilon}' &:= (\vec{d}_{\mathcal{I}_\nu}^T, \vec{0}_{\mathcal{I}_\nu}^T, \vec{0}_{\mathcal{I}_\nu}^T, \vec{0}_F^T, \vec{0}_F^T, \vec{0}_T^T)^T,
\end{align*}
\]
where the subscripts indicate the omitted rows. Multiplying (4.12) by \( \mathbf{B}' \) and using (4.13), (4.14) we see that \( \vec{\epsilon}' = \mathbf{B}' \mathbf{K}^\dagger(\vec{f} - \mathbf{B}^\top \vec{\lambda}) + \mathbf{B}' \mathbf{R} \vec{\alpha} \). As \( \mathbf{B}' \mathbf{R} \) has the full column-rank, we obtain
\[
\vec{\alpha} = (\mathbf{R}^\top \mathbf{B}'^\top \mathbf{B}' \mathbf{R})^{-1} \mathbf{R}^\top \mathbf{B}'^\top (\vec{\epsilon}' - \mathbf{B}' \mathbf{K}^\dagger(\vec{f} - \mathbf{B}^\top \vec{\lambda})).
\] (4.15)

The formulas (4.12), (4.15) together with a method for solving \((\mathcal{D}(\vec{g}))\) give us a way how to find the solution to \((\mathcal{P}(\vec{g}))\). In the rest of the section we recall how to solve discrete contact problems with Coulomb type friction.

Let us introduce the mapping \( \Psi : \mathbf{X}_{\nu+} \mapsto \mathbf{X}_{\nu+} \) by:
\[
\Psi : \vec{g} \mapsto \vec{\lambda}_\nu, \quad \vec{g} \in \mathbf{X}_{\nu+},
\]
where \( \vec{\lambda}_\nu \) is the first subvector in \( \vec{\lambda} \) such that \( -\vec{\lambda}_\nu \) corresponds to the discrete normal contact stress. By a discrete solution to contact problems with Coulomb type friction we call any \( \vec{u} \in \mathcal{K} \) such that
\[
\Psi(\vec{\lambda}_\nu) = \vec{\lambda}_\nu,
\]
where \( (\vec{u}, \vec{\lambda}_\nu) \) solves \((\mathcal{M}(\vec{\lambda}_\nu))\). In other words, \( \vec{\lambda}_\nu \) is a fixed-point of \( \Psi \) in \( \mathbf{X}_{\nu+} \). To find it we use the method of successive approximations:
\[
\text{Initialize } \vec{\lambda}_\nu^{(0)} \in \mathbf{X}_{\nu+} : \quad \vec{\lambda}_\nu^{(k+1)} = \Psi(\vec{\lambda}_\nu^{(k)}), \quad k = 0, 1, 2, \ldots
\] (4.16)
It is well-known that the sequence \( \{\vec{\lambda}_\nu^{(k)}\} \) generated by (4.16) converges to the fixed-point \( \vec{\lambda}_\nu \) if \( \Psi \) is contractive in \( \mathbf{X}_{\nu+} \). In addition, such a fixed-point is unique. In the case
of isotropic friction this property holds provided that the coefficient of friction is small enough [13]. The same result has been extended to orthotropic friction assuming that the elements of $\mathbf{F}$ are small [16]. As the evaluation of $\Psi(\lambda^{(k)}_\nu)$ requires to solve $(D(\bar{\mathbf{g}}))$ with $\bar{\mathbf{g}} := (\bar{\lambda}^{(k)}_\nu)$, we arrive at the following algorithm.

**Algorithm 4.1.** Let $\bar{\lambda}^{(0)} \in X_{\nu+}$ and $\varepsilon > 0$ be given. Set $k := 0$.

(i) Solve $\bar{\lambda}^{(k+1)} := \arg \min S(\bar{\mathbf{h}})$ subject to $\bar{\mathbf{h}} \in \Lambda(\bar{\lambda}^{(k)}_\nu)$.

(ii) Set $\text{err}^{(k)} := \|\bar{\lambda}^{(k+1)} - \bar{\lambda}^{(k)}\|/\|\bar{\lambda}^{(k+1)}\|$. If $\text{err}^{(k)} \leq \varepsilon$, go to step (iii), else set $k := k + 1$ and go to step (i).

(iii) Set $\bar{\lambda} := \bar{\lambda}^{(k+1)}$, compute $\bar{\alpha}$ and $\bar{u}$ by (4.15) and (4.12), respectively.

### 5. Algorithms

In this section, we introduce the main ideas of the SMALSE-M algorithm [5] for solving $(D(\bar{\mathbf{g}}))$ which combines the augmented Lagrangian approach in the outer loop with the conjugate gradient method, the gradient projections, and an effective precision control in the inner loop.

Before doing that, we rewrite $(D(\bar{\mathbf{g}}))$ into the equivalent form involving only homogeneous equality constraints. The solution $\bar{\lambda}$ to $(D(\bar{\mathbf{g}}))$ can be uniquely decomposed into $\bar{\lambda}_{lm} \in \text{Im} \mathbf{G}^\top$ and $\bar{\lambda}_{\text{Ker}} \in \text{Ker} \mathbf{G}$:

$$\bar{\lambda} = \bar{\lambda}_{lm} + \bar{\lambda}_{\text{Ker}}. \tag{5.17}$$

It is readily seen that

$$\bar{\lambda}_{lm} = \mathbf{G}^\top \bar{\mathbf{e}}. \tag{5.18}$$

Since $\bar{\lambda}_{lm}$ can be easily evaluated from (5.18), it remains to show how to compute $\bar{\lambda}_{\text{Ker}}$. Inserting (5.17) into $(D(\bar{\mathbf{g}}))$ we obtain the new minimization problem for $\bar{\lambda}_{\text{Ker}} \in \Lambda^\#(\mathbf{g}) \cap \text{Ker} \mathbf{G}$, where

$$\Lambda^\#(\mathbf{g}) := \{ \bar{\mathbf{h}} \in \mathbb{R}^{3m + m_e + m_d} | \bar{\mathbf{h}} + \bar{\lambda}_{lm} \in X(\bar{\mathbf{g}}) \},$$

with the following modification of $S$: $\bar{\mathbf{h}}$ is now replaced by $\bar{\mathbf{h}}^\# := \bar{\mathbf{h}} - \mathbf{F}\bar{\lambda}_{lm}$. Finally we introduce the orthogonal projector

$$P : \mathbb{R}^{3m + m_e + m_d} \mapsto \text{Ker} \mathbf{G}, \quad P = I - \mathbf{G}^\top \mathbf{G}. \tag{5.19}$$

Since $\bar{\lambda}_{\text{Ker}} \in \text{Ker} \mathbf{G}$, it satisfies $\bar{\lambda}_{\text{Ker}} = P\bar{\lambda}_{\text{Ker}}$. Therefore $\bar{\lambda}_{\text{Ker}}$ solves the following projected form of $(D(\bar{\mathbf{g}}))$:

$$\text{Find } \bar{\lambda}_{\text{Ker}} \in \Lambda^\#(\mathbf{g}) \cap \text{Ker} \mathbf{G} \text{ such that } \left\{ \begin{array}{l} S^\#(\bar{\lambda}_{\text{Ker}}) \leq S^\#(\bar{\mathbf{h}}) \quad \forall \bar{\mathbf{h}} \in \Lambda^\#(\mathbf{g}) \cap \text{Ker} \mathbf{G}, \end{array} \right. \tag{D^\#(\bar{\mathbf{g}})}$$

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Algorithm 5.2. (SMALSE-M [5])

Given $\bar{\beta}^{(0)} \in \mathbb{R}^l$, $\varepsilon > 0$, $\rho > 0$, $M_0 > 0$, $\eta > 0$, and $\beta > 1$.

Set $k := 0$ and $\varepsilon_1 = \varepsilon \|P \hat{h}^\#\|.$

(i) Find $\bar{\mu}^{(k)} \in \Lambda^\#(\bar{g})$ as an approximate solution to

$$\text{minimize } \mathcal{L}^\#(\bar{\mu}, \bar{\beta}^{(k)}) \quad \text{subject to } \bar{\mu} \in \Lambda^\#(\bar{g})$$

(5.19)

satisfying $\|\nabla_{\bar{\mu}}^{\text{proj}} \mathcal{L}^\#(\bar{\mu}^{(k)}, \bar{\beta}^{(k)})\| \leq \min\{M_k \|G \bar{\mu}^{(k)}\|, \eta\}$.

(ii) If $\|\nabla_{\bar{\mu}}^{\text{proj}} \mathcal{L}^\#(\bar{\mu}^{(k)}, \bar{\beta}^{(k)})\| \leq \varepsilon_1$ and $\|G \bar{\mu}^{(k)}\| \leq \varepsilon_1/\|PFP\|$, then $\bar{\lambda}_{\text{Ker}} := \bar{\mu}^{(k)}$ and stop.

(iii) Compute: $\bar{\beta}^{(k+1)} = \bar{\beta}^{(k)} + \rho G \bar{\mu}^{(k)}$.

(iv) Update the precision control for (5.19) as follows: if $k > 0$ and

$$\mathcal{L}^\#(\bar{\mu}^{(k)}, \bar{\beta}^{(k)}) < \mathcal{L}^\#(\bar{\mu}^{(k-1)}, \bar{\beta}^{(k-1)}) + \frac{\rho}{2} \|G \bar{\mu}^{(k)}\|^2,$$

then $M_{k+1} = M_k / \beta$, else $M_{k+1} = M_k$.

(v) Set $k := k + 1$ and go to step (i).

This algorithm can be viewed as a generalization of the recent SMALBE algorithm (Semi-Monotonic Augmented Lagrangian algorithm for Bound and Equality constrained problems [2, 3]). Its convergence analysis in [5] shows that it requires only $O(1)$ matrix-vector multiplications independently of the problem size if applied to a class of problems with uniformly bounded spectrum of the matrix $PFP$. The assumption on the spectrum is satisfied if the ratio between the maximal diameter of the subdomains $\Omega_i^k$ and the norm
of $T_{hi}^k$, $i = 1, \ldots, s$, $k = 1, 2$, is fixed and the matrix $PP$ is normalized in the spectral norm (see [7]). Thus the choice of a method which is able to treat efficiently separable quadratic constraints in the inner loop (5.19) plays the key role in our algorithm. Such method called KPRGP (KKT Proportioning with Reduced Gradient Projections) has been proposed in [25] and analyzed in [26]. As its use is crucial also for treating the orthotropic Coulomb friction law, we mention its main ideas below. Alternatively one can use the MPGP (Modified Proportioning with Gradient Projections) algorithm introduced in [6].

Let $M = \{1, \ldots, 3m + m_e + m_d\}$ and

$$\vec{r} := \vec{r}(\vec{\mu}) = \nabla_{\vec{\mu}} \mathcal{L}^\#(\vec{\mu}, \vec{\beta}^{(k)})$$

denote the partial gradient of $\mathcal{L}^\#$ at $\vec{\mu} \in \Lambda^\#(\vec{g})$. Let $A \subseteq M$ be the active set containing all indices for which the constraints at $\vec{\mu} \in \Lambda^\#(\vec{g})$ are active, i.e.:

$$\mathcal{A} := \mathcal{A}(\vec{\mu}) = \{r | \mu_{vr} = -\lambda_{lm, r}, 1 \leq r \leq m\}$$

$$\bigcup \{q | q = r + m, \|F_r^{-1}(\vec{\mu}_r + \vec{X}_{lm,r})\| = g_r, 1 \leq r \leq m\}$$

$$\bigcup \{q | q = r + 2m, \|F_r^{-1}(\vec{\mu}_r + \vec{X}_{lm,t})\| = g_r, 1 \leq r \leq m\}.$$ 

Recall that the vectors $\vec{\mu}$ and $\vec{X}_{lm}$ are decomposed as in (4.10) and $\vec{X}_{lm,t} \in \mathbb{R}^2$ consists of the $r$th components of $\vec{X}_{lm,t_1}$ and $\vec{X}_{lm,t_2}$. Since $\Lambda^\#(\vec{g})$ is convex, the projection $P_{\Lambda^\#(\vec{g})} : \mathbb{R}^{3m + m_e + m_d} \mapsto \Lambda^\#(\vec{g})$ is well-defined and one can introduce the so-called reduced gradient at $\vec{\mu} \in \Lambda^\#(\vec{g})$ for a fixed $\alpha > 0$ by:

$$\vec{r}^R := \vec{r}^R(\vec{\mu}) = \frac{1}{\alpha}(\vec{\mu} - P_{\Lambda^\#(\vec{g})}(\vec{\mu} - \vec{r}(\vec{\mu}))).$$

The reduced gradient enables us to write the alternative optimality condition satisfied by the solution $\vec{\mu}(\vec{\beta}^{(k)}) \in \Lambda^\#(\vec{g})$ of (5.19) in the form $\vec{r}^R(\vec{\mu}(\vec{\beta}^{(k)})) = 0$. The KPRGP algorithm is based on the fact that the non-zero components of $\vec{r}^R(\vec{\mu})$ at $\vec{\mu} \neq \vec{\mu}(\vec{\beta}^{(k)})$, $\vec{\mu} \in \Lambda^\#(\vec{g})$, determine the descent directions which change appropriately the active set $\mathcal{A}$. To this end, we split $\vec{r}^R(\vec{\mu})$ into the free reduced gradient $\vec{\phi} := \vec{r}^F(\vec{\mu})$ and the chopped reduced gradient $\vec{\psi} := \vec{r}^C(\vec{\mu})$ as follows:

$$\vec{\phi}_A = 0, \quad \vec{\phi}_{M \setminus A} = \vec{r}^R_{M \setminus A},$$

$$\vec{\psi}_A = \vec{r}^R_A, \quad \vec{\psi}_{M \setminus A} = 0,$$

where $\vec{\phi}_A$ and $\vec{\phi}_{M \setminus A}$ denote the sub-vectors of $\vec{\phi}$ with entries determined by the indices of $A$ and $M \setminus A$, respectively (similarly for $\vec{r}^R$ and $\vec{\psi}$).

We combine the following three steps to generate a sequence $\{\vec{\mu}^{(l)}\}$ that approximates the solution to (5.19):

- **the expansion step:** $\vec{\mu}^{(l+1)} = \vec{\mu}^{(l)} - \alpha \vec{r}(\vec{\mu}^{(l)})$,
• the proportioning step: \( \tilde{\mu}^{(l+1)} = \tilde{\mu}^{(l)} - \tilde{\alpha}_c \bar{\psi}(\tilde{\mu}^{(l)}) \),

• the conjugate gradient step: \( \tilde{\mu}^{(l+1)} = \tilde{\mu}^{(l)} - \alpha_c \tilde{p}^{(l)} \), where the step-length \( \alpha_c \) and the conjugate gradient directions \( \tilde{p}^{(l)} \) are computed recurrently [12]; the recurrence starts at \( \tilde{\mu}^{(s)} \) which is generated by the last expansion or the proportioning step and satisfies \( \mathcal{A}(\tilde{\mu}^{(l+1)}) = \mathcal{A}(\tilde{\mu}^{(s)}) \).

The expansion step may add while the proportioning step may release indices to/from the current active set. The conjugate gradient steps are used to carry out efficiently the minimization of \( \mathcal{L}'(\tilde{\mu}, \beta^{(l)}) \) in the interior of the face \( \mathcal{W}(\tilde{\mu}^{(s)}) = \{ \tilde{\mu} \in \Lambda^{#}(\tilde{g}) | \tilde{\mu}_A := \tilde{\mu}^{(s)}, \mathcal{A} = \mathcal{A}(\tilde{\mu}^{(s)}) \} \). Moreover, the algorithm exploits a given constant \( \gamma > 0 \) and the proportioning criterion

\[
\bar{\psi}(\tilde{\mu}^{(l)}) \bar{r}^R(\tilde{\mu}^{(l)}) = \gamma \bar{\phi}(\tilde{\mu}^{(l)}) \bar{r}^R(\tilde{\mu}^{(l)})
\]

(5.20) to decide which of the steps will be performed.

Algorithm 5.3. (KPRGP [26, 5])

Let \( \tilde{\mu}^{(0)} \in \Lambda^{#}(\tilde{g}) \), \( \gamma > 0 \), \( \tilde{\alpha} \in (0, 2\|PP + \rho GG^\top\|^{-1}) \), and \( \varepsilon_\lambda \geq 0 \) be given. For \( \tilde{\mu}^{(l)} \), \( \tilde{\mu}^{(s)} \) known, \( 0 \leq s \leq l \), where \( \tilde{\mu}^{(s)} \) is computed by the last expansion or proportioning step, choose \( \tilde{\mu}^{(l+1)} \) by the following rules:

(i) If \( \|\bar{r}^R(\tilde{\mu}^{(l)})\| \leq \varepsilon_\lambda \), return \( \bar{X} := \tilde{\mu}^{(l)} \).

(ii) If \( \tilde{\mu}^{(l)} \) fulfills (5.20), try to generate \( \tilde{\mu}^{(l+1)} \) by the conjugate gradient step. If \( \tilde{\mu}^{(l+1)} \in \text{Int} \mathcal{W}(\tilde{\mu}^{(s)}) \), accept it, otherwise generate \( \tilde{\mu}^{(l+1)} \) by the expansion step.

(iii) If \( \tilde{\mu}^{(l)} \) does not fulfills (5.20), generate \( \tilde{\mu}^{(l+1)} \) by the proportioning step.

Convergence of this algorithm does not depend on the type of convex constraints [26], however its implementation requires to realize the projection \( P_{\Lambda^{#}(\tilde{g})} \). As the constraints in \( \Lambda^{#}(\tilde{g}) \) are separable, we split \( P_{\Lambda^{#}(\tilde{g})} \) into the single projections \( P_{\Lambda^{#r}_{\mu,\nu}} : \mathbb{R} \mapsto \Lambda^{#r}_{\mu,\nu} \) and \( P_{\Lambda^{#r}_{\mu,\nu}(\tilde{g})} : \mathbb{R}^2 \mapsto \Lambda^{#r}_{\mu,\nu}(\tilde{g}) \), \( 1 \leq r \leq m \), where

\[
\Lambda^{#r}_{\mu,\nu} = \{ \mu_{\nu r} \in \mathbb{R} | \mu_{\nu r} \geq -\lambda_{\mu,\nu, r} \},
\]

\[
\Lambda^{#r}_{\mu,\nu}(\tilde{g}) = \{ \tilde{\mu}^r \in \mathbb{R}^2 | \| \mathcal{F}^{-1}_r(\tilde{\mu}^r_l + \tilde{X}^r_{\mu,\nu}) \| \leq g_r \},
\]

respectively. It is readily seen that

\[
P_{\Lambda^{#r}_{\mu,\nu}}(\mu_{\nu r}) = \begin{cases} 
\mu_{\nu r} & \text{for } \mu_{\nu r} \geq -\lambda_{\mu,\nu, r}, \\
-\lambda_{\mu,\nu, r} & \text{otherwise}.
\end{cases}
\]

The projection \( P_{\Lambda^{#r}_{\mu,\nu}(\tilde{g})} \) is more involved:

\[
P_{\Lambda^{#r}_{\mu,\nu}(\tilde{g})}(\tilde{\mu}^r_l) = \begin{cases} 
\tilde{\mu}^r_l & \text{for } \| \mathcal{F}^{-1}_r(\tilde{\mu}^r_l + \tilde{X}^r_{\mu,\nu}) \| \leq g_r, \\
\tilde{\sigma}^r_l & \text{otherwise},
\end{cases}
\]
where $\vec{\sigma}_t^r \in \mathbb{R}^2$ is given by the explicit formula solely for isotropic friction. Indeed, if $\mathcal{F}_{r,1} = \mathcal{F}_{r,2} =: \mathcal{F}_r$, then

$$\vec{\sigma}_t^r = -\vec{\lambda}_{lm,t} + \frac{\mathcal{F}_r \vec{g}_r}{\|\vec{\mu}_t + \vec{\lambda}_{lm,t}\|}(\vec{\mu}_t + \vec{\lambda}_{lm,t}).$$

The simplest way how to find $\vec{\sigma}_t^r$ when $\mathcal{F}_{r,1} \neq \mathcal{F}_{r,2}$ is based on the following parametric representation of the elliptical boundary of $\Lambda_{lm,t}(\vec{g})$:

$$\vec{\chi}_t^r(\tau) = -\vec{\lambda}_{lm,t} + g_r \mathcal{F}_r \left( \begin{array}{c} \cos \tau \\ \sin \tau \end{array} \right), \quad \tau \in [0, 2\pi).$$

Let $\tau^*$ be such that $\vec{\sigma}_t^r = \vec{\chi}_t^r(\tau^*)$. Then $\tau^*$ is characterized by the orthogonality condition:

$$(\vec{\mu}_t - \vec{\chi}_t^r(\tau^*))^\top \frac{d}{d\tau} \vec{\chi}_t^r(\tau^*) = 0. \quad (5.21)$$

Note that (5.21) is equivalent to the fourth degree polynomial equation with at most four roots. Fortunately, the projection is easily recognized by the fact that it belongs to the same quadrant as the projected point (in the local coordinate system). In order to solve (5.21) we combine the Newton with the bisection methods in $\mathbb{R}^1$. The resulting algorithm may benefit from the fast convergence of the Newton iterations while the bisection steps check that it converges to $\tau^*$. Note that a long sequence of bisection iterations may be generated in situations when the root $\tau^*$ is also the inflection point.

6. Numerical examples

Let us consider a 3D contact problem of two cantilever beams of sizes 2000x1000x1000 [mm] in mutual contact with different coefficients of friction in two orthogonal directions to describe specially milled contact surface. The geometry, the prescribed boundary conditions and material properties are specified in Fig. 6.1. The milled surface is depicted in Fig. 6.2. Finally, the volume forces are neglected and the coefficients of friction $\mathcal{F}_1$ and $\mathcal{F}_2$ on the contact interface are chosen in four different ways:

(a) Frictionless case - friction is neglected (Example 6.1);

(b) Isotropic case - $\mathcal{F}_1 = \mathcal{F}_2 = 0.3$ (Example 6.2);

(c) Anisotropic case - $\mathcal{F}_1 = 0.5$ in the direction $\mathbf{t}_1 = (1, 0, 0)^\top$ and $\mathcal{F}_2 = 0.1$ in $\mathbf{t}_2 = (0, 1, 0)^\top$ (Example 6.3);

(d) Anisotropic case - $\mathcal{F}_1 = 0.5$ in the direction $\mathbf{t}_1 = (\sqrt{2}/2, -\sqrt{2}/2, 0)^\top$ and $\mathcal{F}_2 = 0.1$ in $\mathbf{t}_2 = (\sqrt{2}/2, \sqrt{2}/2, 0)^\top$ (Example 6.4).
Case (d) corresponds to the real measurements, case (c) to incorrectly chosen tangential directions, case (b) to averaged coefficients which is a routinely used approach in engineering practise, and case (a) is for comparison purposes.

Each brick is divided into the same number of cubic subdomains with the decomposition step \(H\) and each subdomain is then decomposed into hexahedrons with the discretization step \(h\); see Fig. 6.3. To demonstrate the behavior of our algorithms, we resolved the problem with varying discretizations and decompositions keeping \(H/h = 10\).

We use an inexact implementation of Algorithm 4.1 (method of successive approximations) which performs only one iteration of SMALSE-M in each step (i). In other words the resulting algorithm can be viewed as the modification of SMALSE-M in which the feasible set \(\Lambda^\#(\bar{g})\) in (5.19) is updated after each (outer) step by \(\bar{g} := \bar{\lambda}^{(k)}\), where \(\bar{\lambda}^{(k)}\) is the first subvector in \(\bar{\lambda}^{(k)} := \bar{\mu}^{(k-1)} + \bar{\lambda}_{lm}\) for \(k > 0\) and \(\bar{\lambda}^{(0)} := 0\). Finally we have to specify the choice of the parameters in SMALSE-M and KPRGP. The analysis in [26] and [5] indicates that the optimal values of \(\gamma, \tilde{\alpha}, \rho\), and \(\eta\) are \(\gamma := 1, \tilde{\alpha} \approx 2\|PFP\|^{-1}, \rho \approx 1\).
\( \rho \approx \| \mathbf{PFP} \| \), and \( \eta := \| \mathbf{Ph}^\# \| \), respectively, where \( \| \mathbf{PFP} \| \) is computed by a few iterations of the Lanczos method. The remaining parameters were \( \varepsilon := 10^{-4} \), \( M_0 = 1 \), and \( \beta := 10 \). The initial iterate is \( \beta^{(0)} := \mathbf{0} \).

The examples were computed by MatSol library [22] developed in Matlab environment and parallelized by Matlab Distributed Computing Engine. For all computations we used 24CPUs of the HP Blade system, model BLc7000. Since convergence of SMALSE-M is influenced by conditioning of the matrix \( \mathbf{PFP} \) (on \( \text{Ker} \mathbf{G} \)), the total efficiency of our solver depends on the stable computation of the generalized inverse \( \mathbf{K}^\dagger \) to the stiffness matrix \( \mathbf{K} \) [28]. To evaluate actions of \( \mathbf{K}^\dagger \), we used the modified Cholesky factorization proposed in [1]. Finally note that the use of the T-FETI domain decomposition method simplifies considerably the construction of the matrix \( \mathbf{R} \) made of the basis of \( \text{Ker} \mathbf{K} \). Since the algorithm treats all subdomains as floating, each one possesses six rigid body motions. Moreover, these rigid body motions are known à-priori so that \( \mathbf{R} \) is not affected by any round-off error.

**Example 6.1.** We start with the frictionless case. The solution characteristics are summarized in Table 6.1. We can observe that the number of matrix–vector multiplications increases only moderately in agreement with the theory [5]. The distribution of the normal contact stress along the contact interface is depicted in Fig. 6.4.

<table>
<thead>
<tr>
<th>Table 6.1: Frictionless problem</th>
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<tbody>
<tr>
<td>Number of subdomains</td>
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<tr>
<td>Primal variables</td>
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<tr>
<td>Dual variables</td>
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<tr>
<td>Equality constraints</td>
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<tr>
<td>Bound constraints (active)</td>
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<tr>
<td>Outer iterations</td>
</tr>
<tr>
<td>Hessian multiplications</td>
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<tr>
<td>Solution time [sec]</td>
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</tbody>
</table>

**Example 6.2.** Let us consider the isotropic case (b). This choice corresponds to the averaged friction coefficients of the real measurements for the case depicted in Fig. 6.2. The solution characteristics are summarized in Table 6.2. One can see that the number of outer iterations increases modestly with the size of the problem and the solution is more expensive than in the previous example. The distribution of the normal contact stress along the contact interface is depicted in Fig. 6.5. Moreover, we depict also the distributions of the Euclidean norm of the tangential contact stress and displacement as well as behavior inside the contact zone; see Figs. 6.6, 6.7, and 6.8.
<table>
<thead>
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<th>Table 6.2: Frictional problems</th>
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</thead>
<tbody>
<tr>
<td>Number of subdomains</td>
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<tr>
<td>Primal variables</td>
</tr>
<tr>
<td>Dual variables</td>
</tr>
<tr>
<td>Equality constraints</td>
</tr>
</tbody>
</table>

Isotropic case (Example 6.2)

| | | | | |
| Bound constraints (active) | 231 (11) | 861 (64) | 1891 (135) | 3321 (246) |
| Circular constraints (active) | 231 (220) | 861 (830) | 1891 (1847) | 3321 (3270) |
| Outer iterations | 11 | 15 | 19 | 22 |
| Hessian multiplications | 121 | 222 | 415 | 721 |
| Solution time [sec] | 17.6388 | 69.7405 | 344.738 | 1341.97 |

Orthotropic case, circular constraints (Example 6.3)

| | | | | |
| Bound constraints (active) | 231 (11) | 861 (63) | 1891 (155) | 3321 (281) |
| Circular constraints (active) | 231 (211) | 861 (796) | 1891 (1771) | 3321 (3141) |
| Outer iterations | 11 | 11 | 14 | 15 |
| Hessian multiplications | 149 | 262 | 487 | 665 |
| Solution time [sec] | 16.573 | 82.9541 | 442.431 | 1236.83 |

Orthotropic case, elliptical constraints (Example 6.3)

| | | | | |
| Bound constraints (active) | 231 (11) | 861 (63) | 1891 (156) | 3321 (284) |
| Elliptical constraints (active) | 231 (213) | 861 (798) | 1891 (1784) | 3321 (3151) |
| Outer iterations | 11 | 10 | 16 | 17 |
| Hessian multiplications | 121 | 221 | 363 | 469 |
| Solution time [sec] | 112.601 | 592.747 | 2240.3 | 5389.77 |

Orthotropic case, circular constraints (Example 6.4)

| | | | | |
| Bound constraints (active) | 231 (13) | 861 (47) | 1891 (113) | 3321 (203) |
| Circular constraints (active) | 231 (229) | 861 (847) | 1891 (1869) | 3321 (3301) |
| Outer iterations | 10 | 12 | 11 | 12 |
| Hessian multiplications | 126 | 315 | 332 | 344 |
| Solution time [sec] | 13.9187 | 96.3434 | 283.024 | 718.197 |

Orthotropic case, elliptical constraints (Example 6.4)

| | | | | |
| Bound constraints (active) | 231 (8) | 861 (41) | 1891 (102) | 3321 (188) |
| Elliptical constraints (active) | 231 (220) | 861 (846) | 1891 (1860) | 3321 (3301) |
| Outer iterations | 15 | 24 | 28 | 29 |
| Hessian multiplications | 208 | 376 | 617 | 738 |
| Solution time [sec] | 264.096 | 1621.58 | 5592.17 | 11753.4 |
Example 6.3. In this example we consider the orthotropic case (c) with the coefficients of friction $\mathcal{F}_1 = 0.5$ and $\mathcal{F}_2 = 0.1$ in the incorrectly chosen tangential directions $t_1 = (1, 0, 0)\top$ and $t_2 = (0, 1, 0)\top$, respectively. The results in Table 6.2 show that the computations using the circular constraints are more expensive than using the original elliptical constraints. This may be due to worse spectral properties of the matrix $PFP$ which increase the bound on the number of iterations in SMALSE-M as it has been shown in [5]. Further we depict the distributions of the normal contact stress, the standard and scaled (by coefficients of friction) Euclidean norm of the tangential contact stress, displacement along the contact interface, and behavior inside the contact zone in Figs. 6.9, 6.10, 6.11, 6.12, 6.13, and 6.14, respectively.
Example 6.4. Finally, let us consider the orthotropic case (d) with the coefficients of friction $F_1 = 0.5$ and $F_2 = 0.1$ in the directions $t_1 = (\sqrt{2}/2, -\sqrt{2}/2, 0)^T$ and $t_2 = (\sqrt{2}/2, \sqrt{2}/2, 0)^T$, respectively. This setting corresponds to the milled surface depicted in Fig. 6.2. We can see in Table 6.2 that the circular constraints require less computations than the elliptical ones. An heuristic argument explaining this fact is that the orientation of the elliptical constraints exhibits an opposite character with respect to the elliptical spectral properties of $PFP$. Again we depict the distributions of the normal contact stress, the standard and scaled (by coefficients of friction) Euclidean norm of the tangential contact stress and displacement along the contact interface, and behavior inside the contact zone in Figs. 6.15, 6.16, 6.17, 6.18, 6.19, and 6.20, respectively.

From the above examples and Table 6.3 which compares displacements for different friction models one can see the significant dependence of the results on the chosen friction model. Using the orthotropic friction law with correctly chosen tangential directions we get the results which are closer to the reality. An industrial application for the isotropic case may be found in [7].

| Friction model | max($|u_1|$) | max($|u_2|$) | max($|u_3|$) | max($\|u\|$) |
|----------------|-------------|-------------|-------------|-------------|
| Case (a)       | 1.93765     | 0.13952     | 3.64374     | 4.12538     |
| Case (b)       | 1.83178     | 0.12977     | 3.02874     | 3.53673     |
| Case (c)       | 1.80919     | 0.12659     | 2.85510     | 3.37758     |
| Case (d)       | 1.89417     | 0.30280     | 3.29204     | 3.80103     |

7. Conclusions

The paper is devoted to the numerical solution of contact problems of linear elasticity with orthotropic Coulomb friction. The presented approach uses the method of successive approximations that requires to solve auxiliary problems with orthotropic Tresca friction in each iterative step. In order to increase the computational efficiency, we use the finite element discretization of these problems based on the T-FETI domain decomposition method. The algebraic dual formulation of the Tresca problem leads to the minimization of the quadratic functional subject to separable quadratic constraints and simple bounds. For this purpose we use the recently proposed optimization algorithm SMALSE-M based on the augmented Lagrangian method in the outer loop combined with the inner loop implemented by KPRGP [5]. The important property of this algorithm is that the number of iterations needed to get a solution with a given accuracy is uniformly bounded (with respect to the size of the problem) provided the spectrum of the Hessian is confined in a given interval. The assumption on the spectrum is satisfied if the ratio between the maximal diameter of the subdomains $\Omega^k_i$ and the norm of $T^{ki}_h$, $i = 1, \ldots, s$, $k = 1, 2$, is fixed and
Figure 6.9: Normal contact stress

Figure 6.10: The norm of the tangential contact stress

Figure 6.11: The scaled norm of the tangential contact stress

Figure 6.12: The norm of the tangential contact displacement

Figure 6.13: Contact zone

Figure 6.14: Contact zone zoom
Figure 6.15: Normal contact stress

Figure 6.16: The norm of the tangential contact stress

Figure 6.17: The scaled norm of the tangential contact stress

Figure 6.18: The norm of the tangential contact displacement

Figure 6.19: Contact zone

Figure 6.20: Contact zone zoom
the Hessian is normalized in the spectral norm (see [7]). Let us recall that the scalability can be proved only for the frictionless case and Tresca friction but we observed it experimentally also for Coulomb fiction. Obviously, the number of outer iterations stays almost constant and the number of the matrix-vector multiplications increases only moderately with the size of the problem. Therefore, the overall approach seems to be the promising method for solving real-life contact problems.

References


