Continuation of the static contact problem with Coulomb friction

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1 Discrete static contact problems with Coulomb friction

Let \( \Omega \subset \mathbb{R}^2 \) be a linearly elastic body supported by a rigid foundation along the contact boundary \( \Gamma_C \). On \( \Gamma_N \) and \( \Gamma_D \), Neumann and Dirichlet boundary conditions are prescribed. We consider the static contact problems with Coulomb friction, see e.g. [1]. In particular, we will investigate a discrete version of this problem, see e.g. [2, 3]. This may be understood as a \emph{FEM-approximation} of the continuous mechanical problem.

Let integers \( n \) and \( p \) define the degrees of freedom of the body \( \Omega \) and the number of contact nodes on \( \Gamma_C \), \( n \geq 2p \). Let \( f \in \mathbb{R}^n \) and \( F \) be the given distributed volume force and the friction coefficient. We seek for

- nodal displacement field \( u \in \mathbb{R}^n \)
- nodal normal and tangential stress components \( \lambda_\nu \in \mathbb{R}^p \) and \( \lambda_t \in \mathbb{R}^p \)

such that

\[
(Au, v)_n = (f, v)_n + (\lambda_\nu, Nv)_p + (\lambda_t, Tv)_p \quad \forall v \in \mathbb{R}^n,
\]

\[
(\mu_\nu - \lambda_\nu, Nu)_p + (\mu_t - \lambda_t, Tu)_p \geq 0 \quad \forall (\mu_\nu, \mu_t) \in \Lambda_\nu \times \Lambda_t(F, -\lambda_\nu).
\]

Here, \( A \in \mathbb{R}^{n \times n} \) is a positive definite stiffness matrix. The full-rank matrices \( N \in \mathbb{R}^{p \times n} \) and \( T \in \mathbb{R}^{p \times n} \) associate \( u \in \mathbb{R}^n \) with its normal and tangential component at the contact nodes. The convex sets of Lagrange multipliers are

\[
\Lambda_\nu = \mathbb{R}^p_+ \quad \text{and} \quad \Lambda_t(F, -\lambda_\nu) = \{ \mu_t \in \mathbb{R}^p : |\mu_t,i| \leq -F \lambda_\nu,i \quad \forall i = 1, \ldots, p \}.
\]

It is worth noticing that the second set in (3) depends on the solution component \( \lambda_\nu \).

Let \( r > 0 \) be a fixed parameter. The variational inequality (2) is equivalent to the equations

\[
\lambda_\nu = P_{\Lambda_\nu}(\lambda_\nu - rNu), \quad \lambda_t = P_{\Lambda_t(F, -\lambda_\nu)}(\lambda_t - rTu),
\]

see e.g. [4, 5]. Here \( P_{\Lambda_\nu} \) and \( P_{\Lambda_t(F, -\lambda_\nu)} \) are the orthogonal projections of \( \mathbb{R}^p \) onto \( \Lambda_\nu \) and \( \Lambda_t(F, -\lambda_\nu) \), see (3).

Under generic assumptions, there exists a solution of (1)&(2) for any data \( f \in \mathbb{R}^n \) and \( F > 0 \). If \( F \) is sufficiently small, the solution is unique. See e.g. [2, 7].
2 Continuation of the static solutions

Solving (1)&(2) for \((u, \lambda_\nu, \lambda_t) \in \mathbb{R}^n \times \Lambda_\nu \times \Lambda_t(F, -\lambda_\nu)\) is equivalent to finding roots of a nonlinear mapping (1)&(4).

The static Coulomb friction model depends on parameters. For example, we may prescribe a smooth loading path \(\alpha \in \mathbb{R} \mapsto f(\alpha) \in \mathbb{R}^n\) and ask for a continuous response of the body. Then the above mentioned roots depend on the parameter \(\alpha\). We will define

\[
\begin{pmatrix}
u \\
\lambda_\nu \\
\lambda_t \\
\alpha
\end{pmatrix} \in \mathbb{R}^{n+2p+1} \mapsto H(\mathbf{z}) = \begin{pmatrix} A u - f(\alpha) - N^T \lambda_\nu - T^T \lambda_t \\ \lambda_\nu - P_{\lambda_a}(\lambda_\nu - rN^T u) \\ \lambda_t - P_{\lambda_{\lambda_t}}(\lambda_t - rT u) \end{pmatrix} \in \mathbb{R}^{n+2p}.
\] (5)

The mapping \(H : \mathbb{R}^{n+2p+1} \to \mathbb{R}^{n+2p}\) is continuous, piecewise smooth, see [7]. Hence, the set \(H(u, \lambda_\nu, \lambda_t, \alpha) = 0 \in \mathbb{R}^{n+2p}\) defines generically a continuous, piecewise smooth curve in \(\mathbb{R}^{n+2p+1}\). The objective is to trace the curves (5) numerically using path-following (i.e. continuation) techniques. Note that the standard continuation techniques require the curve to be smooth. The idea is:

1. Continue the smooth pieces by a classical path-following software, see e.g. [6].
2. Join the smooth pieces continuously, preserving the orientation.

For details, see [7, 8].

3 Case study: \(n = 1320, p = 30\)

For the geometry of the example, see Figure 1: It is understood that each nodal mesh point has two degrees of freedom for the vertical and horizontal displacement. The indicated surface traction depend on a scalar parameter \(\alpha\); we omit the particular formulae. The contact boundary \(\Gamma_C\) is approximated by \(p = 30\) points. The contact data \(\lambda_\nu, \lambda_t, u_\nu, u_t\) are changed with \(\alpha\). A snapshot as \(\alpha = 3.6\) is shown in Figure 2.

We consider continuation of the curve (5) in the parameter range \(-0.5 \leq \alpha \leq 1.5\), starting at \(\alpha = -0.5\). The curve is continuous, piecewise smooth. Hence, the curve is smooth up to transition points. There were detected 14 transition points on the path: E.g., at the six-th transition point which is related to \(\alpha = 0.28019791259766\), the contact nodal point \(i = 13\) changes its classification from no contact to contact, slip. At the seven-th transition point which is related to \(\alpha = 0.42934036865234\), the contact nodal point \(i = 3\) changes its classification from contact, slip to contact, stick. At the eight-th transition point which is related to \(\alpha = 0.60403706054688\), the contact nodal point \(i = 14\) changes its classification from no contact to contact, slip.

In fact, if we know transition points, we can cheaply compute the solution for any given \(\alpha\), see Figure 3.

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Figure 1: FEM approximation: Case Study \( n = 1320, \ p = 30; \) the mesh on the rectangular domain \( \Omega. \) The loading is due to the surface traction.

Figure 2: Contact data \( \lambda_\nu, \lambda_t, u_\nu, u_t \) at the contact points \( i = 1, 2, \ldots, 30 \) for \( \alpha = 3.6. \) Contact classification: circle ... no contact, diamond ... contact-stick, square ... contact-slip. Here, \( u_\nu \) and \( u_t \) are the normal and tangential displacement components at particular contact points.
Figure 3: Profiles of the normal stress components: Contact points \(i = 1, 2, \ldots, 30\) vs \(\lambda_n\) for selected parameter values \(\alpha = -0.5, 0, 0.5, 1, 1.5\). Contact classification: circle ... no contact, diamond ... contact-stick, square ... contact-slip.

References


