

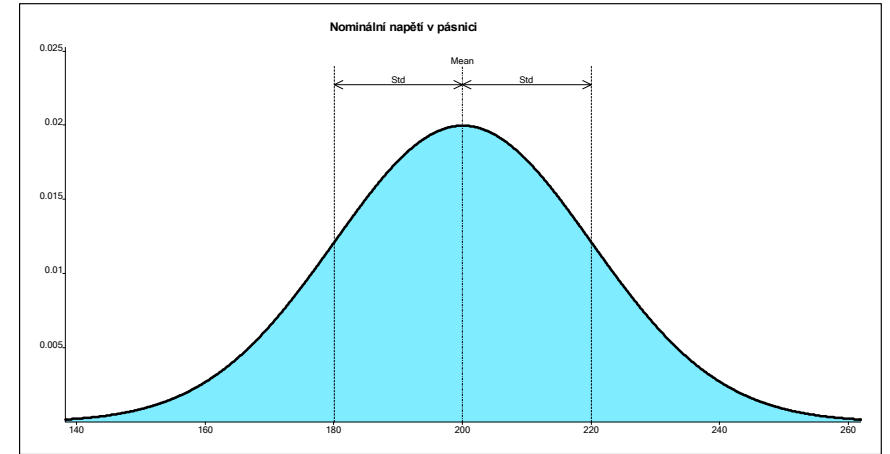
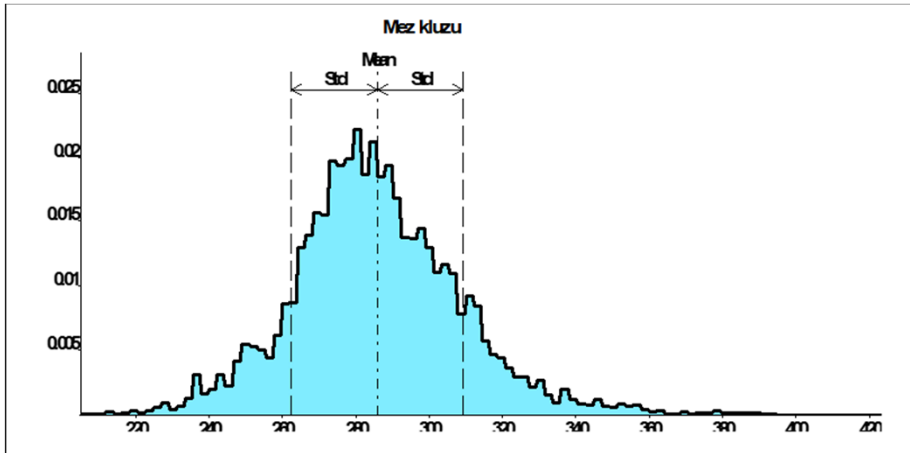
# Parametric probability distributions

- Parametric probability distributions of a continuous random variable
- An overview of important continuous probability distributions
- Histogram creation with parametric probability distribution

# Probability distribution

**Parametric probability distribution** - probabilities defined by **analytical function** – e.g., common expression of **normal (Gaussian) probability distribution**:

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



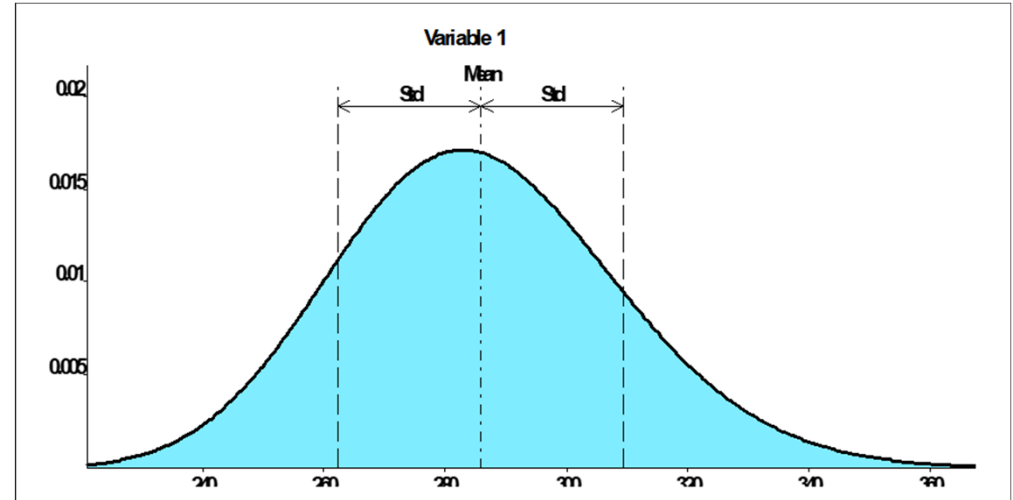
Parameters - characteristics of random variable probability distribution (e.g.,  $\mu$  mean value and  $\sigma$  standard deviation)

**Non-parametric (empirical) probability distribution** - definition based on measurements (often long-term)

# Parametric probability distributions

Important parametric probability distributions for continuous random variables:

- Uniform distribution
- Normal (Gaussian) distribution
- Exponential distribution
- Laplace distribution
- Logistic distribution
- Maxwell distribution
- Student's  $t$ -distribution
- Fisher's  $z$ -distribution
- Chi-square distribution

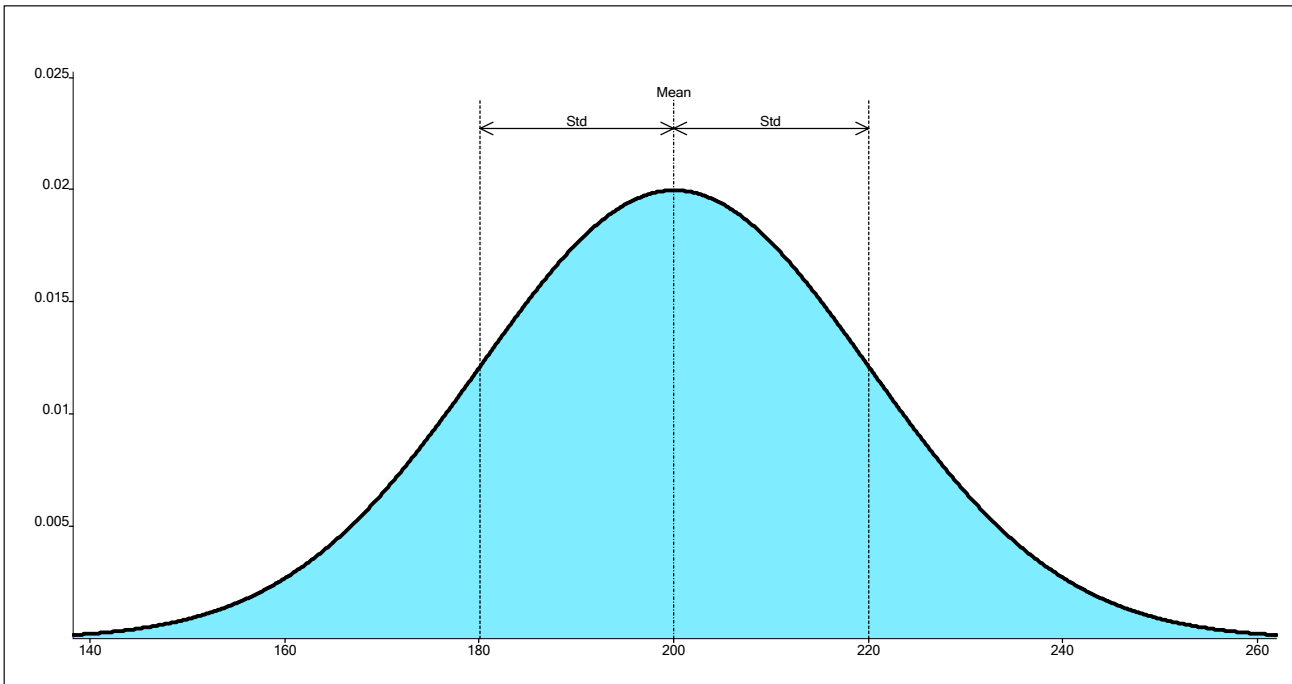


Parameters - characteristics of random variable probability distribution  
(e.g.,  $\mu$  mean value and  $\sigma$  standard deviation)

# Normal (Gaussian) probability distribution

Common expression of **normal (Gaussian) probability distribution**:

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



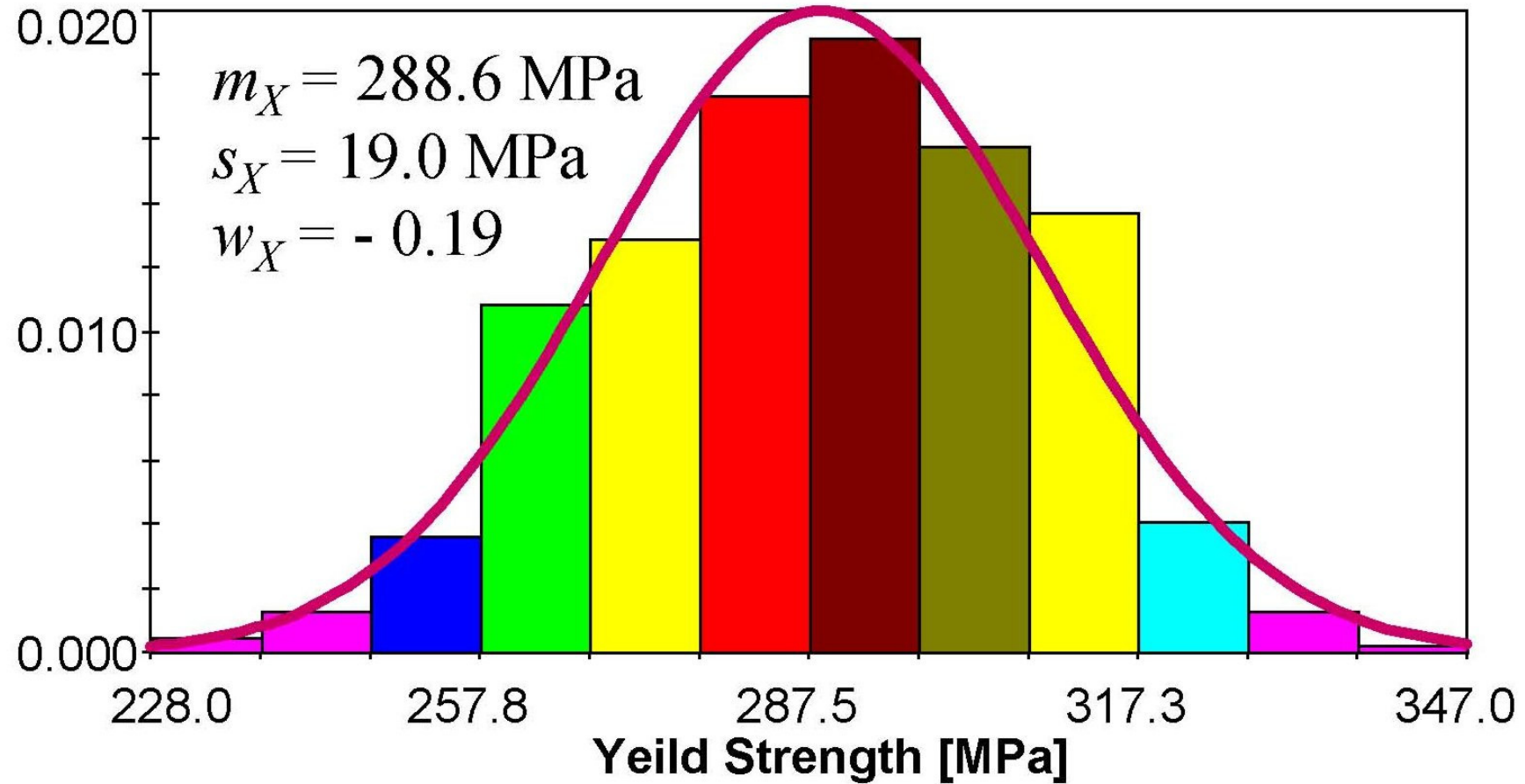
$\mu$  ... mean value

$$\mu = \frac{1}{n} \sum_{i=1}^n x_i$$

$\sigma$  ... standard deviation

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2}$$

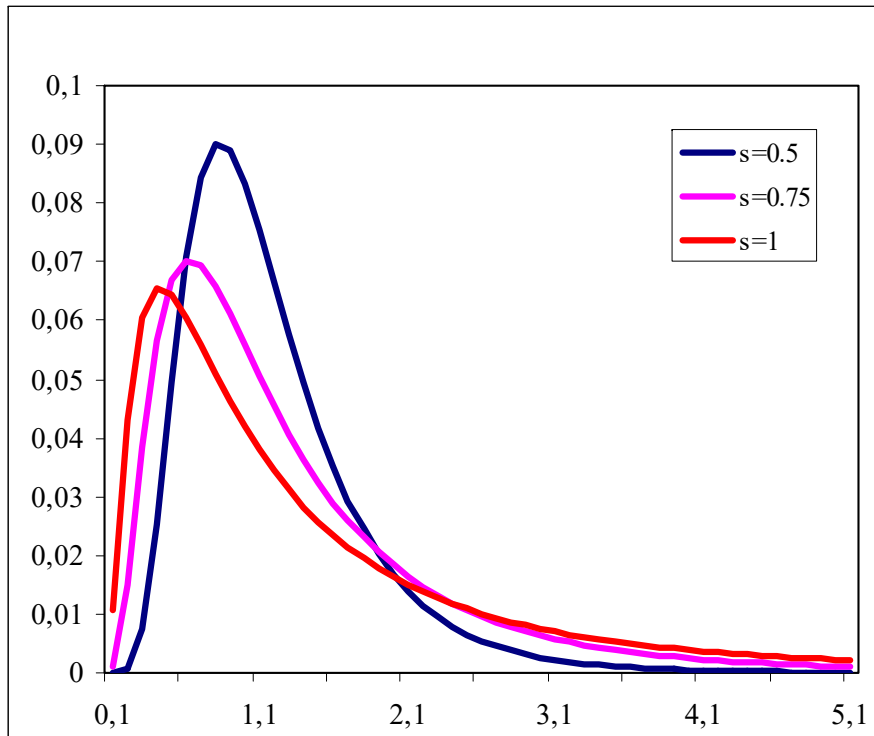
# Yield stress of the S235 steel



# Log-normal probability distribution

Common expression of **log-normal probability distribution**:

$$f(x|\mu,\sigma) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$



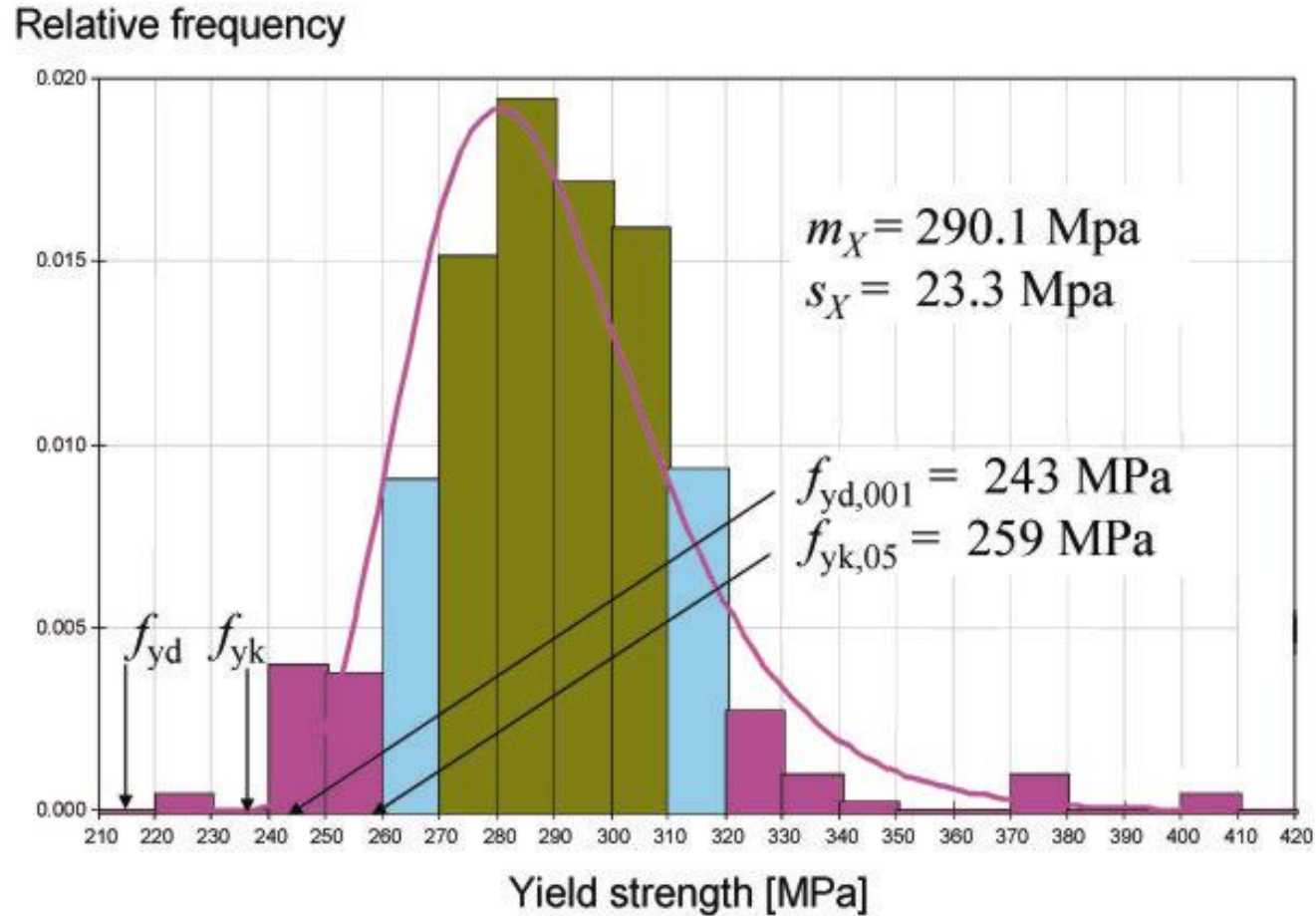
$\mu$  ... **mean value**

$$\mu = \frac{1}{n} \sum_{i=1}^n \ln(x_i)$$

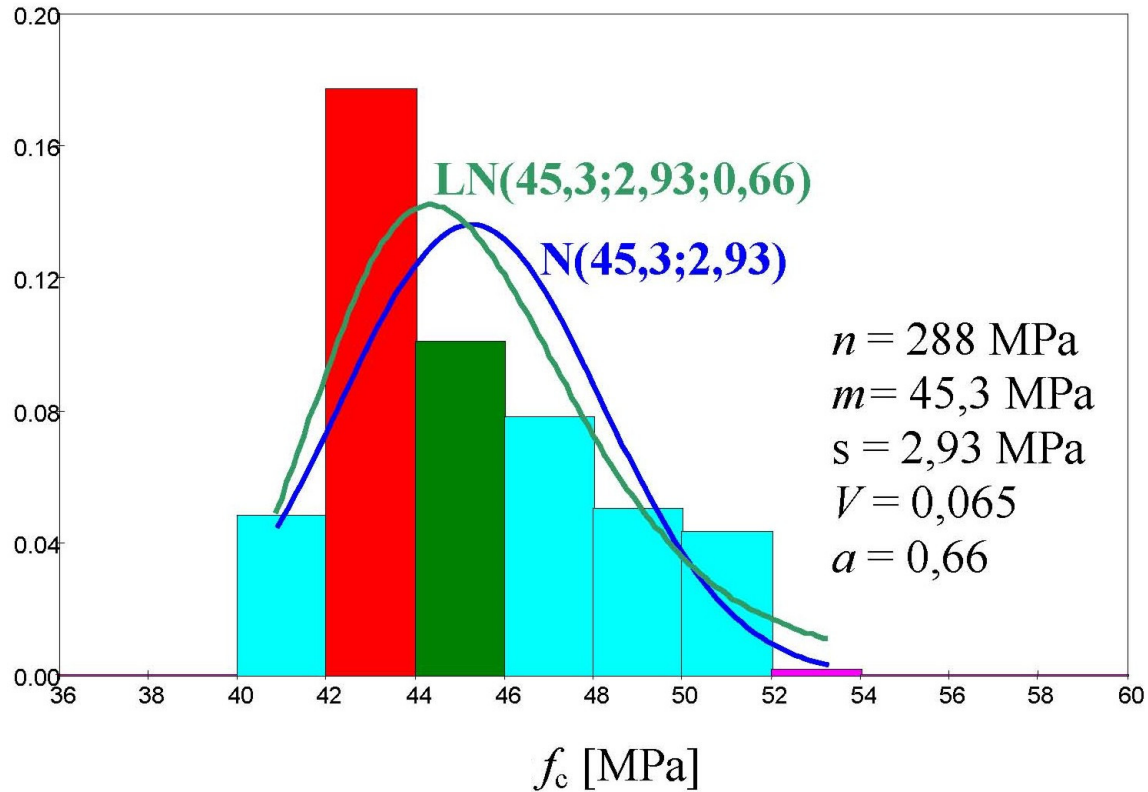
$\sigma$  ... **standard deviation**

$$\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (\ln(x_i) - \mu)^2}$$

# Yield stress of the S235 steel

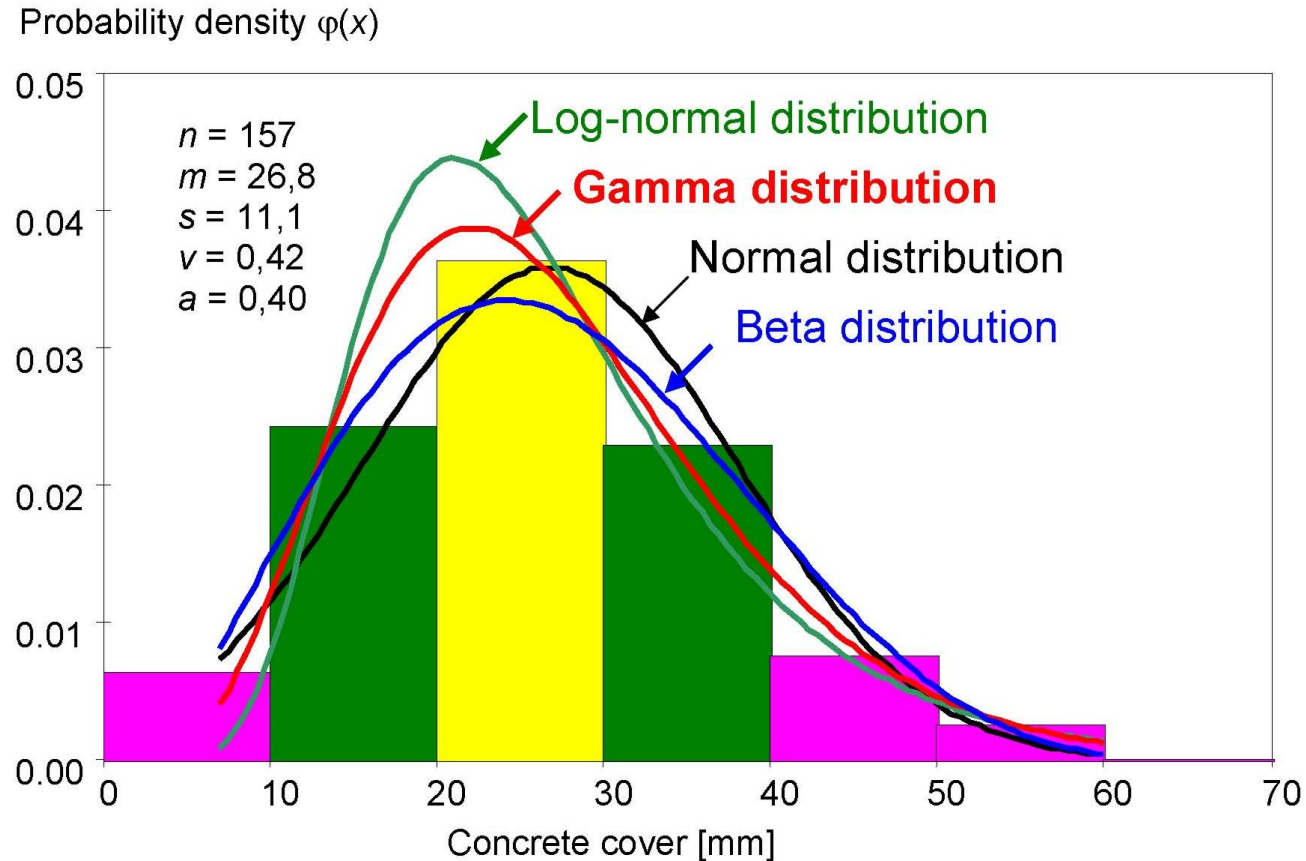


# Compressive strength of concrete

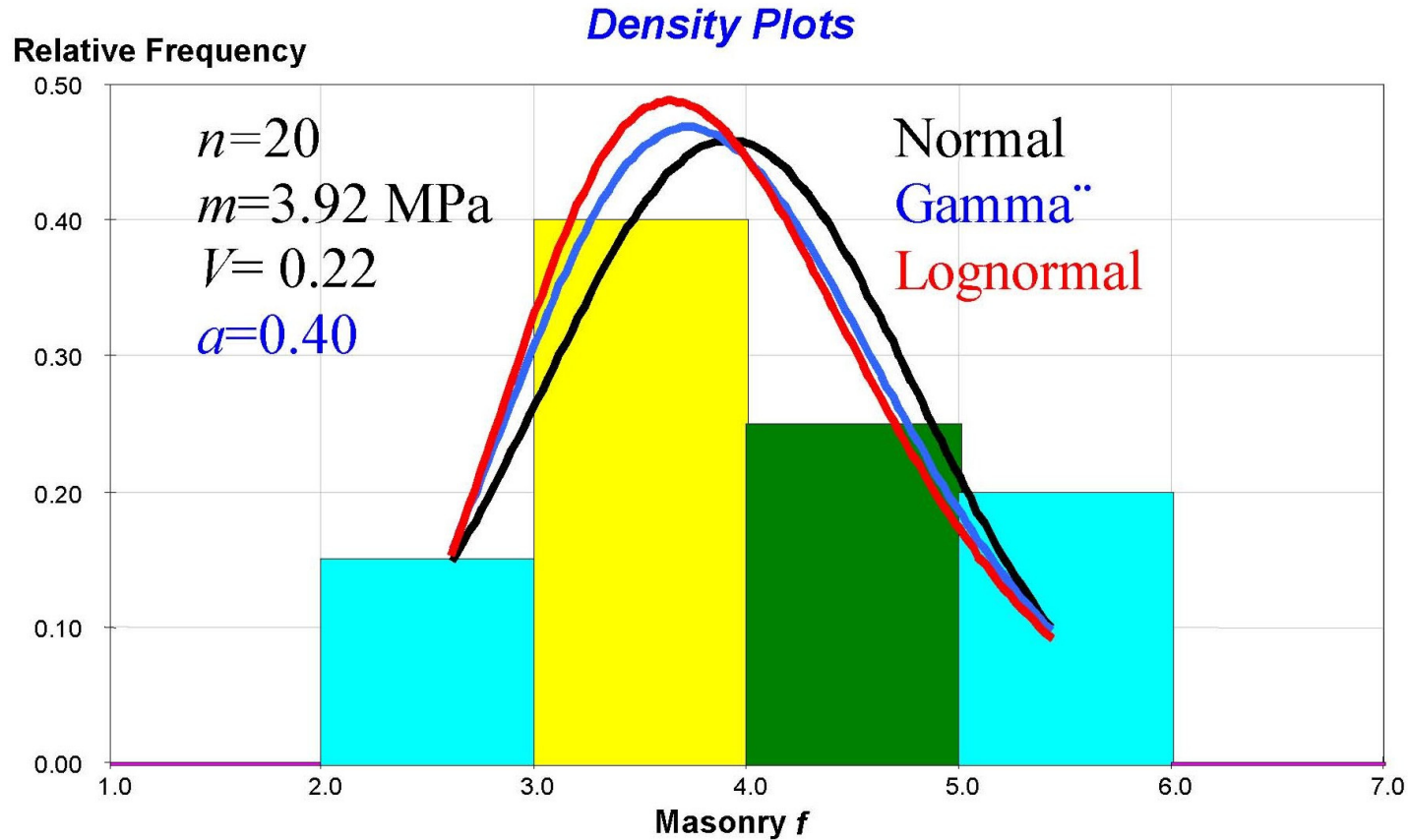




# Concrete cover layer thickness



# Compressive strength of masonry



# Parametric probability distributions

Distribution	Abbreviation	PDF	Parameters
Kappa	KAP	$F(x) = \left\{ 1 - \gamma_2 [1 - \gamma_1 (x - \alpha) / \beta]^{1/\gamma_1} \right\}^{1/\gamma_2}$ $f(x) = \beta^{-1} [1 - \gamma_1 (x - \alpha) / \beta]^{(1/\gamma_1) - 1} \times [F(x)]^{1 - \gamma_2}, \beta > 0$	4
Generalized extreme value type III	GEV	$f(x) = \frac{1}{\beta} \left( 1 + \gamma \frac{x - \alpha}{\beta} \right)^{-1/\gamma - 1} \exp \left[ - \left( 1 + \gamma \frac{x - \alpha}{\beta} \right)^{-1/\gamma} \right]$	3
Generalized Logistic	GLO	$f(x) = \frac{\gamma \exp \left( - \frac{x - \alpha}{\beta} \right)}{\beta \left( 1 + \exp \left( - \frac{x - \alpha}{\beta} \right) \right)^{\gamma + 1}}$	3
Generalized Pareto	GPA	$f(x) = \frac{1}{\beta} \left( 1 + \frac{\gamma(x - \alpha)}{\beta} \right)^{-1/\gamma - 1}$	3
Lognormal	LN3	$f(x) = \frac{1}{(x - \gamma)\sqrt{2\pi}\beta} \exp \left[ - \frac{1}{2} \left( \frac{\ln(x - \gamma) - \alpha}{\beta} \right)^2 \right]$	3
Pearson Type III	P3	$f(x) = \frac{1}{\beta^\gamma \Gamma(\gamma)} (x - \alpha)^{\gamma - 1} \exp \left( - \frac{x - \alpha}{\beta} \right)$	3
Exponential	E	$f(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geq 0 \\ 0, & x < 0 \end{cases}$	2
Gumbel	G	$f(x) = \frac{1}{\beta} \exp \left[ \frac{x - \alpha}{\beta} - \exp \left( \frac{x - \alpha}{\beta} \right) \right]$	2
Normal	N	$f(x) = \frac{1}{\sqrt{2\pi}\beta^2} \exp \left( - \frac{(x - \alpha)^2}{2\beta^2} \right)$	2
Logistic	L	$f(x) = \frac{\exp \left( - \frac{x - \alpha}{\beta} \right)}{\beta \left( 1 + \exp \left( - \frac{x - \alpha}{\beta} \right) \right)^2}$	2
Uniform	U	$f(x) = \begin{cases} \frac{1}{b - a}, & a < x < b \\ 0, & x < a \text{ or } x > b \end{cases}$	1

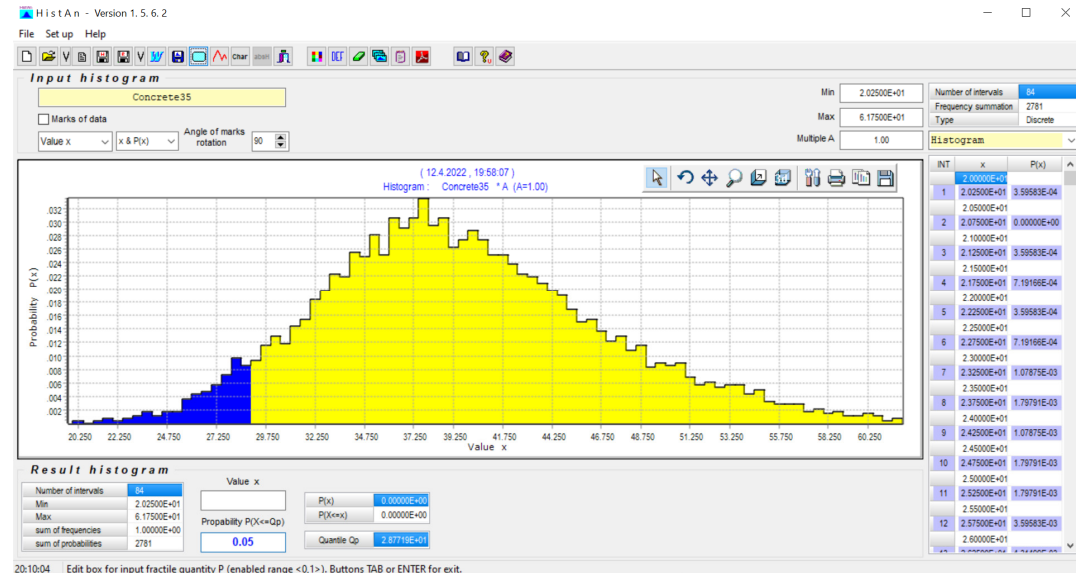
Distribution	Abbreviation	PDF	Parameters
Gamma	G2	$f(x) = \frac{x^{\gamma - 1} \exp \left( - \frac{x}{\beta} \right)}{\Gamma(\gamma) \beta^\gamma}$	2
Generalized Pareto	GP2	$f(x) = \frac{1}{\beta} \left( 1 + \frac{\gamma x}{\beta} \right)^{-(1/\gamma - 1)}$	2
Lognormal	LN2	$f(x) = \frac{1}{x\beta\sqrt{2\pi}} \exp \left( - \frac{(\ln x - \alpha)^2}{2\beta^2} \right)$	2
Weibull	W2	$f(x) = \frac{\gamma}{\beta} \left( \frac{x}{\beta} \right)^{\gamma - 1} \exp \left[ - \left( \frac{x}{\beta} \right)^\gamma \right], x \geq 0$	2

# HistAn software tool

Program for more detailed **analysis of input histograms**:

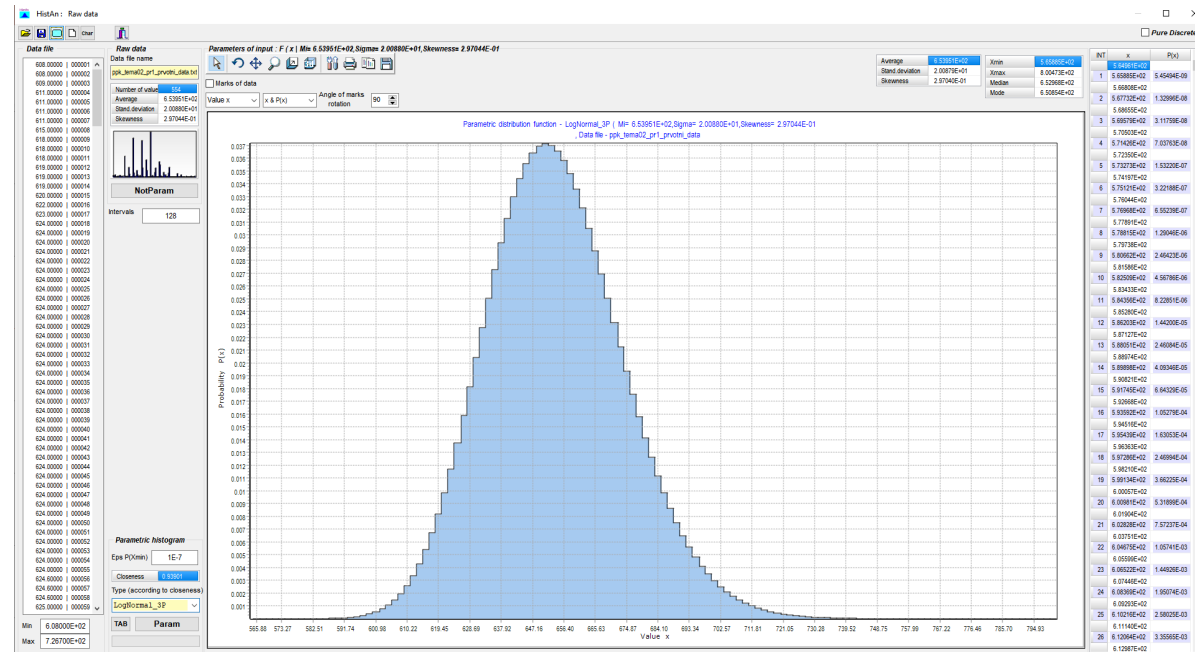
- **Minimum** and **maximum values** of a random variable
- **Number of histogram classes** (intervals) and frequencies defined in them
- **Simple probabilistic calculations** with histograms (determination of  $p$ -quantile and probability of exceeding the determined value of a random variable)
- Determining the **combination of several input histograms**
- Creation of **histograms with parametric distribution**
- Processing of **measured raw data**

Desktop of the **HistAn software tool**



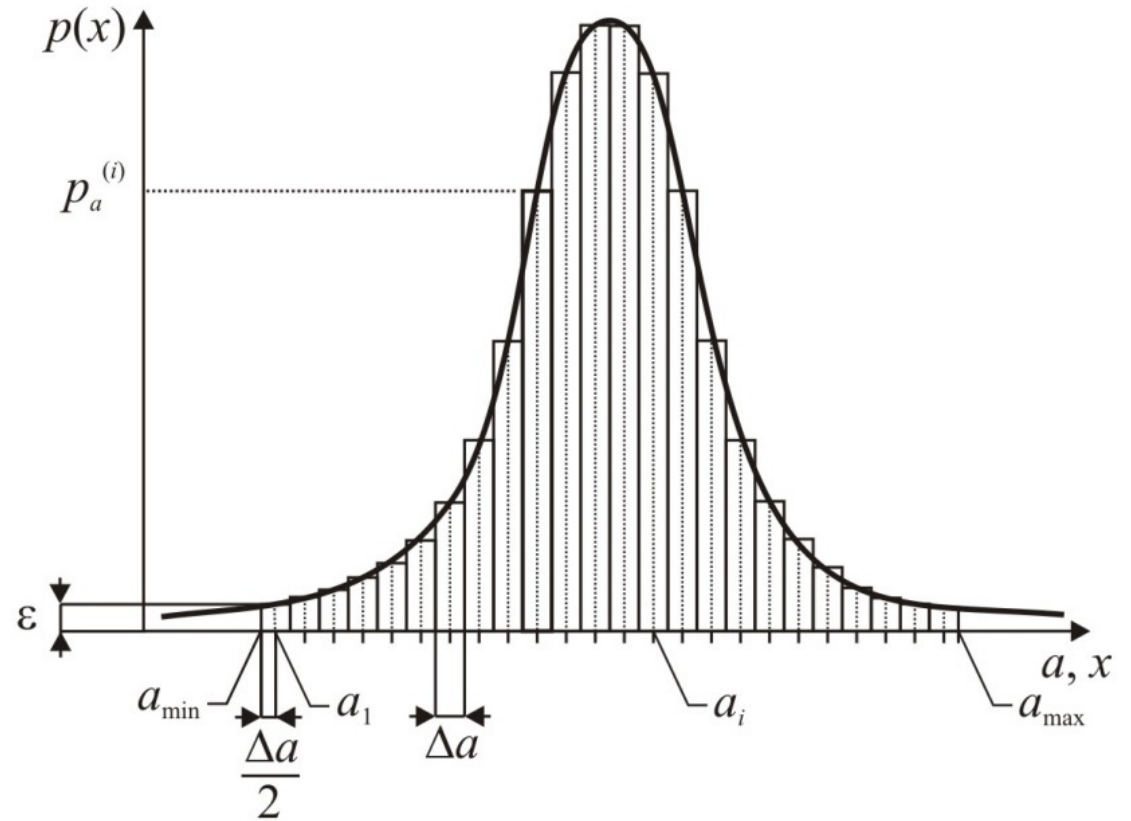
# Parametric distributions in HistAn software tool

- Implementation of a module for entering set of measured data and for their evaluation.
- Creation of histograms with non-parametric and parametric distribution (23 types) with the possibility of choosing the number of intervals.
- Choice of appropriate probability distribution according to **coefficient of determination**.



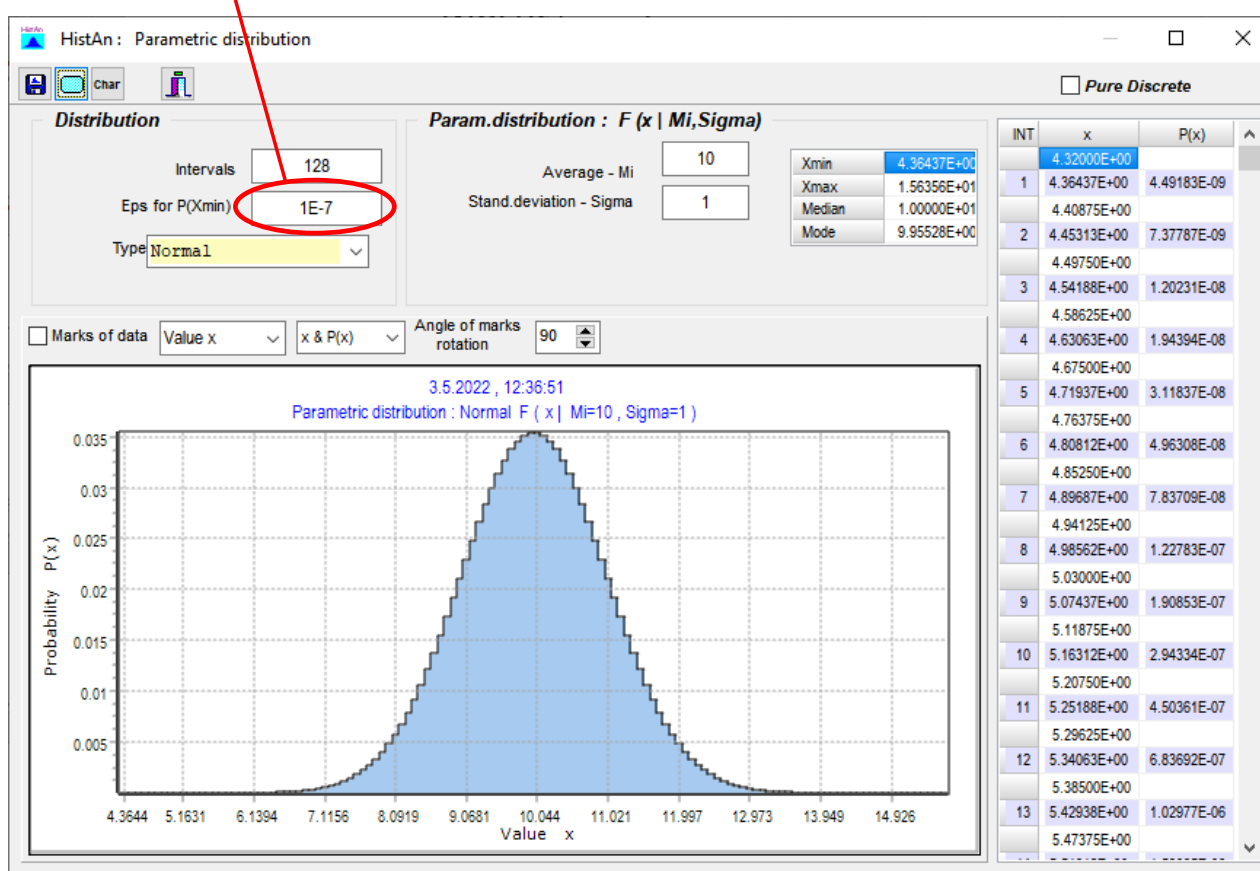
# Histogram of random variable

**Histogram** of discretized continuous random variable with parametric probability distribution



# Parametric distributions in HistAn

Probability for cutting of the probability distribution



- Normal
- Log-Normal
- Gumbel I and II
- Raised-Cosine
- Cauchy
- Fischer-Tippett
- Laplace
- Logistic
- Weibull
- Rayleigh
- Lévy
- Student
- Beta
- Gama
- Snedecor's  $F$  distribution
- Pareto
- Uniform
- Triangular
- Exponential
- Chi-square
- Half-Logistic

# Parametric distributions in HistAn

Characteristics of derived parametric data

HistAn: Table of applicable histograms with parametric probability distribution

**Statistical moments of raw data**

Average	Standard deviation	Skewness
6.53951E+02	2.00880E+01	2.97044E-01

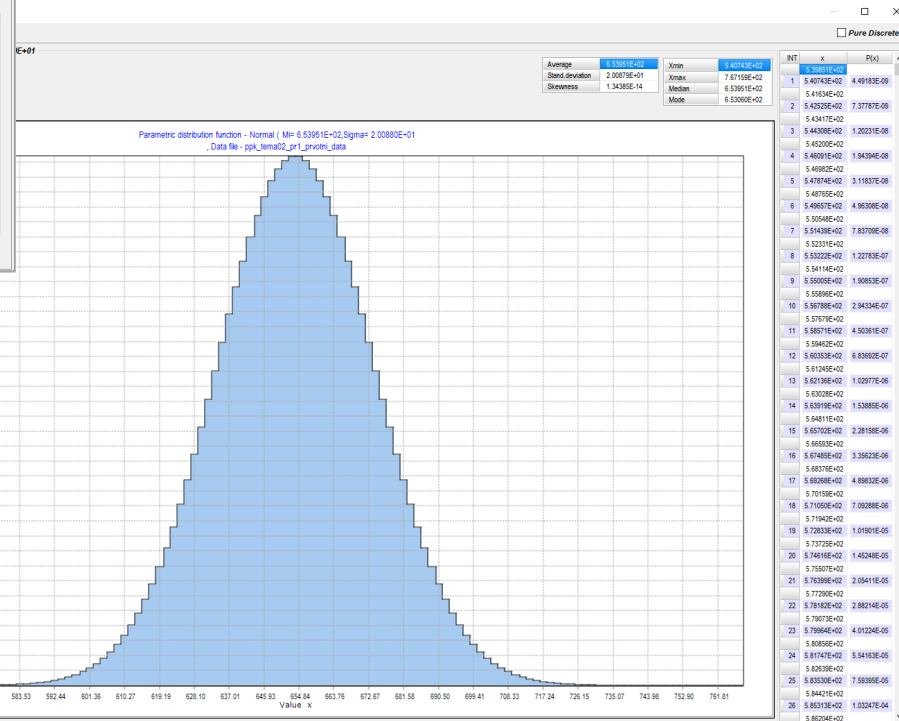
**Statistical moments and other characteristics of the histogram with non-parametric probability distribution**

i	Type	Closeness	Median	Mode	Average	Dispersion	Standard deviation	Variation (%)	Skewness	Kurtosis
1	Prvomi data	1.00000E+00	6.51431E+02	6.62250E+02	6.54026E+02	4.03288E+02	1.62641E+05	6.16624E+01	3.01057E-01	2.24183E-03

**Statistical moments and other characteristics of the histograms with parametric probability distribution**

i	Type	Closeness	Median	Mode	Average	Dispersion	Standard deviation	Variation (%)	Skewness	Kurtosis	Parametric distribution function -
1	Laplace	9.42289E-01	6.53951E+02	6.55742E+02	6.53951E+02	4.04594E+02	1.53696E+05	6.19891E+01	-7.13091E-14	2.98199E+00	M= 6.53951E+02, Beta=15.66255
2	Fisher-Tippett	9.40551E-01	6.59701E+02	6.54942E+02	6.62992E+02	4.03529E+02	1.62832E+05	6.08642E+01	1.13947E+00	2.39879E+00	M= 6.53951E+02, Beta=15.66255
3	Qumbelevov_J	9.40443E-01	6.50635E+02	6.46144E+02	6.53955E+02	4.03529E+02	1.62833E+05	6.17055E+01	1.13947E+00	2.39899E+00	M= 6.53951E+02, Sigma= 2.00880E+01
4	Logistic	9.39980E-01	6.53951E+02	6.52436E+02	6.53951E+02	4.03529E+02	1.62833E+05	6.17056E+01	-4.5892E-14	1.19949E+00	M= 6.53951E+02, s=11.07509
5	Normal	9.39008E-01	6.53951E+02	6.53060E+02	6.53951E+02	4.03524E+02	1.62832E+05	6.17056E+01	1.34385E-14	-1.28920E-05	M= 6.53951E+02, Sigma= 2.00880E+01
6	LogNormal_3P	9.39007E-01	6.52688E+02	6.50854E+02	6.53951E+02	4.03525E+02	1.62833E+05	6.17057E+01	2.97044E-01	1.57241E+01	M= 6.53951E+02, Sigma= 2.00880E+01, Skewness= 2.97044E-01
7	LogNormal_2P	9.38880E-01	6.53843E+02	6.53151E+02	6.53951E+02	4.03527E+02	1.62834E+05	6.17060E+01	9.21816E-02	1.50999E+02	M= 6.53951E+02, Sigma= 2.00880E+01
8	Raised-cosine	9.37705E-01	6.53951E+02	6.53517E+02	6.53951E+02	4.03520E+02	1.62829E+05	6.17050E+01	-2.82292E-14	-5.93763E-01	M= 6.53951E+02, s=55.56651
9	Students	9.34391E-01	0.00000E+00	-2.36250E+00	6.92821E+14	2.16511E+01	4.68770E+02	3.12509E+16	1.08332E-15	3.88770E+02	n=2.00497
10	Snedecorovo	9.34299E-01	1.60535E+01	1.22626E+01	4.08678E+01	1.94428E+04	3.78021E+08	4.73750E+04	1.14262E-01	1.68709E+02	n1=0.46282, n2=2.00306

Sort at: Closeness Laplace



Selection of a suitable distribution according to the coefficient of determination



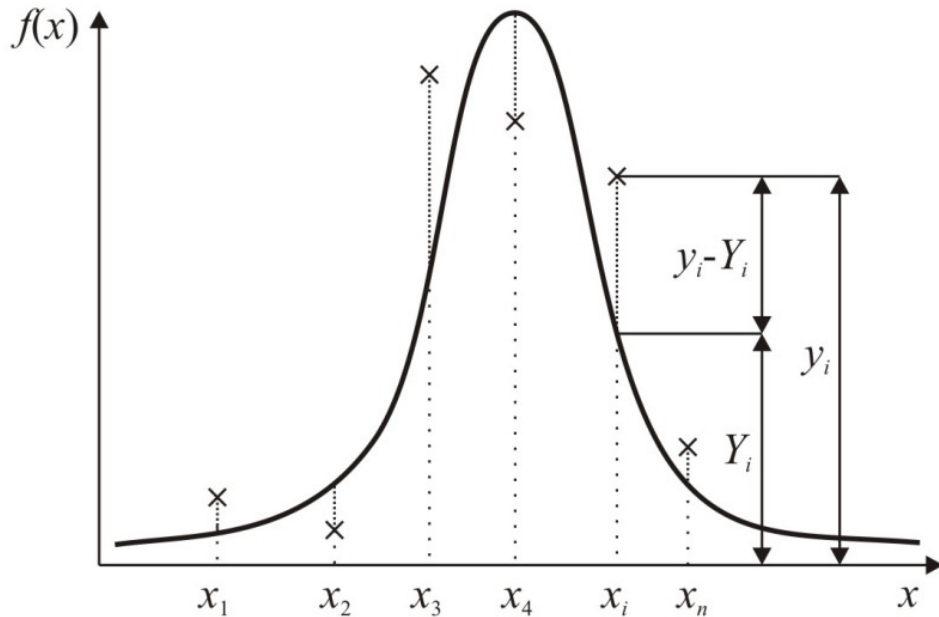
# Coefficient of determination

$$\frac{s_Y^2}{s_y^2} = 1 - \frac{s_{y,x}^2 + \frac{2}{n} \cdot \sum_i (y_i - Y_i) \cdot (Y_i - \bar{y})}{s_y^2}$$

$$\frac{s_Y^2}{s_y^2} \in \langle 0,1 \rangle$$

$$\left\{ \begin{array}{l} s_y^2 = \frac{1}{n} \cdot \sum_i (y_i - \bar{y})^2 \\ s_Y^2 = \frac{1}{n} \cdot \sum_i (Y_i - \bar{y})^2 \\ s_{y,x}^2 = \frac{1}{n} \cdot \sum_i (y_i - Y_i)^2 \end{array} \right.$$

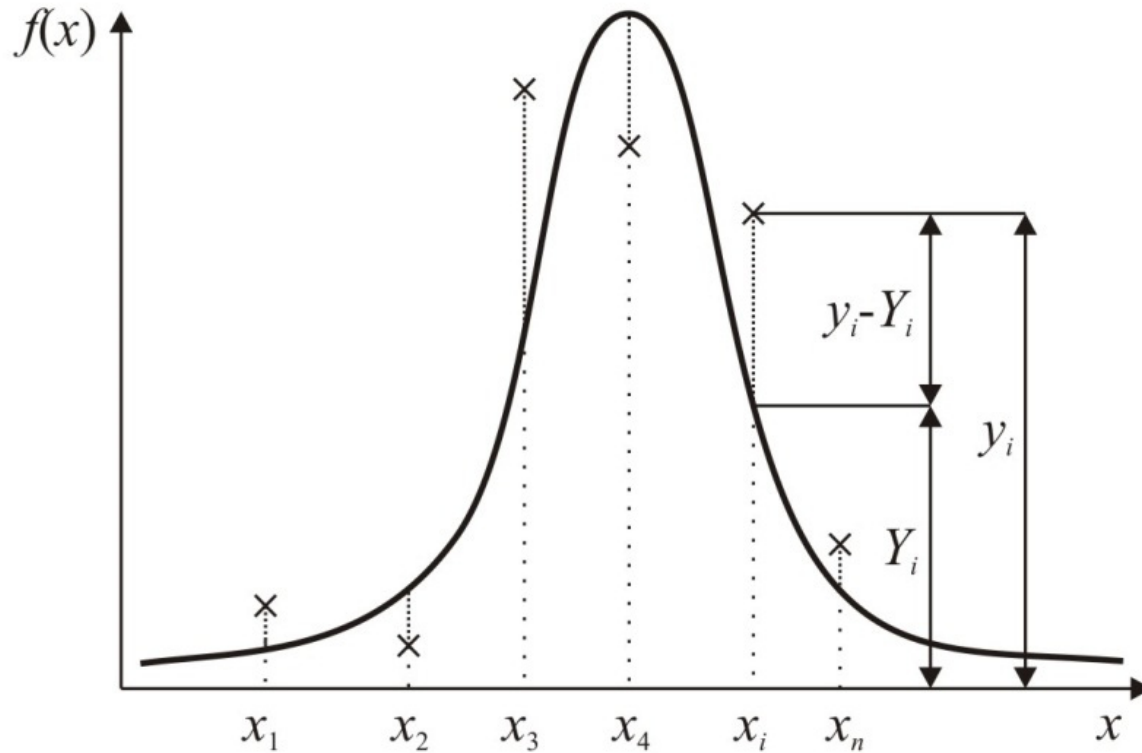
variances for  $n$  intervals



$Y_i$  ... the value of the probability density function of the parametric distribution at the respective value  $x_i$

$\bar{y}$  ... mean value from all  $y_i$

# Residual sum of squares



Scatter

$$s_{y,x}^2 = \frac{1}{n} \cdot \sum_i (y_i - Y_i)^2$$

Desired minimum value

$Y_i$  ... the value of the probability density function of the parametric distribution at the respective value  $x_i$

# Use of parametric distributions in HistAn

