1.1) $\int \frac{\cos^5 x}{\sin^4 x}\, dx = \int \frac{\cos^4 x}{\sin^4 x} \cdot \cos x\, dx = \int \frac{(1-\sin^2 x)^2}{\sin^4 x} \cdot \cos x\, dx =$

$= \left\{ \begin{array}{l} t = \sin x \\ dt = \cos x\, dx \end{array} \right\} = \int \frac{(1-t^2)^2}{t^4}\, dt = \int \frac{1}{t^4} - \frac{2}{t^2} + 1\, dt =$

$= \cancel{-\frac{t^3}{3}} - \frac{1}{3t^3} + \frac{2}{t} + t = -\frac{1}{3\sin^2 x} + \frac{2}{\sin x} + \sin x$

1.2) $\int \cos^2(x-2)\, dx = \left\{ \begin{array}{l} \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \\ \cos^2 x = \frac{1+\cos 2x}{2} \end{array} \right\} =$

$= \frac{1}{2}\int 1 + \cos(2x-4)\, dx = \left\{ \begin{array}{l} t = 2x-4 \\ dt = 2\, dx \end{array} \right\} = \frac{1}{4}\int 1 + \cos t\, dt =$

$= \frac{1}{4}\left(t + \sin t\right) = \frac{1}{4}\left(2x-4 + \sin(2x-4)\right) = \frac{x-2}{2} + \sin(2x-4)$

2.1) $\int \frac{\sqrt{\ln x} + 4}{x \ln x} = \left\{ \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\, dx \end{array} \right\} = \int \frac{\sqrt{t} + 4}{t}\, dt =$

$= \int \frac{1}{\sqrt{t}} + \frac{4}{t}\, dt = 2\sqrt{t} + 4 \ln|t| = 2\sqrt{\ln x} + 4 \ln|\ln x| =$

$= \underline{\underline{2\sqrt{\ln x} + 4 \ln(\ln x)}}$ (protože integrovaná funkce je definovaná jen pro $\ln x > 0$)

2.2) $\int \sin(\sqrt{x})\, dx = \left\{ \begin{array}{l} t = \sqrt{x} \\ x = t^2 \\ dx = 2t\, dt \end{array} \right\} = 2\int \sin t \cdot t\, dt =$

$= \left\{ \begin{array}{ll} u'(t) = \sin t & v(t) = t \\ u(t) = -\cos t & v'(t) = 1 \end{array} \right\} = 2\left(t \cdot (-\cos t) + \int \cos t\, dt\right) =$

$= 2(-t \cdot \cos t + \sin t) = \underline{\underline{2(-\sqrt{x} \cdot \cos\sqrt{x} + \sin\sqrt{x})}}$

3.1)

$$\int (2x^2-2x+1)\cdot \cos x \, dx = \left\{\begin{matrix} u(x)=2x^2-2x+1 & v'(x)=\cos x \\ u'(x)=4x-2 & v(x)=\sin x \end{matrix}\right\} =$$

$$\overset{\text{P.P.}}{=} (2x^2-2x+1)\cdot \sin x - \int (4x-2)\cdot \sin x \, dx =$$

$$= \left\{\begin{matrix} u(x)=4x-2 & v'(x)=\sin x \\ u'(x)=4 & v(x)=-\cos x \end{matrix}\right\} \overset{\text{P.P.}}{=}$$

$$= (2x^2-2x+1)\cdot \sin x - \left((4x-2)(-\cos x) + \int 4\cdot \cos x \, dx \right) =$$

$$= (2x^2-2x+1)\cdot \sin x + (4x-2)\cos x - 4\sin x =$$

$$= \underline{\underline{(2x^2-2x-3)\cdot \sin x + (4x-2)\cdot \cos x}}$$

3.2)

$$\int (x^2-1)\, e^{2x} \, dx = \left\{\begin{matrix} u(x)=x^2-1 & v'(x)=e^{2x} \\ u'(x)=2x & v(x)=\int e^{2x}dx = \frac{1}{2}e^{2x} \end{matrix}\right\} =$$

$$\overset{\text{P.P.}}{=} \frac{1}{2}(x^2-1)\, e^{2x} - \int x\cdot e^{2x} \, dx = \left\{\begin{matrix} u(x)=x & v'(x)=e^{2x} \\ u'(x)=1 & v(x)=\frac{1}{2}e^{2x} \end{matrix}\right\} =$$

$$\overset{\text{P.P.}}{=} \frac{1}{2}(x^2-1)\cdot e^{2x} - \left(\frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx \right) =$$

$$= \frac{1}{2}(x^2-1)\, e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{2}\,\frac{e^{2x}}{2} = \underline{\underline{\cancel{1}\, e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right)}}$$

4.1) $\displaystyle\int \frac{1}{(1+\sqrt[4]{x})^3\cdot\sqrt{x}}\,dx = \left\{\begin{matrix} 4 = n.s.n(2,4) & x = t^4 \\ t = \sqrt[4]{x} & dx = 4t^3\,dt \end{matrix}\right\} =$

$$= \int \frac{1}{(1+t)^3\cdot t^2}\cdot(4t^3)\,dt = 4\int \frac{t}{(1+t)^3}\,dt = 4\int \frac{t+1-1}{(1+t)^3}\,dt =$$

$$= 4\int \frac{1}{(1+t)^2} - \frac{1}{(1+t)^3}\,dt = \left\{\begin{matrix} u = t+1 \\ du = dt \end{matrix}\right\} = 4\int \frac{1}{u^2} - \frac{1}{u^3}\,du =$$

$$= 4\left(-\frac{1}{u} + \frac{1}{2u^2}\right) = 4\left(-\frac{1}{t+1} + \frac{1}{2(t+1)^2}\right) = \underline{\underline{-\frac{4}{\sqrt[4]{x}+1} + \frac{2}{(\sqrt[4]{x}+1)^2}}}$$

4.2) $\displaystyle\int \sqrt{\frac{2-x}{2+x}}\,dx = \left\{\begin{matrix} t = \sqrt{\frac{2-x}{2+x}} & x = \frac{2-2t^2}{1+t^2} \\ t^2 = \frac{2-x}{2+x} & dx = \frac{-4t(1+t^2)-(2-2t^2)\cdot 2t}{(1+t^2)^2}\,dt = \\ & = -\frac{8t}{(1+t^2)^2}\,dt \end{matrix}\right\} =$

$$= -8\int t\cdot\frac{t}{(1+t^2)^2}\,dt = -8\int \frac{t^2+1-1}{(1+t^2)^2}\,dt = -8\int \frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\,dt =$$

$$= -8\Big(\operatorname{arctg} t - \underbrace{\frac{t}{2(1+t^2)} - \frac{1}{2}\operatorname{arctg} t}_{\text{rekurentní vzorec pro } \int \frac{1}{(1+x^2)^n}dx}\Big) = -4\operatorname{arctg}\sqrt{\frac{2-x}{2+x}} +$$

$$+ \frac{4\sqrt{\frac{2-x}{2+x}}}{1+\frac{2-x}{2+x}} = \underline{\underline{-4\operatorname{arctg}\sqrt{\frac{2-x}{2+x}} + \sqrt{4-x^2}}}$$

5.1) $\displaystyle\int \frac{3x^4-3x^3-7x+4}{x^3-x^2-x-2}\,dx = ⊛$

$$3x^4-3x^3-7x+4 : x^3-x^2-x-2 = 3x + \frac{3x^2-x+4}{x^3-x^2-x-2}$$

$$\frac{-\left(3x^4-3x^3-3x^2-6x\right)}{3x^2-x+4}$$

rozklad jmenovatele:

$$x^3-x^2-x-2 = \underbrace{(x-2)}_{x=2 \text{ je kořen}}(x^2+x+1)$$

$$x^3-x^2-x-2 : x-2 = x^2+x+1$$

$$\frac{-\left(x^3-2x^2\right)}{x^2-x-2}$$

$$\frac{-\left(x^2-2x\right)}{\cancel{x}x-2}$$

rozklad na parciální zlomky:

$$\frac{3x^2-x+4}{x^3-x^2-x-2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

$$3x^2-x+4 = A(x^2+x+1)+(Bx+C)(x-2)$$

$x=2$: $\quad 14 = 7A \Rightarrow \underline{A=2}$

$A=2$: $\quad x^2-3x+2 = Bx^2+(C-2B)x-2C \Rightarrow \underline{B=1,\ C=-1}$

$$⊛ = \int 3x + \frac{2}{x-2} + \frac{x-1}{x^2+x+1}\,dx$$

$$\left\{ \int \frac{x-1}{x^2+x+1}\,dx = \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{-3}{x^2+x+1}\,dx\right) = \right.$$

$$= \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$$

$$= \frac{1}{2} \ln(x^2+x+1) - \frac{3}{2} \cdot \frac{4}{3} \int \frac{1}{\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)^2+1}\, dx =$$

$$= \left\{ \begin{array}{l} t = \frac{2x+1}{\sqrt{3}} \\ dt = \frac{2}{\sqrt{3}}\, dx \end{array} \quad \text{integrace + zpětná substituce} \right\} =$$

$$= \frac{1}{2} \ln(x^2+x+1) - \sqrt{3}\, \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) \Big\} =$$

$$= \frac{3x^2}{2} + 2 \ln|x-2| + \frac{1}{2} \ln(x^2+x+1) - \sqrt{3}\, \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$$

5-2)

$$\int \frac{5x-4}{x^3-3x^2+4}\,dx = ⊛$$

rozklad jmenovatele:

$$x^3-3x^2+4 = (x+1)(x\mp 2)(x-2) = (x+1)(x-2)^2$$

kořeny $-1, 2$

$$x^3-3x^2+4 : \underbrace{x^2-x-2}_{(x+1)(x-2)} = x-2$$

$$\begin{array}{l} -\left(x^3-x^2-2x\right) \\ \hline -2x^2+2x+4 \\ -2x^2+2x+4 \\ \hline \qquad\qquad 0 \end{array}$$

rozklad na parciální zlomky:

$$\frac{5x-4}{x^3-3x^2+4} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$5x-4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$x=-1$: $\quad -9 = 9A \Rightarrow A=-1$

$x=2$: $\quad 6 = 3C \Rightarrow C=2$

$A=-1, C=2$: $\quad 5x-4 = -x^2+4x\mp 4 + B(x^2-x-2) + 2x+2$

$\Rightarrow B=+1$ (např. porovnáním koeficientů u x^2)

$$⊛ = \int \frac{-1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}\,dx =$$

$$= -\ln|x+1| + \ln|x-2| - \frac{2}{x-1}$$