Sample Exam - solution

1. Determine the limit

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x}.$$

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \left[\frac{0}{0}\right] \overset{LP}{=} \lim_{x\to 0} \frac{+\sin x}{\cos x} = \frac{0}{1} = \underline{\underline{0}}$$

2. On the set $M = [-\pi; 2\pi]$, determine the global extremes of the function

$$f(x) = 4\sin\frac{x}{2} + (2\pi - 2x)\cos\frac{x}{2}.$$

$D(f) = \mathbb{R}$

$$f'(x) = 4\cdot\left(\cos\frac{x}{2}\right)\cdot\frac{1}{2} + (-2)\cos\frac{x}{2} - (2\pi-2x)\left(\sin\frac{x}{2}\right)\cdot\frac{1}{2} =$$

$$= 2\cdot\cancel{\cos}\frac{x}{2} - 2\cancel{\cos}\frac{x}{2} + (x-\pi)\left(\sin\frac{x}{2}\right) =$$

- stationary pts.

$f'(x) = 0 \iff (x-\pi)\left(\sin\frac{x}{2}\right) = 0 \iff x = \pi \lor x = 0 + 2k\pi,\ k\in\mathbb{Z}$

- pts. where $f'(x) \nexists$ - no such pts.

Global extrema on $[-\pi, 2\pi]$

$f'(x)$: $+$ on $(-\pi, 0)$, $-$ on $(0, \pi)$, $+$ on $(\pi, 2\pi)$; M

$-\pi$ 0 π 2π

$f(x)$: $\nearrow$ Loc. MAX $\searrow$ Loc MIN $\nearrow$

Compare fc. values.

$$\left.\begin{aligned} f(-\pi) &= \underline{-4} \\ f(0) &= \underline{2\pi} \\ f(\pi) &= 4 \\ f(2\pi) &= \underline{2\pi} \end{aligned}\right\} \Rightarrow$$

f has the global Minimum -4 at $-\pi$

f has the global Maximum 2π at 0 and 2π.

3. Find all asymptotes of the function

$$f(x) = \frac{(x+2)^2}{x+1}.$$

$D(f) = \mathbb{R} \setminus \{-1\}$

discontinuity pt $x = -1$

Vertical asymptote

$$\lim_{x \to -1^-} \frac{(x+2)^2}{x+1} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to -1^+} \frac{(x+2)^2}{x+1} = \frac{1}{0^+} = +\infty$$

$\Rightarrow$ Vertical as. has equation $\underline{x = -1}$

slant asymptotes $x \to \pm\infty$

$y = ax + b$

$$a_{+\infty} = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{(x+2)^2}{x(x+1)} = \lim_{x \to \infty} \frac{x^2+4x+4}{x^2+1} = \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2}\left(1 + \overset{\to 0}{\frac{4}{x}} + \overset{\to 0}{\frac{4}{x^2}}\right)}{\cancel{x^2}\left(1 + \underset{\to 0}{\frac{1}{x^2}}\right)} = 1$$

The same computation

$$a_{-\infty} = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{(x+2)^2}{x(x+1)} = \cdots = \underline{\underline{1}}$$

$$b_{+\infty} = \lim_{x \to \infty} f(x) - a \cdot x = \lim_{x \to \infty} \frac{(x+2)^2}{(x+1)} - x = \lim_{x \to \infty} \frac{x^2+4x+4-x^2-x}{x+1}$$

$$= \lim_{x \to \infty} \frac{3x+4}{x+1} \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x \to \infty} \frac{3}{1} = \underline{\underline{3}}$$

the same computation

$$b_{-\infty} = \lim_{x \to -\infty} f(x) - ax = \cdots = 3$$

One slant asymptote for $x \to \pm\infty$ $\boxed{y = x + 3}$

4. Write the Taylor polynomial of the second degree of the function

$$f(x) = \tan x$$

at the point $x_0 = \frac{\pi}{4}$.

$$D(f) = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \right\}$$

$$T_2(x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$f'(x) = \frac{1}{\cos^2 x} \Big/_{x_0 = \frac{\pi}{4}} = \left(\frac{\sqrt{2}}{2}\right)^{-2} = 2 \qquad f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$f''(x) = \left((\cos x)^{-2}\right)' = -2(\cos x)^{-3} \cdot (-\sin x) = \frac{2\sin x}{\cos^3 x} \Big/_{x_0 = \frac{\pi}{4}} =$$

$$= \frac{2 \cdot \frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^3} = \frac{\sqrt{2}}{\frac{2\sqrt{2}}{8}} = 4$$

$$T_2\left(\frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2}\left(x - \frac{\pi}{4}\right)^2 =$$

$$= 1 + 2x - \frac{\pi}{2} + 2\left(x - \frac{\pi}{4}\right)^2 = \dots\dots =$$

$$= 2x^2 + (2 - \pi)x + 1 - \frac{\pi}{2} + \frac{\pi^2}{8}$$

5. Solve the indefinite integral

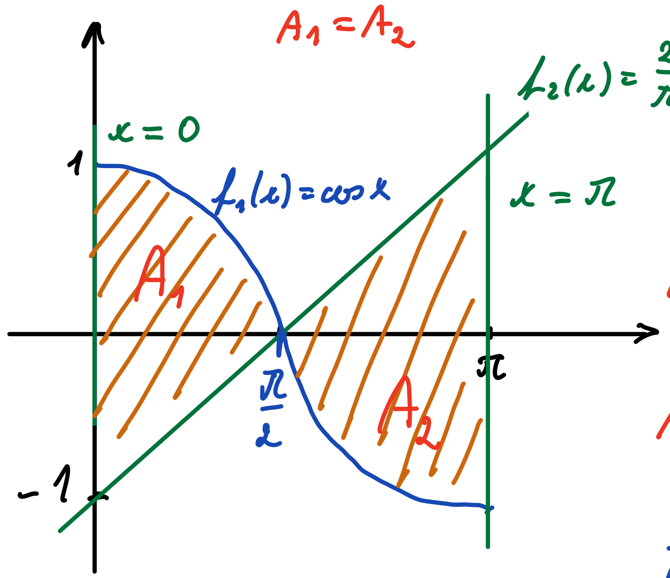
$$\int \frac{\ln^2 x}{2x}\,\mathrm{d}x.$$

$$\int \frac{\ln^2 x}{2x}\,dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \end{array} \right| = \int \frac{t^2}{2}\,dt = \frac{1}{2}\frac{t^3}{3} =$$

$$= \frac{1}{6}(\ln x)^3 + C$$

6. What is the area of the plane region bounded by the curves

$$y = \cos x,\ x = 0,\ x = \pi,\ y = \frac{2}{\pi}x - 1.$$



$$A = \int_0^{\pi} |f_1(x) - f_2(x)|\,dx$$

$$A = A_1 + A_2 = 2 \cdot A_1$$

$$A_1 = \int_0^{\frac{\pi}{2}} \cos x - \left(\frac{2x}{\pi} - 1\right) dx =$$

$$= \int_0^{\frac{\pi}{2}} \cos x - \frac{2x}{\pi} + 1\,dx =$$

$$= \left[\sin x - \frac{x^2}{\pi} + x\right]_0^{\frac{\pi}{2}} = 1 - \frac{\pi}{4} + \frac{\pi}{2} = 1 + \frac{\pi}{4}$$

$$A = 2 \cdot A_1 = 2 \cdot \left(1 + \frac{\pi}{4}\right) = \underline{\underline{2 + \frac{\pi}{2}}}$$

7. Determine which of the following statements are True or False.

(a) The function $\cos x$ is the inverse function to the trigonometric function $\sin x$.
(b) If f is strictly monotonic function, then f is injective.
(c) If $\lim_{x \to -1^+} f(x) = -\infty$, then f has asymptote $x = -1$.
(d) If $f''(x) = x - 1$ for any $x \in (2, 5)$, then the function f is strictly convex in the interval $(2, 5)$.
(e) Each increasing sequence is not bounded.

a) False

b) True

c) True

d) True

e) False