Solved exercises in Discrete mathematics
Sample problems

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Introduction

This file contains an English version of exercises in the course of *Discrete mathematics*. Most of the problems were prepared by Michael Kubesa, Tereza Kovářová, and Petr Kovář. The English version was prepared by Tereza Kovářová and Petr Kovář.

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1 Sets

1.1. Determine the set $A \times B$, if $A = \left\{\lfloor \frac{n}{2} \rfloor : n \in \mathbb{N},\ 5 \leq n \leq 11\right\}$ and $B = \{\pi, e\}$. Are the ordered pairs $(6, \pi)$ and $(e, 2)$ elements of $A \times B$?

Since $A = \{2, 3, 4, 5\}$, the cartesian product $A \times B = \{(2, \pi), (2, e), (3, \pi), (3, e), (4, \pi), (4, e), (5, \pi), (5, e)\}$. Neither of the pairs is an element of the set $A \times B$. [None of the pairs belongs to $A \times B$.]

1.2. Is always $A \times B = B \times A$? Justify your answer!

From the solution of the previous example we can obsereve, that the ordered pair $(e, 2)$ is not in $A \times B$, but it is in $B \times A$. Therefore, in general $A \times B \neq B \times A$. [In general $A \times B \neq B \times A$.]

1.3. Is always $|A \times B| = |B \times A|$? Give reasons for your answer!

The equality holds. We know $|A \times B| = |A| \cdot |B|$ and $|B \times A| = |B| \cdot |A|$. Since both $|A| \cdot |B|$ and $|B| \cdot |A|$ are products of natural numbers, the order of multiplication is not important, $|A| \cdot |B| = |B| \cdot |A|$ and thus $|A \times B| = |B \times A|$. [$|A \times B| = |B \times A|$]

1.4. Determine the set A, if $A = 2^X$, where $X = \{u, v, w\}$.

The set A is the so called "power set of the set X". Power set A contains all the subsets of set X.

$$A = 2^X = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}\,.$$

[$|A| = 8,\ A = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}$]

1.5. Determine the set A^2, if $A = 2^X$ and $X = \{0, 1\}$.

$A = 2^X = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Therefore

$$A^2 = A \times A = \{(\emptyset, \emptyset), (\emptyset, \{0\}), (\emptyset, \{1\}), (\emptyset, \{0, 1\}), (\{0\}, \emptyset), (\{0\}, \{0\}), (\{0\}, \{1\}), (\{0\}, \{0, 1\}),$$

$$(\{1\}, \emptyset), (\{1\}, \{0\}), (\{1\}, \{1\}), (\{1\}, \{0, 1\}), (\{0, 1\}, \emptyset), (\{0, 1\}, \{0\}), (\{0, 1\}, \{1\}), (\{0, 1\}, \{0, 1\})\}\,.$$

$\left[|A^2| = 16,\ A^2 = A \times A = \ldots\right]$

1.6. Is always $|A \cup B| = |A| + |B|$? Justify your answer!

The equality does not hold in general. The equality is true only for disjoint sets A and B. [No.]

1.7. Is always $|A \cap B| = |A \cup B| - |A \setminus (A \cap B)| - |B \setminus (A \cap B)|$? Justify your answer!

Yes, the equality holds. We know that $A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B)$ and further we know that $A \setminus (A \cap B), B \setminus (A \cap B),\ A \cap B$ are disjoint. (This can be verified with the aid of Venn diagrams). Therefore $|A \cup B| = |A \setminus (A \cap B)| + |B \setminus (A \cap B)| + |A \cap B|)$. From here the equality follows. [Yes.]

1.8. Determine the sets $\bigcap_{i=1}^{4} A_i$ and $\bigcup_{i=1}^{4} A_i$, if $A_i = \{i \cdot n : n \in [3, 7]\}$.

Because $A_1 = \{3, 4, 5, 6, 7\}$, $A_2 = \{6, 8, 10, 12, 14\}$, $A_3 = \{9, 12, 15, 18, 21\}$, $A_4 = \{12, 16, 20, 24, 28\}$, we have $\bigcup_{i=1}^{4} A_i = \{3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 28\}$ and $\bigcap_{i=1}^{4} A_i = \emptyset$.

$\left[\bigcap_{i=1}^{4} A_i = \emptyset,\ \bigcup_{i=1}^{4} A_i = \{3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 28\}\right]$

2 Sums and Products

2.1. Evaluate $\prod_{i=1}^{5} i^2$*?*
$\prod_{i=1}^{5} i^2 = (\prod_{i=1}^{5} i)^2 = (5!)^2 = (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)^2 = (120)^2 = 14400$ [14400]

2.2. Evaluate the sum $\sum_{i \in J} i$*, where* $J = \{2^n : n \in \mathbb{N}, 1 \leq n \leq 8\}$.
$\sum_{i \in J} i = 2 + 4 + 8 + \cdots + 256$. It is the sum of the first eight terms of a geometric progression, where $a_1 = 2$ and common ratio $q = 2$. Therefore $\sum_{i \in J} i = 2\frac{1-2^8}{1-2} = -2 \cdot (-255) = 510$. [510]

2.3. Does the equality $\sum_{i=1}^{n}(i-3) = \sum_{i=1}^{n} i - 3$ *hold? If not, adjust the right side in a non-trivial way, so that the equality holds. Justify your decision!*
Equality does **not** hold. $\sum_{i=1}^{n}(i-3) = (1-3)+(2-3)+\cdots+(n-3) = (1+2+\cdots+n)-3n = \sum_{i=1}^{n} i - \sum_{i=1}^{n} 3$.
Another solution:
We will show, that in general the equality is false.

$$\begin{aligned} \sum_{i=1}^{n}(i-3) &= \sum_{i=1}^{n} i - 3 \\ \sum_{i=1}^{n} i - \sum_{i=1}^{n} 3 &= \sum_{i=1}^{n} i - 3 \\ -\sum_{i=1}^{n} 3 &= -3 \\ -3n &= -3 \\ n &= 1. \end{aligned}$$

Equality holds only for $n = 1$. Otherwice the equality has to be modified. For instance as $\sum_{i=1}^{n}(i-3) = \sum_{i=1}^{n} i - 3 - (n-1)3$. [No]

2.4. Does the equality $\prod_{i=1}^{n}(i-1) = \prod_{i=1}^{n} i - 1$ *hold? Justify your answer!*
The first factor of the product $\prod_{i=1}^{n}(i-1)$ is zero and so $\prod_{i=1}^{n}(i-1) = 0$. The product $\prod_{i=1}^{n} i - 1 = n! - 1$, which is zero only for $n = 1$. For $n > 1$ is the product on the right side positive, on the left side zero. For $n < 1$ the products on both sides are empty, and so the left side equals 1 and the right side equals zero. The equality does not hold. [No]

2.5. Evaluate the sum $\sum_{i=1}^{n} j$.
$\sum_{i=1}^{n} j = \underbrace{j + j + \cdots + j}_{n} = nj$. [$nj$]

2.6. Evaluate the product $\prod_{i=1}^{n} a$.
$\prod_{i=1}^{n} a = \underbrace{a \cdot a \cdots a}_{n} = a^n$. [$a^n$]

2.7. Evaluate $\sum_{i \in J} i$*, where* $J = \{5n : n \in \mathbb{N},\ 1 \leq n \leq 6\}$*?*
$\sum_{i \in J} i = 5 + 10 + 15 + 20 + 25 + 30$. It is the sum of the first six terms of an arithmetic progression, with first term $a_1 = 5$ and difference $d = 5$. Therefore $\sum_{i \in J} i = \frac{1}{2} \cdot 6 \cdot (5 + 30) = 3 \cdot 35 = 105$. [105]

2.8. What is the expression $\sum_{i=1}^{n}(4i-1)$ *equal to? Evaluate the sum.*
$\sum_{i=1}^{n}(4i-1) = 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 4 \cdot \frac{1}{2}n(n+1) - n = 2n^2 + 2n - n = 2n^2 + n = n(2n+1)$. [$n(2n+1)$]

2.9. What is the expression $\prod_{i=1}^{n}(a+2)i$ *equal to? Evaluate the product.*
$\prod_{i=1}^{n}(a+2)i = \prod_{i=1}^{n}(a+2) \cdot \prod_{i=1}^{n} i = (a+2)^n \cdot n!$. [$(a+2)^n n!$]

3 Selections and Arrangements Without Repetition.

3.1. A hockey coach has a team of 5 fullback players and 7 forward players. In how many ways can he set up a formation of 2 fullbacks and 3 forwards?

The coach has to make an unordered selection of 2 fullbacks out of 5 and 3 forwards out of 7. Any player can appear only once in a single selection, players can't be repeated. The coach has $\binom{5}{2}\binom{7}{3} = 10 \cdot 35 = 350$ ways to set up a formation. [350]

3.2. A hockey coach has a team of 5 fullback players and 6 forward players. In how many different ways can he set up a formation of 2 fullbacks and 3 forwards, if one particular forward player can play as a fullback too?

Again setting up a formation is an unordered selection without repetition. We will distinguish two cases.

(i) The exceptional forward player plays as a forward or is out of the game: $\binom{5}{2}\binom{6}{3} = 200$ ways.

(ii) The exceptional forward player plays as a fullback: $\binom{5}{1}\binom{5}{3} = 50$ ways.

All together 250 ways. [250]

3.3. A hockey coach has a team of 9 men. Three of the men are good forward players. In how many different ways can the coach set up a formation of 2 fullbacks and 3 forwards, if at least one of the three good forwards has to be on the formation?

To set up a formation is again an unordered selection without repetition.

(i) The number of all ways to set up a formation of 5 players is $\binom{9}{5} = 126$.

(ii) The number of ways to set up a formation without any of the three good forwards included is $\binom{9-3}{5} = \binom{6}{5} = \binom{6}{1} = 6$.

All together there are $126 - 6 = 120$ ways to set up the required formation.

Another solution

The coach can select i players, where $i = 1, 2, 3$, out of the good forwards and $5-i$ players out of remaining 6 members of the team. The number of ways to select a formation of 5 players with at least one good forward is:

$$\binom{3}{1} \cdot \binom{6}{4} + \binom{3}{2} \cdot \binom{6}{3} + \binom{3}{3} \cdot \binom{6}{2} = 3 \cdot 15 + 3 \cdot 20 + 1 \cdot 15 = 45 + 60 + 15 = 120.$$

[120]

3.4. There are 12 children in the kindergarten. The teacher has 12 different toys available. In how many ways can the teacher distribute the toys among the children so that each child gets at least one toy. No two children share a toy.

The selection in this case is ordered (an arrangement), since we have to distinguish to which child which toy is given. We distribute all the toys, therfore the number of ways is given by the permutation of the set of toys. $P(12) = 12! = 479001600$ [479001600]

3.5. There are 12 children in the kindergarten. The teacher has 18 different toys available. In how many ways can the teacher distribute the toys among the children so that each child gets one toy. No two children share a toy.

Again the selection is ordered (an arrangement), we distinguish to which child which toy is given. Out of 18 toys 12 will be selected and distributed. The number of options is given by 12-permutation out of 18 objects without repetition of toys. $\binom{18}{12}12! = \binom{18}{6}12! = \frac{18!}{6!} = 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 8892185702400.$
[8892185702400]

3.6. How many injective mappings of a 3-element set to a 5-element set exist?

We have $\binom{5}{3}$ options to select 3 elements out of five element set onto which elements of the three element set will be mapped injectively. There are 3! ways to realize the injective mapping of 3 elements of one set to the 3 selected elements of the other set. Overall number of injective mappings is $\binom{5}{3}3! = 10 \cdot 6 = 60$.

Another solution

It is an arrangement, when we choose three objects out of five element set without repetition. $V(5,3) = \frac{5!}{2!} = 60$. [60]

3.7. How many injective mappings of a 5-element set to a 3-element set exist?

An injective mapping assigns to each element of the first set exactly one distinct element of the second set. Therefore, no injective mapping of 5 elements to 3 elements does exist. [None]

3.8. In how many ways can I eat my favorite biscuits for breakfast, if there are 10 different biscuits in the bag and I will eat 6 of them (4 biscuits will be left for a snack)? Consider two cases. First case – do not distinguish the order of eaten biscuits, second case – distinguish the order. How is the number of ways changed if the order matters?

If we do not consider the order in which the biscuits are eaten, the number of ways corresponds to an unordered selection of 6 objects out of 10 element set (without repetition). The number of ways is $\binom{10}{6} = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2} = 210$.

If the order of the eaten biscuits matters, the number of ways corresponds to an arrangement of 6 objects out of 10 element set (without repetition). The number of ways is $V(10,6) = \frac{10!}{(10-6)!} = 10\cdot9\cdot8\cdot7\cdot6\cdot5 = 151200$.

Or we can just realize, that it is enough to order in $P(6)$ ways the six already chosen biscuits, which leads to $\binom{10}{6}6! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151200$ ways.

The number of options is $6! = 720$ times greater. [210, 151200]

3.9. How many different lines can lead through 8 points of the plane? Assume that no three points are collinear (do not lay in a line).

Through each pair of points we can draw one line. It is an unordered selection of two points out of set of eight points without repetition. The number of options is $\binom{8}{2} = \frac{8\cdot7}{2} = 28$. We can draw 28 different lines. [28]

3.10. In how many ways can 6 friends be seated in a theatre row of 6 seats so that Theofil sits next to Angelina?

We distinguish two cases: Angelina sits next to Theofil to the left or to the right. In both cases, we consider the couple Angelina-Theofil as one person and compute the number of options to seat 5 persons in a row. The total number of options is $2 \cdot P(5) = 2 \cdot 5! = 2 \cdot 120 = 240$. [240]

3.11. Compute. Is there more ways to:

(a) pick 3 cards out of 10 card deck without considering the order,

(b) or to arrange 5 different cards in a row?

We compute the number of ways in both cases:

(a) $C(10,3) = \binom{10}{3} = \frac{10\cdot9\cdot8}{3\cdot2} = 10 \cdot 3 \cdot 4 = 120$.

(b) $P(5) = 5! = 120$.

The number of ways is the same. [The same.]

4 Selections and Arrangements with Repetition, Compound Arrangements or Selections

4.1. How many different car license plates can be issued in our Moravian-Silesian region? (A license plate has the form ?T? ????, where ? stands for an integer?)

A license plate consists of an arrangements of six digits selected from 0 up to 9, repetition of digits allowed. We choose 6 times (for each digit position) always out of 10 options (among 10 digits). $V^*(10,6) = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^6$. $\left[10^6\right]$

4.2. At a gas station, there is a row of 12 flagpoles with 3 blue, 2 green, 4 red, and 3 yellow flags. In how many different arrangements can the flags be hanged up? Is it possible to have different ordering of flags every day in 700 years?

It is an ordered arrangement of 12 flags of 4 different colors, while the number of flags of a specific color is given. The number of options how to set up such an arrangement is the permutation with repetition: $P^*(3,2,4,3) = \frac{(3+2+4+3)!}{3!2!4!3!} = \frac{12!}{4!(3!)^2 2!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 6 \cdot 2} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 5 = 277200$.

Further, we know that $277200/365 \doteq 759$, and so a different ordering of flags is possible every day in 700 years.

If we consider leap-years, we can say that a year has less than 365.25 days (not every 4th year is a leap-year). Then $277200/365.25 > 758$, and different orderings of flags on every day are possible even longer than for 700 years. [277200, Yes.]

4.3. Compute how many anagrams can be formed of the letters of the words "KUALA LUMPUR" (space including), that have two words.

Anagrams contain 1x space, 2x A, 1x K, 2x L, 1x M, 1x P, 1x R, and 3x U. It is an ordered arrangement of 12 letters with given number of repeats. The number of anagrams is given by the number of permutation with repetitions:

$$P^*(1,2,1,2,1,1,1,3) = \frac{12!}{(2!)^2(3!)} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{24} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 2 = 19958400.$$

Out of all anagrams

$$P^*(2,1,2,1,1,1,3) = \frac{11!}{(2!)^2(3!)} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{24} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 2 = 1663200$$

have the space at the beginning and the same number 1663200 have the space at the end. Overall there exist $19958400 - 2 \cdot 1663200 = 16632000$ anagrams. [16632000]

4.4. In how many ways can four chessboard squares can be selected, if no two can be from the same column?

We solve this problem as a compound selection. (i) In the first step we compute the number of options to select four columns. It is an unordered selection of 4 columns out of 8. $\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 2 \cdot 7 \cdot 5 = 70$.

(ii) In the second step, in each of the four choosen columns we select one square arbitrarily. Since it matters in which column a square is choosen this is an arrangement. The number of options can be counted as four independent choices of 1 square out of 8 $\binom{8}{1}^4 = 8^4 = 4096$ or ordered arrangement with repetition $V^*(8,4) = 8^4$.

All together we have $70 \cdot 4096 = 286720$ ways to select four squares.

$\left[C(8,4) \cdot V^*(8,4) = 286720\right]$

4.5. In how many ways can 11 be written as the sum of (a) five nonnegative integers; (b) four positive integers? Suppose, we distinguish the order of the summands, i.e. $3+1+4+0+3 = 11$ *and* $4+1+0+3+3 = 11$ *are different sums.*

We count the number of ways as the number of options to distribute 11 ones into five or four boxes (a box corresponds to a summand) with repetition (the choice of a box can be repeated).

- (a) $C^*(5,11) = \binom{11+5-1}{5-1} = \binom{15}{4} = \frac{15\cdot14\cdot13\cdot12}{24} = 15\cdot7\cdot13 = 1365$ ways.
- (b) First we place one 1 into each box, so that each box is non-empty (a positive summand). Further, we distribute remaining seven 1s: $C^*(4,7) = \binom{7+4-1}{4-1} = \binom{10}{3} = \frac{10\cdot9\cdot8}{3\cdot2} = 10\cdot3 = 120$ ways.

[(a) $C^*(5,11) = 1365$; (b) $C^*(4,7) = 120$]

4.6. A vending machine dispenses three kinds of beverages: Pepsi, Fanta, and Sprite. During a break between classes six beverages were sold. What is the number of possibilities which beverage brands were sold?

We compute the number of possibilities as an unordered selection of 6 objects out of 3 kinds with repetition of kinds allowed. $C^*(3,6) = \binom{6+3-1}{3-1} = \binom{8}{2} = \frac{8\cdot7}{2} = 28.$ [$C^*(3,6) = 28$]

4.7. How many different mappings of a 3-element set to a 5-element set do exist?

For each of 3 elements (pre-images) we select one of 5 elements (images) with repetition. It is an arrangement with repetition $V^*(5,3) = 5^3 = 125.$ [$V^*(5,3) = 125$]

4.8. We need to place 5 girls and 7 boys in a row so that no girl stands next to another girl. (They would chatter all the time.) In how many ways can it be done?

First we stand 7 boys in a row, i.e. $P(7) = 7! = 5040$ options. Then we need to fit the girls into eight spaces between boys (including edges). Into each space at most one girl can be fitted. It is an unordered selection of 5 spaces out of 8, there is $\binom{8}{5}$ options. For each choice of spaces for girls, there exist $P(5) = 5!$ options how to arrange the girls.

The total number of ways to stand the boys and girls is $P(7)\cdot C(8,5)\cdot P(5) = 7!5!\binom{8}{5} = 7!5!(56) = 120\cdot5040\cdot56 = 33\ 868\ 800.$ [33 868 800]

4.9. In the USA senate there are 100 senators, where each state is represented by two senators. (USA has 50 states.) In how many ways a 4 senator committee for economy and management tender can be formed, if at least one pair of senators on the committee must be from the same state.

We have $C(50,1) = \binom{50}{1} = 50$ options to choose a pair of senators from one state of Union. For each choosen pair there is $C(98,2) = \binom{98}{2}$ ways to complete the four member committee. However, in a total $50\binom{98}{2}$ of options, the number of committees with two pairs of senators from two different states, $\binom{50}{2}$ options, is counted twice. So the correct total number of options to set up the required committee is $50\binom{98}{2} - \binom{50}{2} = 50\cdot49\cdot97 - 25\cdot49 = 237650 - 1225 = 236425.$

Another solution

We choose a state from which the pair of senators will be on committee, it is 50 options. We choose the third member out of 98 possibilities and the forth member out of 96 possibilities. We do not distiguish the order of the third and fourth member so it is together $C(50,1)C(98,1)C(96,1)\frac{1}{2} = 50\cdot\frac{98\cdot96}{2}$ options. Committees with two pairs of senators from two different states are not counted yet, there are $\binom{50}{2}$ options. So the total number of options to set up the required committee is $C(50,1)C(98,1)C(96,1)\frac{1}{2} + C(50,2) = 50\cdot\frac{98\cdot96}{2} + \binom{50}{2} = 50\cdot49\cdot96 + 25\cdot49 = 2352001225 = 236\ 425.$

[236 425]

4.10. Matthew had 7 white club-T-shirts with numbers $2, 4, 7, 22, 68, 77,$ *and* 88*. He wants to color three shirts red, two shirts blue, and leave two shirts white. In how many different ways can he color his T-shirts?*

Matthew can choose any 3 shirts out of 7 to color them blue, there are $C(7,3)$ options. Then, from the remaining 4 shirts he can choose any 2 to color them blue, there are $C(4,2)$ options. The remaining two shirts are left white. Total number of options to color the shirts is $\binom{7}{3}\binom{4}{2} = \frac{7\cdot6\cdot5}{6}\cdot6 = 210.$

Another solution
Matthew can sort his T-shirts according to their numbers (increasing order). Then each coloring of T-shirts can be characterize by a "word" with letters representing the colors since there will always be 3 r, 2 b and 2 w. For instance, the word "wrrbwbr" says that shirt 2 is white, 4 is red, 7 is red, 22 is blue, 68 is white, 77 is blue, and 88 is red. It means that the number of colorings of T-shirts is the same as the number of anagrams of that word. Therefore, Matthew has $P^*(3,2,2) = \frac{7!}{3!2!2!} = \frac{7\cdot6\cdot5\cdot4}{2\cdot2} = 210$ options to color his T-shirts.

Notice that $\binom{7}{3}\cdot\binom{4}{2} = \frac{7!}{3!4!}\cdot\frac{4!}{2!2!} = \frac{7!}{3!2!2!}$.

[210]

4.11. What is the number of all six digit positive integers divisible by 5? (Naturally, the first digit cannot be 0.)

Always, the numbers that are divisible by 5 have the last digit 0 or 5. According to the combinatorial product rule we obtain the total number of options as a product of three sub-selections: first digit out of 9 options (not 0), next four digits each out of 10 options, and the last digit out of 2 options (0 or 5). $V^*(9,1)V^*(10,4)V^*(2,1) = 9\cdot10^4\cdot2 = 180\ 000$.
Another solution
We can count all the 6 digit numbers, that is $(999999 - 100000 + 1)$. Divisible by 5 is each fifth integer starting at 1, so the total number of numbers we count is $(999999 - 100000 + 1)/5 = 900000/5 = 180\ 000$.
$\left[9\cdot10^4\cdot2 = 180\ 000\right]$

4.12. Suppose, we have 8 identical balls and four different colors. We want to color each ball with one of these four colors. In how many ways can we do that?

It is an unordered selection of 8 balls (objects) out of 4 colors (kinds, boxes), where colors (kinds, boxes) can be repeated. The number of ways to color the balls corresponds to the number of ways to distribute 8 objects into 4 boxes with repetition (each box can contain more objects). There are $C^*(4,8) = \binom{8+4-1}{4-1} = \binom{11}{3} = 165$ possibilities. $[C^*(4,8) = 165]$

4.13. We consider the expression $(4x^2 - 7y)^9$. After taking the power, what is the coefficient at the term x^8y^5?

The binomial theorem $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$ yields, that for the term containing $x^8y^5 = (x^2)^4y^5$ is $k = 4$ and $n-k = 9-4 = 5$. Therefore, after taking the 9th power of the binomial $(4x^2 - 7y)$ the 4th term is $\binom{9}{4}4^4x^8(-7)^5y^5$ and the coefficient we are looking for is $\binom{9}{4}4^4(-7)^5 = 126\cdot256\cdot(-16807) = -542126592$.
$[-542126592]$

5 Probability

5.1. George has 8 pairs of socks in a drawer. There are two pairs of blue socks, two pairs of black socks, two pairs of brown socks, and two pairs of green socks. If he pulls out two socks at random

(a) what is the probability, that he has a pair of brown socks?

(b) what is the probability, that he has two socks of the same color?

We consider the sample space, that models pulling two socks out of the drawer. Then, elementary events are all possible pairs of socks, while the order of socks in a pair is unimportant.

$$\omega = \{\{p_1, p_2\} : p_1, p_2 \text{ are two socks}\}.$$

Now $|\omega| = C(16, 2) = \binom{16}{2} = \frac{16\cdot 15}{2} = 120$. There are 16 socks in the drawer and we suppose that any pair of socks is equally likely to be drawn. Therefore, the corresponding sample space (ω, P) is uniform.

(a) By A we denote the event "George draws a pair of brown socks", that is the subset $A \subset \omega$ containing all possible pairs of brown socks.

$$A = \{\{p_1, p_2\}, p_1, p_2 \text{ are two brown socks}\}.$$

The number of such pairs is $|A| = C(4, 2) = 6$. Because the sample space is uniform, we count $P(A) = \frac{6}{120} = \frac{1}{20}$.

(b) By B we denote the event "George draws a pair of socks of the same color", that is the subset $B \subset \omega$ containing all possible pairs of socks of the same color. The pairs are black, blue, brown or green.

$$B = \{\{p_1, p_2\}, p_1, p_2 \text{ are two socks of the same color}\}.$$

The number of such pairs is $|B| = 4C(4, 2) = 4 \cdot 6 = 24$. Because the sample space is uniform, we count $P(A) = \frac{24}{120} = \frac{1}{5}$.

$\left[\text{(a) } \frac{1}{20}, \quad \text{(b) } \frac{1}{5}\right]$

5.2. George has 8 pairs of socks disorganized in a drawer. There are two blue pairs, three black pairs, three brown pairs, and one green pair of socks. If he pulls out two socks at random

(a) what is the probability, that he has the brown pair of socks?

(b) what is the probability, that he has two socks of the same color?

We consider the sample space, that models pulling two socks out of the drawer. Then, elementary events are all possible pairs of socks, while the order of socks in a pair is unimportant.

$$\omega = \{\{p_1, p_2\} : p_1, p_2 \text{ are two socks}\}.$$

It is $|\omega| = C(18, 2) = \binom{18}{2} = \frac{18\cdot 17}{2} = 153$. There are 16 socks in the drawer and we suppose that any pair of socks is equally likely to be drawn. Therefore the corresponding sample space (ω, P) is uniform.

(a) By A we denote the event "George draws a pair of brown socks", that is the subset $A \subset \omega$ containing all possible pairs of brown socks.

$$A = \{\{p_1, p_2\}, p_1, p_2 \text{ are two brown socks}\}.$$

The number of such pairs is $|A| = C(6, 2) = 15$. Because the sample space is uniform, we count $P(A) = \frac{|A|}{|\omega|} = \frac{15}{153} = \frac{5}{51}$.

(b) By B we denote the event "George draws a pair of socks of the same color", that is the subset $B \subset \omega$ containing all possible pairs of socks of the same color. The pairs are black, blue, brown, or green.

$$B = \{\{p_1, p_2\}, p_1, p_2 \text{ are two socks of the same color}\}.$$

The number of such pairs is (acording to the number of socks of each color): $|B| = C(4,2) + C(6,2) + C(6,2) + C(2,2) = 6 + 15 + 15 + 1 = 37$. Because the sample space is uniform, we count $P(B) = \frac{|B|}{|\omega|} = \frac{37}{153}$.

$\left[\text{(a) } \frac{5}{51}, \quad \text{(b) } \frac{37}{153}\right]$

5.3. Consider a randomly shuffled deck of 32 cards. What is the probability that

(a) the first four cards are aces?

(b) the first four cards have values 7,8,9,10 (not necessarily in this order) and are of the same suit?

(c) first four cards can be ordered in a sequence Jack, Queen, King, and Ace? They can be of different suits.

(d) cards in the whole the deck are ordered in alternating colors as black, red, black, red, ... and so on?

We set up the sample space Ω, that models the choice of the first four cards. Elementary events are all possible unordered fourtuples of cards and each fourtuple is equally likely to appear on the first four positions of the randomly schuffled deck.

$$|\Omega| = \{\{k_1, k_2, k_3, k_4\} : k_1, k_2, k_3, k_4 \text{ are distinct cards.}\}$$

Such a sample space is uniform and $|\Omega| = C(32,4) = \binom{32}{4} = 35960$.

(a) Event A contains only one option, that is one choice of four aces out of the deck of 32 cards, $|A| = 1$. The probability is $P(A) = \frac{|A|}{|\Omega|} = \frac{1}{\binom{32}{4}} = \frac{1}{35960}$.

(b) Event B includes all possible fourtuples of cards with numbers 7, 8, 9, 10 and of the same suit. There is one such fourtuple of each of the four suits. And so $|B| = \binom{4}{1}$. The probability is $P(B) = \frac{|B|}{|\Omega|} = \frac{4}{\binom{32}{4}} = \frac{4}{35960} = \frac{1}{8990}$.

(c) Event C includes all possible fourtuples of cards J,Q,K,A not necessarily of the same suit. Each card value can be independently selected out of the four different suits. The number of such fourtuples is $|C| = \binom{4}{1}^4$. The probability is $P(C) = \frac{|C|}{|\Omega|} = \frac{\binom{4}{1}^4}{\binom{32}{4}} = \frac{4^4}{35960} = \frac{32}{4495}$.

(d) We set up a different sample space Ω' that contains all different orderings of red and black cards, while we do not distinguish the card values. In a sequance of 32 cards there are $\binom{32}{16}$ options to select 16 positions for the red cards, and so $|\Omega'| = C(32,16) = \binom{32}{16}$. Only one option correspond to the assigned ordering, therefore $P(D) = \frac{|D|}{|\Omega'|} = \frac{1}{\binom{32}{16}} = \frac{1}{601080390}$.

Another solution
We set up the sample space Ω, that is significantly larger and models schuffled deck as a whole. Elementary events are all possible orderings of the deck of 32 cards and each of this orderings is obtaind with the same probability $\frac{1}{32!}$.

$$|\Omega| = \{\{k_1, k_2, \ldots, k_{32}\} : k_1, k_2, \ldots, k_{32} \text{ are distinct cards.}\}$$

Such a sample space is uniform and holds $|\Omega| = P(32) = 32!$.

(a) Four aces can be ordered at the first four positions in $P(4) = 4!$ ways and the remaining 28 cards can be shouffled in $P(28) = 28!$ ways. The number of orderings in which aces are at the first four positions is $|A| = 4! \cdot 28!$ and so $P(A) = \frac{|A|}{|\Omega|} = \frac{4! \cdot 28!}{32!} = \frac{1}{35960}$.

(b) We have $|B| = P(4) \cdot 4 \cdot P(28)$ of different orderings of cards, in which at the first four positions are cards $7, 8, 9, 10$ in any order but of the same color (suit). And so $P(B) = \frac{|B|}{|\Omega|} = \frac{4! \cdot 4 \cdot 28!}{32!} = \frac{1}{8990}$.

(c) We have $V^*(4,4) = 4^4$ options how form a sequence J, Q, K, A at the first four positions. For each sequance, we have $P(28) = 28!$ ways how to order the remaining 28 cards and also $P(4) = 4!$ ways how to reorder the first 4 cards of the sequence. In a whole the probability is $P(C) = \frac{|C|}{|\Omega|} = \frac{4^4 \cdot 28! \cdot 4!}{32!} = \frac{96}{13485} = \frac{32}{4495}$.

(d) The event D contains all the orderings (shufflings) of the deck, such that black and red cards regularly alternate starting with black. First we place black cards on odd positions where we can order them in $P(16) = 16!$ ways. Analogically red cards we place on even positions with all $P(16) = 16!$ possible orderings. For each of 16! orderings of black cards we have 16! orderings of red cards, and so $P(D) = \frac{|D|}{|\Omega|} = \frac{16! \cdot 16!}{32!} = \frac{1}{601080390}$.

$\left[\text{(a) } \frac{1}{35960}, \quad \text{(b) } \frac{1}{8990}, \quad \text{(c) } \frac{32}{4495}, \quad \text{(d) } \frac{1}{601080390}\right]$

5.4. Suppose a fair six sided die is rolled four times. I bet that precisely one 6 is rolled. My opponent bets that no 6 is rolled. Who of us has the better chance to win?

We set up the finite uniform sample space Ω, that contains all possible outcomes obtained when rolling a dice four times. Then $\Omega = \{(x_1, x_2, x_3, x_4) \in \mathbb{N}^4 | 1 \le x_i \le 6, i \in \{1,2,3,4\}\}$ and so $|\Omega| = 6^4$. The number of ordered fourtuples that do not contain 6 is $|B| = 5^4$ and the number of those that have exactly one 6 is $|A| = 4.5^3$. The probability that the oponent wins is $P(B) = \frac{|B|}{|\Omega|} = \frac{5^4}{6^4}$ and the probability that I win is $P(A) = \frac{|A|}{|\Omega|} = \frac{4 \cdot 5^3}{6^4}$. We see that $\frac{5^4}{6^4} > \frac{4.5^3}{6^4}$ so $P(B) > P(A)$ so the oponent has greater chance to win. [opponent]

5.5. If three six sided dice are rolled, what is the probability that the sum of points is

(a) 6?

(b) an odd number?

(c) an even number?

We set up the finite uniform sample space, that contains all possible outcomes obtained when rolling three dice. Then $\Omega = \{(x_1, x_2, x_3) \in \mathbb{N}^3 | 1 \le x_i \le 6, i = 1,2,3\}$ and so $|\Omega| = 6^3$.

(a) We denote A the event when the sum of points is 6. Now $A \subset \Omega$ containing all ordered triples such that $x_1 + x_2 + x_3 = 6$. All possible combinations of three values are $1,1,4$ or $1,2,3$ or $2,2,2$, out of which $\frac{3!}{2!} + 3! + 1 = 10$ ordered triples can be formed. Probability of rolling the sum 6 is $P(A) = \frac{10}{6^3} = \frac{5}{2^2 \cdot 3^3} = \frac{5}{108}$.

(b) By B we denote event, that the sum $x_1 + x_2 + x_3$ of the points rolled is odd. Then $B \subset \Omega$ contains all the ordered triples such that either all the summands are odd or one of the summands is odd.

First we count the number of triples with one of the summands odd. There is $\binom{3}{1}$ ways to choose which of the summands is odd, to choose the value of this summand there are 3 options and we have $V^*(3,2) = 3^2$ options to fill even number in positions of remaining two summands. So the number of ordered triples with one number odd is $|B| = \binom{3}{1} \cdot 3 \cdot 3^2 = 3^4$.

It should not be difficult to observe that the number of triples containing only odd numbers is $V^*(3,3) = 3^3$.

Finaly, the number of triples (x_1, x_2, x_3), where $x_1+x_2+x_3$ is an odd number, is $|B| = 3^4+3^3 = 4\cdot 3^3$ and so $P(B) = \frac{|B|}{|\Omega|} = \frac{4\cdot 3^3}{6^3} = \frac{2^2\cdot 3^3}{2^3\cdot 3^3} = \frac{1}{2}$.

Another solution

We consider the uniform sample space of all rolles $\Omega = \{(i,j,k) : 1 \le i,j,k \le 6\}$. Obviously the set E of all even rolles and the set O of all odd rolles form partition of Ω, i.e. $E \cup O = \Omega$ and $E \cap B = \emptyset$. We observe that $|E| = |O|$, because to each roll with odd summ on top of dice we have even sum on bottom of dice. Therefore, because O nad E are complementary events, we write

$$P(O) + P(E) = 1 \quad \wedge \quad P(O) = P(E).$$

From here by substituting $P(O) + P(O) = 2P(O) = 1$ and so $P(O) = \frac{1}{2}$. Consequaently $P(E) = \frac{1}{2}$.

(c) This event is complementary to event B, which is denoted $\overline{B}$. It holds that $P(\overline{B}) = 1 - P(B)$, therefore $P(\overline{B}) = 1 - \frac{1}{2} = \frac{1}{2}$.

$\big[$(a) $\frac{5}{108}$, (b) $\frac{1}{2}$, (c) $\frac{1}{2}\big]$

5.6. A six sided die is rolled two times. Event A is "the sum of points rolled is 6", the event B is "the product of points rolled is 8", the event C is "on the first dice 1 point or 3 points are rolled", the event D is "on the first dice 1 point, or 2 points, or 4 points are rolled". Explain:

(a) Are the events A and B independent?

(b) Are the events C and D dependent?

(c) Are the events A and C independent?

(d) Are the events B and C dependent?

We set up a uniform sample space, that contains all possible outcomes obtained when rollong two 6-sided dice. It is $\Omega = \{(x_1, x_2) \in \mathbb{N}^2 | 1 \le x_i \le 6, i = 1, 2\}$, and so $|\Omega| = 6^2$, therefore the probability of every elementery event is $\frac{1}{36}$.

(a) Event $A = \{(1,5),(5,1),(2,4),(4,2),(3,3)\}$. Therefore $P(A) = \frac{5}{6^2} = \frac{5}{36}$. The event $B = \{(2,4),(4,2)\}$ and therefore $P(B) = \frac{2}{6^2} = \frac{1}{18}$. Next $A \cap B = \{(2,4),(4,2)\}$ and so $P(A \cap B) = \frac{1}{18}$. We see, that $P(A \cap B) = \frac{1}{18} \neq \frac{1}{18} \cdot \frac{5}{36} = P(A) \cdot P(B)$. It means that events A, B are not independent, we call them *dependent*.

(b) Event $C = \{(1,1),(1,2),\ldots,(1,6),(3,1),(3,2),\ldots,(3,6)\}$. Therefore $P(C) = \frac{12}{6^2} = \frac{1}{3}$. Event $D = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(2,6),(4,1),(4,2),\ldots,(4,6)\}$, which implies $P(D) = \frac{18}{6^2} = \frac{1}{2}$. Further $C \cap D = \{(1,1),(1,2),\ldots,(1,6)\}$, and so $P(C \cap D) = \frac{6}{6^2} = \frac{1}{6}$. From previous, we see that $P(C \cap D) = \frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2} = P(C) \cdot P(D)$. It meanst that events C and D are independent.

(c) We know already that $P(A) = \frac{5}{36}$, $P(C) = \frac{1}{3}$. Further $A \cap C = \{(1,5),(3,3)\}$, and so $P(A \cap C) = \frac{2}{6^2} = \frac{1}{18}$. That yields $P(A \cap C) = \frac{1}{18} \neq \frac{5}{36} \cdot \frac{1}{3} = P(A) \cdot P(C)$, which implies that events A, C are dependent.

(d) We know already $P(B) = \frac{1}{18}$, $P(C) = \frac{1}{3}$, and $B \cap C = \emptyset$. It holds that $P(B \cap C) = 0$. We see, that $P(B \cap C) = 0 \neq \frac{1}{18} \cdot \frac{1}{3} = P(B) \cdot P(C)$. Therefore, events B, C are dependent.

[(a) dependent, (b) independent, (c) dependent, (d) dependent]

5.7. If an eight sided dice with numbers 1 through 8 is rolled, what is the average number of points obtained?

We consider a random variable X that describes all numbers that can be rolled on the dice, i.e. $X \in [1, 8]$. We denote the probability that the number i is rolled by p_i, where $i \in [1, 8]$. Since we suppose that the dice is a fair dice $p_i = \frac{1}{8}$ for each i. And so the everage number of points rolled is $EX = \sum_{i=1}^{8} p_i \cdot i = \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 2 + \cdots + \frac{1}{8} \cdot 8 = \frac{1}{8}(1+2+\cdots+8) = \frac{1}{8} \cdot 4 \cdot 9 = \frac{9}{2} = 4.5$. [4.5]

5.8. If a six and an eight sided dice are rolled, what is the average value of the sum of points rolled? What is the average value of the product of points rolled?

We define random variables X, Y as follows: X is the value rolled on the six sided dice and Y is the value rolled on the eight sided dice. Therefore $X = [1,6]$ and $Y = [1,8]$. Obviously X and Y are independent variables, which means $E(X+Y) = E(X) + E(Y)$ and $E(X \cdot Y) = E(X) \cdot E(Y)$, while $E(X) = \frac{1}{6}(1+2+\cdots+6) = 3.5$ and $E(Y) = 4.5$ (see previous Example 7). And so $E(X+Y) = 3.5+4.5 = 8$ and $E(X \cdot Y) = 3.5 \cdot 4.5 = 15.75$. [8, 15.75.]

5.9. Suppose our six sided dice is biased. The chances to roll a small number 1, 2, or 3 are equal. Also chances to roll a big number 4, 5, or 6 are equal, just twice as big as chance to roll a small number. What is the average number of points rolled on this dice?

First we think of a sample space ω that models rolling of this dice. Sample space ω is not uniform. Now we count the probabilities of elementary events. We denote the probability that the value i is rolled by p_i. We know $p_1 = p_2 = p_3 = \frac{1}{2}p_4 = \frac{1}{2}p_5 = \frac{1}{2}p_6$, i.e. for instance $p_4 = 2p_1$. According to the definition of probability function, it must hold $\sum_{i=1}^{6} p_i = 1$. By substituting and simplifying we obtain $3 \cdot p_1 + 2 \cdot 3p_1 = 1$, from where $p_1 = \frac{1}{9}$. Now we compute the remaining probabilities easily.

$$p_1 = p_2 = p_3 = \frac{1}{9}, \quad p_4 = p_5 = p_6 = \frac{2}{9}.$$

Random variable X will represent all numbers that can be rolled, it is $X \in [1,6]$. Using the definition of the expected value

$$E(X) = \sum_{i=1}^{6} i \cdot p_i = \frac{1}{9}(1+2+3) + \frac{2}{9}(4+5+6) = \frac{9}{6+30} = \frac{36}{9} = 4.$$

So the average value rolled on this biased dice is 4. [4]

5.10. We have five coins in our pocket. There is a one 10-crown coin and four 2-crown coins. Suppose we take out two coins at random. How many crowns do we take out on average?

We consider a random variable X that takes on values of coins taken out of the pocket.

The variable X is:

- 4, when two 2-crown coins are taken,
- 12, when a one 10-crown coin along with one 2-crown coin is taken.

Therefore $X = \{4, 12\}$. Next we set up a uniform sample space containing all possibilities when taking two coins out of the pocket. There are $\binom{5}{2} = 10$ options how to select 2 coins out of all five coins in the pocket, and so $|\Omega| = 10$. We denote probabilities corresponding to the values of random variable X by p_4 and p_{12}. Number of possibilities to draw 4 crowns is $\binom{4}{2} = 6$ and to draw 12 crowns is $\binom{4}{1} \cdot \binom{1}{1} = 4$. For the probabilities we get $p_4 = \frac{6}{10} = \frac{3}{5}$ and $p_{12} = \frac{4}{10} = \frac{2}{5}$. Using the definition of the expected value, we get for random variable X

$$E(X) = 4 \cdot p_4 + 12 \cdot p_{12} = 4 \cdot \frac{3}{5} + 12 \cdot \frac{2}{5} = \frac{12+24}{5} = \frac{36}{5} = 7.2.$$

$\left[\frac{36}{5} = 7.2\right]$

5.11. In a drawing drum there is one token of value 4, two tokens of value 3, three tokens of value 2, and four tokens of value 1. If one token is drawn, what is the average value of this token?

We introduce the random variable giving the value of the token drawn. Altogether there are $1+2+3+4 = 10$ tokens in the drum. Value of X can have the following values

- 1 with probability $p_1 = \frac{4}{10} = \frac{2}{5}$,
- 2 with probability $p_2 = \frac{3}{10}$,
- 3 with probability $p_3 = \frac{2}{10} = \frac{1}{5}$,
- 4 with probability $p_4 = \frac{1}{10}$.

Thus $X \in \{1, 2, 3, 4\}$. The probability of each value we obtain similarly as in the previous example $p_1 = \frac{4}{10}$, $p_2 = \frac{3}{10}$, $p_3 = \frac{2}{10}$, and $p_4 = \frac{1}{10}$, . The expected value of X is by the definition

$$E(X) = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + 4 \cdot p_4 = 1 \cdot \frac{2}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{10} = \frac{4+6+6+4}{10} = \frac{20}{10} = 2.$$

[2]

5.12. Suppose four dice are rolled. Are events A "sequence of four consecutive numbers in any order is rolled" and B "sum of rolled numbers is even" independent?

We set up a sample space $\omega = [1, 6]^4$, which is uniform and its size is $|\Omega| = 6^4 = 1296$. The consecuteve numbers, that can be rolled are $1, 2, 3, 4,\ 2, 3, 4, 5,$ or $3, 4, 5, 6$ in any of $P(4) = 4!$ orders. Event A contains all $|A| = 3 \cdot P(4) = 3 \cdot 24 = 72$ ways to roll four consecutive numbers. Because Ω is uniform $P(A) = \frac{|A|}{|\Omega|} = \frac{72}{1296} = \frac{1}{18}$.

Among all ordered fourtuples in Ω there is exactly one half, that gives an even sum. It is because by turning one dice upside down a different fourtuple is obtained and the parity of the sum of the fourtuple is changed. Therefore $|B| = \frac{|\Omega|}{2}$ a $P(B) = \frac{1}{2}$.

Finaly $A \cap B = A$, because the sum of an ordered fourtuple is either $1+2+3+4 = 10$, or $2+3+4+5 = 14$, or $3 + 4 + 5 + 6 = 18$. For the independent events must hold $P(A) \cdot P(B) = P(A \cap B)$. However, $P(A) \cdot P(B) = \frac{1}{18} \cdot \frac{1}{2} = \frac{1}{36}$, and $P(A \cap B) = P(A) = \frac{1}{18}$. Therefore, events A and B are not independent.

[not independent]

6 Permutations and Inclusion–Exclusion principle.

6.1. Suppose we order the elements of the set $[1,9]$ as $6,7,5,4,2,3,1,9,8$. Such an ordering is a permutation of the set $[1,9]$. Write this permutation using the cycle notation. What is the order of the given permutation? (The order of π permutation is the smallest (nonzero) natural number k, such that $\pi^k = \underbrace{\pi \circ \pi \circ \ldots \circ \pi}_{k}$ is identity.)

$\pi = (163527)(4)(89)$. Permutation π is of order $LCM(6,1,2) = 6$. [Order of π is $LCM(6,1,2) = 6$]

6.2. Let's take permutations $\pi_1 = (157)(2436)$ and $\pi_2 = (13)(46)(257)$.

(a) Find permutations $\pi_1 \circ \pi_2$, $\pi_2 \circ \pi_1$ and $(\pi_2)^6$.
Hint: To find $(\pi_2)^6$ use that $(\pi_2)^6 = (\pi_2)^4 \circ (\pi_2)^2 = (\pi_2)^2 \circ (\pi_2)^2 \circ (\pi_2)^2$.

(b) Find permutations π_1^4 and π_1^6. Use cyclic notation.

(c) Find permutations π_1^{25} a π_1^{40}. Use the order of the permutation.

(a) $\pi_1 \circ \pi_2 = (1635)(274)$.
$\pi_2 \circ \pi_1 = (1734)(265)$.
$(\pi_2)^6 = (1)(2)(3)(4)(5)(6)(7)$

[a) $(\pi_2)^6 = (1)(2)(3)(4)(5)(6)(7)$, b) $(\pi_1)^4 = (157)(2)(3)(4)(6)$, $(\pi_1)^6 = (1)(23)(46)(5)(7)$ c) $(\pi_1)^{25} = (\pi_1)$, $(\pi_1)^{40} = (\pi_1)^4$]

6.3. Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

We denote the following sets:

- all students $|A| = 18$,
- students studying math $|M| = 7$,
- students studying science $|S| = 10$,
- students studying programming $|P| = 10$,
- students studying math and science $|M \cap S| = 3$,
- students studying math and programming $|M \cap P| = 4$,
- students studying science and programming $|S \cap P| = 5$,
- students studying all three subjects $|M \cap S \cap P| = 1$.

Students that do not study any subject are not in the union $M \cup S \cup P$. By inclusion-exclusion principle

$$|M \cup S \cup P| = |M| + |S| + |P| - |M \cap S| - |S \cap P| - |M \cap P| + |M \cap S \cap P| = 7 + 10 + 10 - 3 - 4 - 5 + 1 = 16.$$

There are $18 - 16$ students that do not study any subject. [2]

6.4. How many surjective mappings of an n-element set to an $(n-1)$-element set do exist?

We use inclusion-exclusion principle. From the total of $(n-1)^n$ mappings we subtract the number of all mappings, which skip a certain element, we add the number of all mappings that skip certain two elements, we subtract the number of all mappings, which skip certain three elements, ...

$$(n-1)^n - \binom{n-1}{1}(n-2)^n + \cdots + (-1)^{n-2}\binom{n-1}{n-2}1^n$$

and we get

$$\sum_{i=0}^{n-2}(-1)^i\binom{n-1}{i}(n-1-i)^n.$$

$\left[\binom{n}{2}(n-1)!\right]$

6.5. How many surjective mappings of an n-element set to a 2-element set do exist?

We use inclusion-exclusion principle.

$$\sum_{i=0}^{1}(-1)^i\binom{2}{i}(2-i)^n$$

We get

$$\binom{2}{0}2^n - \binom{2}{1}1^n = 2^n - 2.$$

$[2^n - 2]$

7 Simple graphs, Parity principle, degree sequence, Havel-Hakimi Theorem.

7.1. Is it possible to draw a graph with 9 *vertices, such that no two vertices have the same degree?*

The highest possible degree of a vertex in a simple graph with nine vertices is 8, the smallest possible degree is 0. Because each vertex should be of different degree, vertex degrees would have to be of all values $0, 1, 2, \ldots, 8$ (nine different admissible values). This is impossible, because a vertex of degree 8 (adjacent to all remaining 8 vertices) and a vertex of degree 0 (not joined by an edge to any other vertex) cannot be in the graph at the same time.

[No.]

7.2. Is it possible to draw a graph with 11 *vertices, where all the vertices are of degree 3 or of degree 5? Explain your decision.*

According to the Parity Principle $\sum_{v \in V} \deg(v) = 2|E|$ from where $2|E| = \sum_{v \in V} \deg(v) = s \cdot 3 + (11 - s) \cdot 5 = 55 - 2s$. And so the number of edges $|E| = \frac{55}{2} - s$, where $s \in \mathbb{N}_0$, which is not an integer. Therefore, such graph does not exist.

[No.]

7.3. How many edges has a graph G with 150 vertices of degree 3 and with 1000 vertices of degree 4? Explain!

The number of edges of G is $|V(G)| = 150 + 1000 = 1150$. By parity principle is $\sum_{i=1}^{1150} \deg(v_i) = 2|E(G)|$. Since 150 vertices is of degree 3 and 1000 vertices is of degree 4, we substitute and get $3 \cdot 150 + 4 \cdot 1000 = 2|E(G)|$. Therefore the number of edges of G is $|E(G)| = \frac{450+4000}{2} = 2225$. Such graph does exist, for example 25 components $K_{3,3}$ and 200 components K_5.

[2225]

7.4. Suppose graph G has 10 vertices and 26 edges. Moreover, G has vertices of only two different degrees. If 4 vertices are of degree 4, of what degree are the remaining vertices?

We know $|V(G)| = 10$ and $|E(G)| = 26$. We denote by x the unknown degree and substitute to the Parity Principle $\sum_{i=1}^{|V(G)|} \deg(v_i) = 2|E(G)|$. After substituting the known vertex degree and the number of edges we get $4 \cdot 4 + 6 \cdot x = 2 \cdot 26$. Therefore $6x = 36$ and thus $x = 6$. The remaining 6 vertices have degree 6. Using Havel-Hakimi Theorem one can verify that such graph exists. [G has 6 vertices of degree 6.]

8 Isomorphisms of Graphs

8.1. In a graph G given by Figure 8.1 determine

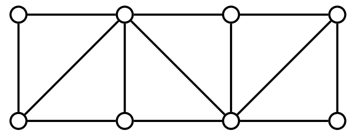

Figure 8.1: Graph G.

(a) the greatest independent set of vertices, $X \subseteq V(G)$. (The set $X \subseteq V(G)$ is the greatest independent set in G, if it contains the maximum number of vertices of G that are not connected by any edge.)

(b) a subgraph that is the longest path and a subgraph that is the longest cycle.

(c) the longest induced path and the longest induced cycle.

(a) Any cycle of the length 8 contains at most 4 independent vertices (see Figure 8.2).

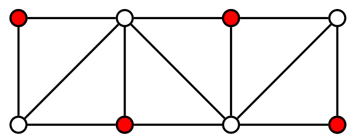

Figure 8.2: *Independent set in G.*

(b) Graph G contains path P_8 and a hamiltonian cycle (the longest cycle possible).

Figure 8.3: *The longest path and the longest cycle in G.*

(c) Out of each triangle, to an induced path, at most one edge can be included. Therefore the longest induced path is of the length 4 (see Figure 8.4).

Each cycle of the length greater than 4 has an edge connecting vertices that are not adjacent on this cycle. Therefore the longest induced cycle is of the length 3.

Figure 8.4: *The longest induced path and the longest induced cycle in G.*

8.2. Graphs G, H, and I are given in Figure 8.5.

(a) Are graphs G and H isomorphic? If yes, write down an isomorphism of these graphs.

(b) Are graphs G and I isomorphic? If yes, write down an isomorphism of these graphs.

(c) Are graphs H and I isomorphic? If yes, write down an isomorphism of these graphs.

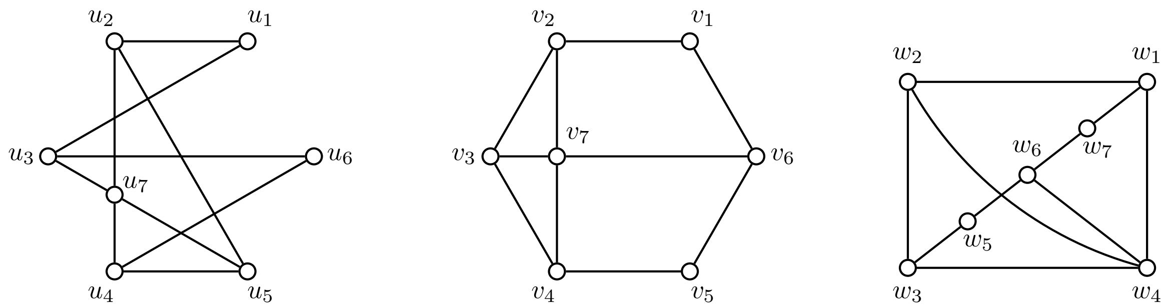

Figure 8.5: Graphs G, H and I.

(a) When searching for an isomorphism of graphs, we try to redraw the graph G, so that it is a copy of the graph H. We start with the vertices of degree four. There is a unique one in G and a unique one in H as well. Therefore, if an isomorphism exists these vertices have to be mapped to each other. Let us number the vertices of G (see Figure 8.6)

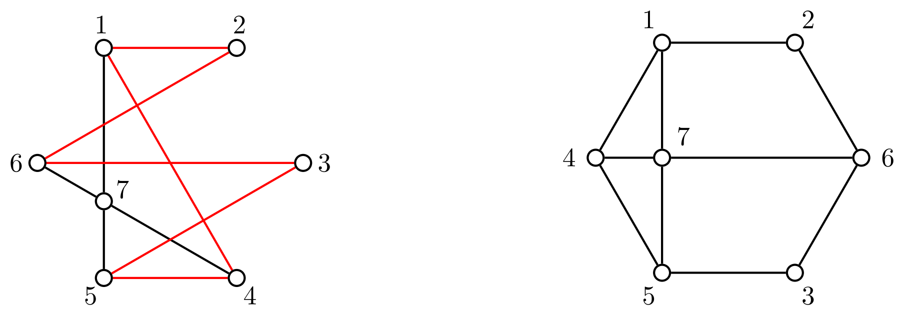

Figure 8.6: *Graphs G a H.*

Then we number the vertices of H, so that the vertices in G have the same numbers as their images in H (see Figure 8.6). The isomorphism is $f : V(G) \to V(H)$, where $f(i) = i$ for $i = 1, 2, \ldots, 7$.

Another solution:
Graphs G and H are isomorphic, two isomorphisms do exist, one of them is $f : V(G) \to V(H)$ given by

$$f(u_1) = v_1, \quad f(u_2) = v_2, \quad f(u_3) = v_6, \quad f(u_4) = v_4, \quad f(u_5) = v_3, \quad f(u_6) = v_5, \quad f(u_7) = v_7.$$

(b) Graphs G and I are isomorphic, two isomorphisms do exist, one of them is $f : V(G) \to V(I)$ given by

$$f(u_1) = w_7, \quad f(u_2) = w_1, \quad f(u_3) = w_6, \quad f(u_4) = w_3, \quad f(u_5) = w_2, \quad f(u_6) = w_5, \quad f(u_7) = w_4.$$

(c) Graphs H and I are isomorphic, two isomorphisms do exist, one of them is $f : V(H) \to V(I)$ given by

$$f(v_1) = w_7, \quad f(v_2) = w_1, \quad f(v_3) = w_2, \quad f(v_4) = w_3, \quad f(v_5) = w_5, \quad f(v_6) = w_6, \quad f(v_7) = w_4.$$

8.3. Graphs G, H, and I are given in Figure 8.7.

(a) Are graphs G and H isomorphic? If yes, write down an isomorphism of these graphs.

(b) Are graphs G and I isomorphic? If yes, write down an isomorphism of these graphs.

(c) Are graphs H and I isomorphic? If yes, write down an isomorphism of these graphs.

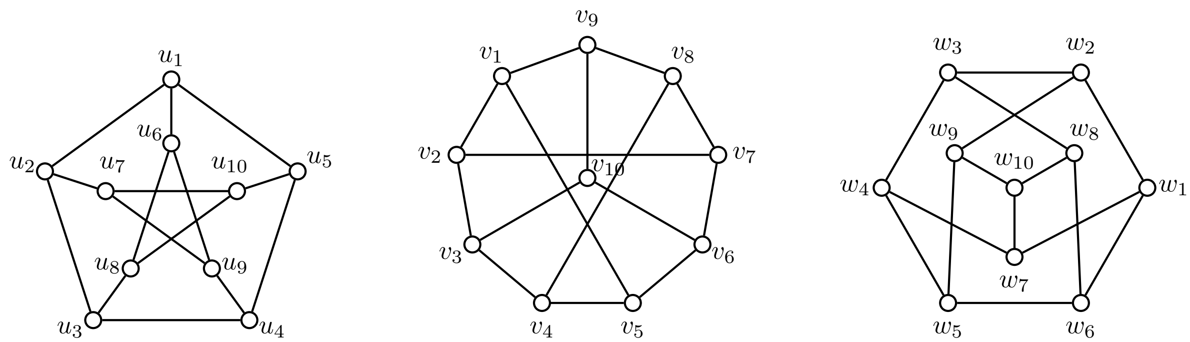

Figure 8.7: Graphs G, H and I.

(a) When searching for an isomorphism of graphs, we try to redraw the graph G, so that it is a copy of graph H. Let us assign numbers to the vertices of G (see Figure 8.8). We number the vertices of H, so that the vertices in G have the same numbers as their images in H. When assigning numbers to the vertices of H we follow some of the cycles while trying to preserve the adjacency of corresponding vertices. Notice that in the Petersen graph the shortest cycles are cycles of the length 5. The isomorphism is $f: V(G) \to V(H)$ where $f(i) = i$ for $i = 1, 2, \ldots, 10$.

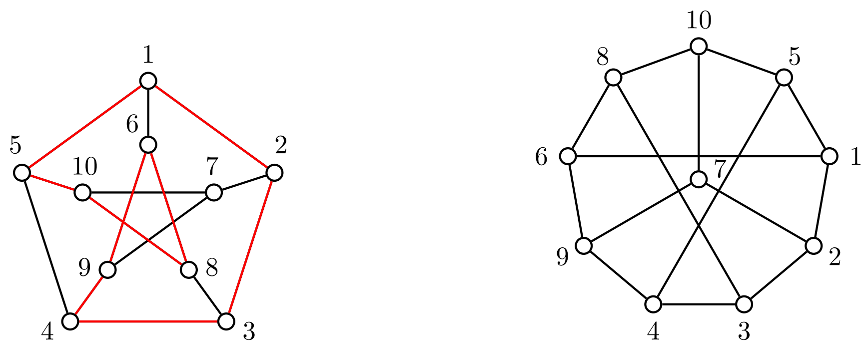

Figure 8.8: *Graphs G a H.*

Another solution:
Graphs G and H are isomorphic, it is possible to show that 120 different isomorphisms exist. One of them is $f: V(G) \to V(H)$ given by

$$f(u_1) = v_1, \quad f(u_2) = v_2, \quad f(u_3) = v_7, \quad f(u_4) = v_8, \quad f(u_5) = v_9,$$

$$f(u_6) = v_5, \quad f(u_7) = v_3, \quad f(u_8) = v_6, \quad f(u_9) = v_4, \quad f(u_{10}) = v_{10}.$$

(b) Graphs G and I are isomorphic, it is possible to show that 120 different isomorphisms exist. One of them is $f: V(G) \to V(I)$ given by

$$f(u_1) = w_3, \quad f(u_2) = w_4, \quad f(u_3) = w_7, \quad f(u_4) = w_1, \quad f(u_5) = w_2,$$

$$f(u_6) = w_8, \quad f(u_7) = w_5, \quad f(u_8) = w_{10}, \quad f(u_9) = w_6, \quad f(u_{10}) = w_9.$$

(c) Graphs H and I are isomorphic, it is possible to show that 120 different isomorphisms exist. One of them is $f: V(H) \to V(I)$ given by

$$f(v_1) = w_3, \quad f(v_2) = w_4, \quad f(v_3) = w_5, \quad f(v_4) = w_6, \quad f(v_5) = w_8,$$

$$f(v_6) = w_{10}, \quad f(v_7) = w_7, \quad f(v_8) = w_1, \quad f(v_9) = w_2, \quad f(v_{10}) = w_9.$$

8.4. If possible, find at least two non-isomorphic graphs with the given degree sequence.

(a) $(3,3,3,3,3,3)$

(b) $(3,3,2,2)$

(c) $(4,4,3,3,3,3)$

(d) $(4,4,4,3,3)$

(e) $(2,2,2,2,2)$

(f) $(3,3,3,3,2,2)$

Give a valid argument, that your graphs are not isomorphic. Is it possible to find two non-isomorphic graphs for each of the given sequences? Explain!

(a) Two non-isomorphic graphs with the given degree sequence do exist. Graphs G and H (see Figure 8.9) are non-isomorphic since H contains C_3 as a subgraphs and G does not.

Another solution:
Graphs G and H are not isomorphic, since the complement of G is $2C_3$, while the complement of H is C_6. Because G and H have different (non-isomorphic) complements, they also are non-isomorphic. (The definition of a graph complement is to be found in graph theory textbooks or on the Internet.)

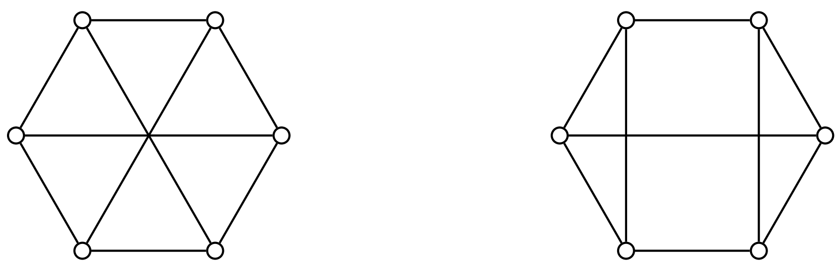

Figure 8.9: *Graphs G a H.*

(b) A graph with the degree sequence $(3,3,3,3)$ is the complete graph K_4 (each vertex is adjacent to all three other vertices). According to the Parity Principle this graph has $\frac{1}{2}\cdot 3\cdot 4 = 6$ edges. In a graph with the degree sequence $(3,3,2,2)$ there are 5 edges. Therefore a graph with the degree sequence $(3,3,2,2)$ is obtained by removing an arbitrary edge from K_4. Such a graph is (up to renumbering vertices) unique. (This can also be clearly observed from the fact that complement of the graph is the only one graph with the degree sequence $(0,0,1,1)$, which is K_2 and two isolated vertices.) No two non-isomorphic graphs with degree sequence $(3,3,2,2)$ exist.

(c) A graph G with the degree sequence $(4,4,3,3,3,3)$ contains according to the Parity principle ten edges. A complement to G has the degree sequence $(2,2,2,2,1,1)$. Several of such complements exist, for inst. P_6, $K_2 \cup C_4$, or $P_3 \cup C_3$. Therefore, also several non-isomorphic graphs with the original degree sequence exist.

(d) A graph with the degree sequence $(4,4,4,3,3)$ is the complete graph K_5 without one (an arbitrary) edge. Such a graph is (up to renumbering vertices) unique. (This can also be clearly observed from the fact that the complement of the graph is the unique graph with the degree sequence $(1,1,0,0,0)$, which is K_2 and three isolated vertices.) No two non-isomorphic graphs with degree sequence $(4,4,4,3,3)$ exist.

(e) Each vertex is adjacent to at least two vertices. Therefore it is possible to start at an arbitrary vertex and always continue along some path to another vertex and after passing through at most five vertices close the cycle. Each graph with vertices of degree at least two therefore contains a cycle. Each cycle has to be of length at least 3. Because our graph has five vertices it cannot contain more than one cycle. Graph G is just the cycle C_5 and is unique.

(f) To find two non-isomorphic graphs with the same degree sequence sometimes Havel-Hakimi theorem can be used. (Beware, not all non-isomorphic graphs can be obtained via construction implied by the Havel-Hakimi theorem.)

First we reduce the given degree sequence:

$$(3,3,3,3,2,2) \sim (2,2,2,2,2) \sim (2,2,2,1,1) \sim (1,1,1,1).$$

Up to the sequence $(2,2,2,2,2)$ is the construction of a graph using steps of reducing in backwards direction unique (we obtain the cycle C_5). In the last step when adding the vertex of degree 3 we can create two non-isomorphic graphs (see Figure 8.10).

Figure 8.10: *Non-isomorphic graphs with the same degree sequence* $(3,3,3,3,2,2)$.

The two graphs are not isomorphic, because in the first graph the two vertices of degree 2 are adjacent while in the second graph they are not (they are independent).

8.5. Find at least two non-isomorphic graphs with the degree sequence $(5,4,3,3,2,2,2,1)$. *Give a valid argument, that your graphs are not isomorphic.*

This time it is possible to find two non-isomorphic graphs by the construction from the proof of Havel-Hakimi Theorem. First we start with reducing the given degree sequence from $(5,4,3,3,2,2,2,1)$ to $(3,2,2,2,1,1,1)$, and then to $(1,1,1,1,1,1)$. By adding a vertex of degree 3 to a graph with the degree sequence $(1,1,1,1,1,1)$ we obtain for instance the graph in Figure 8.11.

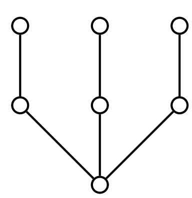

Figure 8.11: *Graph with the degree sequence* $(3,2,2,2,1,1,1)$.

Next, in the last step, by adding a vertex of degree 5, we can create two non-isomorphic graphs (see Figure 8.11).

Figure 8.12: *Graphs with the degree sequence* $(5,4,3,3,2,2,2,1)$.

Notice that in the left graph are two adjacent vertices of degree 2 , while in the right graph adjacent vertices of degree 2 do not exist. Therefore the graphs are not isomorphic.

8.6. Graph G is given in Figure 8.13. Find a graph U with the same degree sequence as G, that is not isomorphic to G. Show, that G and U are not isomorphic.

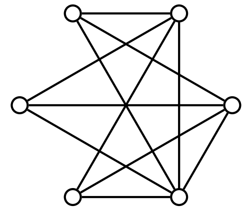

Figure 8.13: Graph G.

The given graph has the degree sequence $(5,4,4,3,3,3)$. By Havel-Hakimi theorem we reduce the sequence as follows:

$$(5,4,4,3,3,3) \sim (3,3,2,2,2) \sim (2,1,1,2) \sim (2,2,1,1) \sim (1,0,1) \sim (1,1,0).$$

By back-wards reconstruction according to reduced sequences we obtain graph H (see Figure 8.14). Graph H has the same degree sequence as the given graph, but it is not isomorphic to G. The vertices of degree 4 are adjacent in H, while in the given graph G they are not.

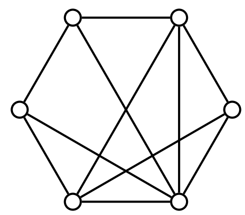

Figure 8.14: *Graph H.*

Another solution:
The complement graph of G has the degree sequence $(2,2,2,1,1,0)$. Such a degree sequence have two non-isomorphic graphs, $P_5 \cup K_1$, which is complement of G and $C_3 \cup P_2 \cup K_1$, which is complement of non-isomorphic graph H we searched for.

8.7. Let $A = \{1,2,3,4\}$. Draw a graph G with the vertices corresponding to all two-element subsets of the set A. By an edge two vertices (two subsets X and Y) are connected if:

(a) the sum of all four numbers from both subsets is odd. Find the longest path in G.

(b) the sum of all four numbers from both subsets is even. Find the longest induced path in G.

(c) the subsets contain the same number (have a nonempty intersection). Find the longest cycle in G.

(d) the subsets do not contain the same number (have an empty intersection). Determine $\delta(G)$, and $\Delta(G)$.

Set A has $\binom{4}{2} = 6$ 2-element subsets. Therefore, graph G has six vertices.

(a) The solution is the graph $K_{2,4}$. The longest path that is a subgraph of $K_{2,4}$ is P_5 (Figure 8.15). A longer path than with four edges does not exist in G, since each vertex $\{1,3\}$ and $\{2,4\}$ is incident with at most two edges of the path and no more edges exist in graph G.

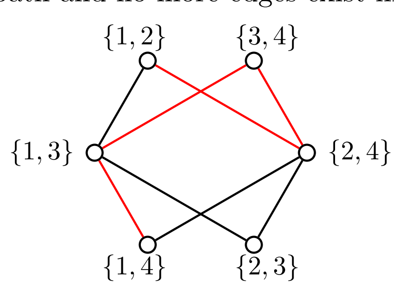

Figure 8.15: *Graph G with a highlighted subgraph P_5.*

(b) The solution is graph $K_4 \cup P_2$. The longest induced path is P_2 (Figure 8.16). The graph has two components – complete graphs – and in a complete graph the longest induced path has length 1.

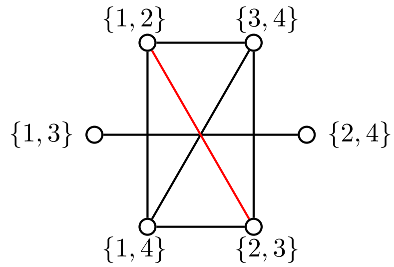

Figure 8.16: *Graph G with a highlighted path P_2.*

(c) The solution is a graph K_6 with a perfect matching removed. The longest cycle is C_6 (Figure 8.17).

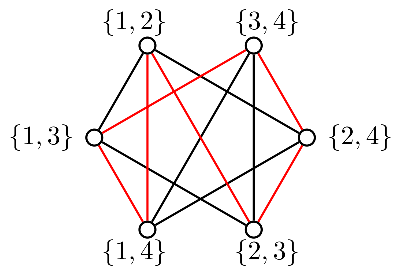

Figure 8.17: *Graph G with a highlighted cycle C_6.*

(d) The solution is a graph with three components K_2. It holds that $\delta(G) = \Delta(G) = 1$ (Figure 8.18).

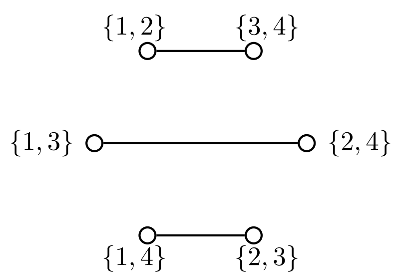

Figure 8.18: *Graph G.*

8.8. How many non-isomorphic graphs with four vertices do exist?

There are 11 such graphs (not distinguishing renumbering of vertices). (See Figure 8.19).

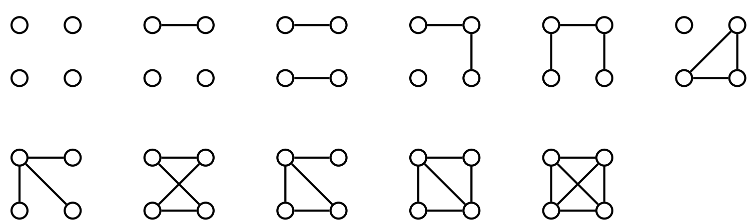

Figure 8.19: *Non-isomorphic graphs on 4 vertices.*

8.9. Are the graphs G and H given by their adjacency matrices isomorphic?

$$(a)\ A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A(H) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(b)\ A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad A(H) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(a) We sketch the graphs G and H so that the numbering of their vertices corresponds to the order of columns (and rows) of adjacency matrices. Graphs G and H are in Figure 8.20. They are not isomorphic. The argument is for instance, that H does not contain any subgraph C_3 while G does.

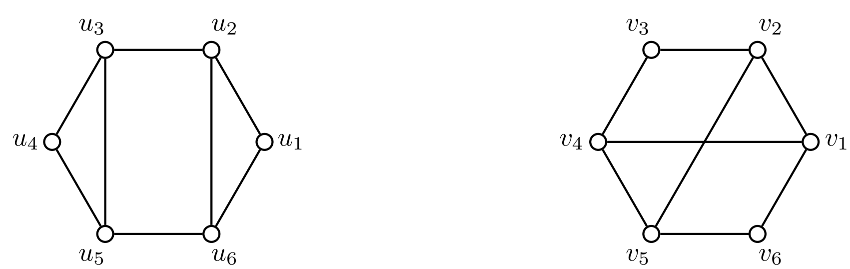

Figure 8.20: *Graphs G and H.*

(b) We sketch the graphs G and H so that the numbering of their vertices corresponds to the order of columns (and rows) of adjacency matrices. Graphs G and H are in Figure 8.21. These graphs are isomorphic. One of the possible isomorphisms is: $f : V(G) \to V(H)$, where $f(v_1) = u_1$, $f(v_2) = u_2$, $f(v_3) = u_5$, $f(v_4) = u_6$, $f(v_5) = u_3$, $f(v_6) = u_4$.

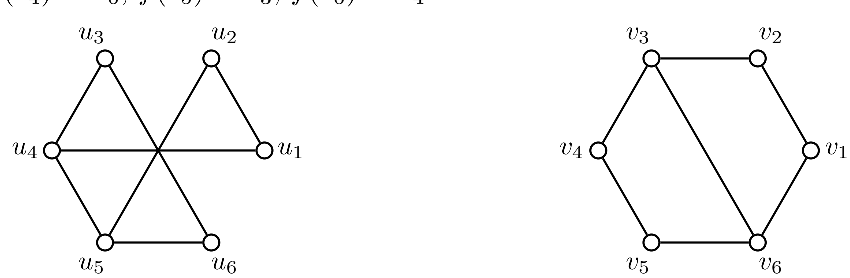

Figure 8.21: *Graphs G and H.*

9 Connectivity of Graphs, Eulerian Graphs, Distances in Graphs

9.1. How many components can a graph have, that is

(a) 4-regular with 30 *vertices,*

(b) or 5-regular with 30 *vertices?*

Justify your answer!

(a) In a 4-regular graph, each vertex has exactly 4 adjacent vertices. Therefore the smallest possible component is the complete graph with 5 vertices K_5. On 30 vertices we can have at most $\frac{30}{5} = 6$ components K_5.

(b) In a 5-regular graph, each vertex has exactly 5 adjacent vertices. Therefore the smallest possible component is the complete graph with 6 vertices K_6. On 30 vertices we can have at most $\frac{30}{6} = 5$ components K_6.

9.2. At most how many components can a graph have, if it is

(a) 4-regular on 34 *vertices,*

(b) or 5-regular on 34 *vertices?*

Justify your answer!

(a) We proceed similarly as in the previous assignment. The smallest 4-regular component is K_5 and additionally we need to construct a 4-regular component with 9 vertices. Let us construct the cycle $C_9 = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_1$ and into this cycle we add edges of the cycle $C_9' = v_1, v_3, v_5, v_7, v_9, v_2, v_4, v_6, v_8, v_1$. In this way we obtain a 4-regular component with 9 vertices. The constructed component is the circulant $C_9(1, 2)$.

(b) In a 5-regular graph, each vertex has exactly 5 adjacent vertices. Therefore the smallest possible component is the complete graph on 6 vertices K_6. On 34 vertices, there can be at most $\lfloor \frac{34}{6} \rfloor = 5$ components. We show, that such a graph exists. It suffices to take 4 components K_6 and on the remaining 10 vertices create the fifth component so that into the cycle $C_{10} = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1$ we add edges of the two cycles $C_5' = v_1, v_3, v_5, v_7, v_9, v_1$ and $C_5'' = v_2, v_4, v_6, v_8, v_{10}, v_2$. Next we add five more edges v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10} to obtain a connected 5-regular component with 10 vertices. (The fifth component is the circulant graph $C_{10}(1, 2, 5)$.)

9.3. How many components can a 2-regular graph with 9 vertices have? (Or equivalently, how many non-isomorphic 2-regular graphs on 9 vertices do exist?) Explain!

Because the graph, which we are supposed to find, is 2-regular, each its component has to be a cycle. Since the smallest cycle is C_3 with three vertices, our graphs will have at most three components. A connected (one component) graph exists only one, namely C_9. Graphs with two components are: C_3, C_6 and C_4, C_5. A graph with three components is again unique and it is C_3, C_3, C_3. Overall we have 4 non-isomorphic graphs.

9.4. At most how many edges can a disconnected graph with 23 vertices have?

A disconnected graph has always at least two components. If such a graph is supposed to have maximum number of edges, then

- the number of component is just two. Otherwise an edge could be added between two vertices picked from any two components and the number of components would decrease by 1, yet it remains disconnected;
- each component is a complete graph otherwise it would be possible to add more edges into such component.

If we have two components K_x and K_y (where $x \le y$), then by removing vertex v from the smaller component we remove $x-1$ edges and by adding vertex v to the greater component we add $y > x-1$ edges. Therefore, the disconnected graph on n vertices will have the maximum number of edges if it has two components K_1 and K_{n-1}. For $n = 23$ we get the maximum number of edges $\binom{22}{2} = 11 \cdot 21 = 231$.

9.5. We set up a graph H as follows: The vertices are all two element subsets of the set $\{1,2,3,4,5\}$. The vertex $\{x_1, x_2\}$ is connected by an edge with the vertex $\{y_1, y_2\}$ if at least one of the next conditions is true.

- *Both sums $x_1 + x_2$ and $y_1 + y_2$ are even,*
- *both sums $x_1 + x_2$ and $y_1 + y_2$ are divisible by 3,*
- *both sums $x_1 + x_2$ and $y_1 + y_2$ are equal.*

How many components does graph H have? Sketch H and also separately every its component.

Graph H has 3 components. Two components are K_2 and the third component is two copies of K_4 sharing two vertices.

9.6. How many components can graph G with 7 vertices and with 15 edges have? Justify your answer carefully.

- G can have one component. An example of such a graph is K_7 from which we remove $\binom{7}{2} - 15 = 21 - 15 = 6$ edges so that it remains connected. For inst. we can remove edges of some C_6.
- G can have two components. It must be components K_6 and K_1 since K_6 has 15 edges and any other two components of a graph on 7 vertices have smaller number of edges (as solved in one of the problems in class).
- G cannot have 3 components. With the two components it already has 15 edges in the structure that allows maximum number of edges. By splitting it to more components, we would have to remove more edges.

[1 or 2 components]

9.7. How many components can graph G with 11 vertices of degree 2 have? Justify your answer carefully. Sketch the components.

Since every vertex of the graph G is of degree 2, every vertex must always have two neighbours. Such property only have graphs that are cycles. With 11 vertices we can have:

a) One component, that is the cycle C_{11}.

b) Two components, either C_3 and C_8, or C_4 and C_7, or C_5 and C_6.

c) At most three components. Because the shortest cycle has three vertices and if there were four components, the graph would need to have at least 12 vertices, while each component is the cycle C_3. With 11 vertices we can have for instance two cycles C_3 and one cycle C_5.

[1, 2, or 3 components]

9.8. What is

a) the edge connectivity

b) the vertex connectivity

of graph $K_{4,2}$? Sketch $K_{4,2}$ and carefully explain your answer.

The smallest vertex degree of $K_{4,2}$ is $\delta(K_{4,2}) = 2$. Edge and vertex connectivity is therefore smaller or equal to 2.

a) When removing any one edge the graph does not disconnect and so it is 2-edge connected. When removing two edges incident to a vertex of degree 2 we obtain disconnected graph and so $K_{4,2}$ is not 3-edge connected. Therefore the edge connectivity of $K_{4,2}$ is 2.

b) When removing any one vertex (There are two non-isomorphic options to remove a vertex. First to remove a vertex of degree 2 and second to remove a vertex of degree 4.) the graph does not disconnect and so $K_{4,2}$ is 2-vertex connected. By removing two vertices of the smaller partite set the graph becomes disconnected and so $K_{4,2}$ is not 3-vertex connected. Therefore the vertex connectivity of $K_{4,2}$ is 2.

[Vertex and edge connectivity are both equal to 2.]

9.9. Sketch a graph with at least 7 and at most 10 vertices, that has the degree of edge connectivity 3 and the degree of vertex connectivity 1. Label the vertices of your graph and list the edges and vertices, that you need to exclude to disconnect the graph.

An example of a such graph is for instance two copies of K_4 sharing a vertex. The one vertex to remove is the shared vertex. The three edges to remove are edges adjacent to a vertex different from v.

9.10. Sketch a graph with at least 6 and at most 9 vertices that has edge connectivity 3 and vertex connectivity 2. Label the vertices of your graph and list the edges and vertices, that you need to exclude to disconnect the graph.

An example of a such graph is for instance two copies of K_4 sharing two vertices. The two vertices to remove are the two shared vertices u, v. The three edges to remove are edges adjacent to a vertex different from u and v.

9.11. Sketch an Eulerian graph G with 8 vertices, where $\delta(G) \geq 2$ and $\Delta(G) = 4$. Label the vertices of your graph and give an example of an Eulerian trail.

As a solution can for instance be a graph that is the cycle C_8, with three edges added to conveniently selected three vertices (so, that they are not adjacent in C_8) creating a cycle C_3.

9.12. What is the greatest distance between two vertices in $K_{m,n}$.

Every two vertices x, y from different partite sets are at distance 1. If the size of each partite set is $m = n = 1$, this is the greatest distance in the graph. If at least one partite set has more than one vertex, then every two vertices x, z from the same partite set are at distance at least 2. Because x and z share a common neighbor y in the second partite set, there exist a path x, y, z in $K_{m,n}$, thus the distance between x and z is 2. This is the highest possible distance of two vertices in graph $K_{m,n}$. [distance 1 for $m = n = 1$, otherwise 2]

10 Rooted Trees, Algorithm to Determine Isomorphism of Trees

10.1. Can every finite binary sequence with the same number of 0s and 1s, that starts with 0 and ends with 1, always be the code of a rooted tree. Justify your answer.

Not each sequence starting with 0 and ending with 1 is a valid binary code of a rooted tree. For instance, the sequence starting 011 is not a valid code. It is, because when setting up the code of a rooted tree, we always connect segments that start with zero and end with one. It means that for each 1 there precedes a 0 somewhere in the code to form a pair. Therefore, any starting part of a code cannot contain more 1s than 0s. Such a binary sequence, that in some starting segment contains more 1s than 0s is not a valid code of a rooted tree.

10.2. Are the following binary sequences minimum codes of some rooted trees?

(a) 00001100111001011100111,

(b) 110010000111011010011001,

(c) 000110011111011010011011,

(d) 000111110001010010011011.

Justify your answers!

(a) 00001100111001011100111 is not a minimum code of a rooted tree, since its length is 23 and so it does not contain the same number of 0s as 1s.

(b) 110010000111011010011011, is not a minimum code of a rooted tree, since it starts with 1.

(c) 000110011111011010011011, is not a minimum code of a rooted tree, since the number of 0s is differs from the number of 1s: there are 14 of 1s and only 10 0s.

(d) 000111110001010010011011 is not a minimum code of a rooted tree, even if the number of 0s is the same as the number of 1s. First five 1s are preceded by only three 0s, therefore it is not a valid code of a rooted tree.

10.3. Sketch the rooted tree with the minimum code:

(a) 00001100111001011100111,

(b) 00001011100100111100111.

(a) See Figure 10.1.

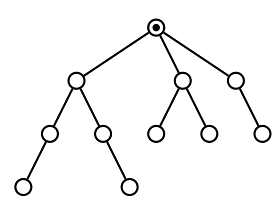

Figure 10.1: *Tree T.*

(b) See Figure 10.2.

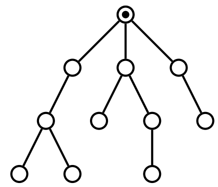

Figure 10.2: *Tree T.*

10.4. Write the minimum code of

(a) the rooted tree (T_1, r_1) on the left in Figure 10.3 ,

(b) the rooted tree (T_2, r_2) on the right in Figure 10.3 .

Figure 10.3: Trees (T_1, r_1) and (T_2, r_2).

(a) The minimum code of rooted tree (T_1, r_1) is 0000011001111001011011.

(b) The minimum code of rooted tree (T_2, r_2) is 000001101101100001110111.

10.5. Determine the minimum codes of the trees T and T' from Figure 10.4 (Set the root in the center!). Use the correct algorithm to decide about their isomorphism.

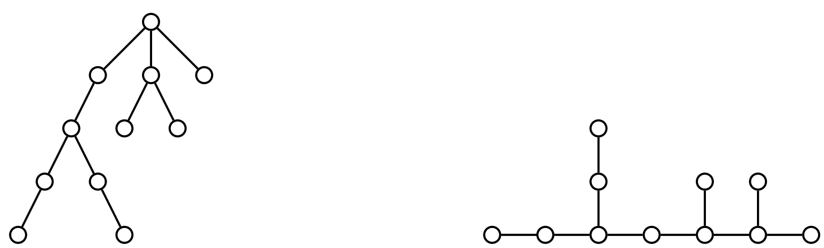

Figure 10.4: Trees T and T'.

First, we check if T and T' have the same number of vertices (This is the first necessary step of the isomorphism algorithm for trees.) Second, we find the centers of trees T and T'. Then we draw T and T' as the rooted trees (T, c) and (T', c') with roots at the centers, see the Figure 10.5.

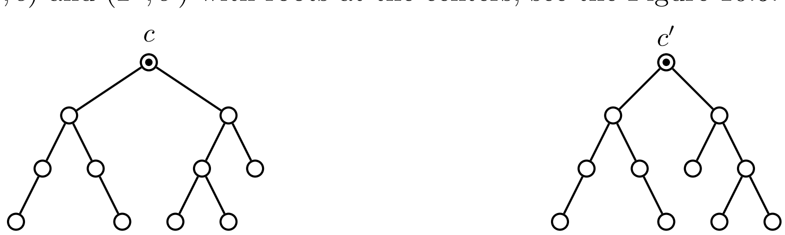

Figure 10.5: *Rooted trees (T, c) and (T', c').*

Next find up the minimum codes. For (T, c) we get 0000101101100011001111 and for (T', c') we get 0000101101100011001111. We observe, that the codes are the same and so T and T' are isomorphic.

10.6. What is the minimum code of a rooted tree (T, r) if:

(a) the rooted tree is a path $T = P_n$ with the root r in one of the end vertices?

(b) the rooted tree is a path $T = P_{2n+1}$ with the root r in the center?

(c) the rooted tree is a path $T = P_{2n}$ with the root r in the center?

(d) the rooted tree T is the graph $K_{1,n}$ with the root r in the center?

(a) The minimum code of the rooted tree (P_n, r), where r is one of the endvertices of P_n is

$$\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{n}$$

(b) The minimum code of the rooted tree (P_{2n+1}, r), where r is in the center of P_{2n+1} is

$$0\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{n}\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{n}1$$

(c) We need to set up the minimum code of the rooted tree (P_{2n}, r), where r is in the center of P_{2n}. Since the center of P_{2n} is an edge, we have to add another vertex between the endvertices of this edge. By this operation we obtain the path P_{2n+1}, the minimum code of which is the same as in the previous case.

$$0\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{n}\underbrace{0\ldots0}_{n}\underbrace{1\ldots1}_{n}1$$

(d) The minimum code of the rooted tree $(K_{1,n}, r)$, where r is in the center of $K_{1,n}$ is

$$0\underbrace{01\ 01\ \ldots\ 01}_{n}1$$

10.7. What is the minimum code of a complete binary tree with 7 vertices? Does there exist such an ordering of vertices of this rooted tree with the root in the center, that its binary code is not minimum? Explain. (In a complete binary tree, each non-leaf vertex has exactly two children and all leafs are at the same distance from the root.)

Because we consider a complete binary tree in which each non-leaf vertex has exactly two children (branches) and all leafs are at the same distance from the root, the code is determined uniquely. The code is 00010110010111.

Because the codes of all children of each parent vertex are identical (leafs have the code 01, their parents have the code 001011 and the root has the code 00010110010111), the ordering of descendant vertices (children in the same generation) does not cause any changes in the code, that is also the minimum code.