Discrete mathematics

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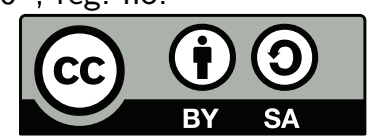
Winter Term 2022/2023
DiM 470-2301/02, 470-2301/04, 470-2301/06

EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research,
Development and Education

MINISTRY OF EDUCATION,
YOUTH AND SPORTS

The translation was co-financed by the European Union and the Ministry of Education, Youth and Sports from the Operational Programme Research, Development and Education, project "Technology for the Future 2.0", reg. no. CZ.02.2.69/0.0/0.0/18_058/0010212.



About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/" (weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Part II Introduction to Graph Theory

Chapter 1. The graph

- motivation
- definition of a graph
- oriented graphs and multigraphs
- degree of a vertex
- subgraphs and isomorphisms
- implementation of graphs

Introduction to Graph Theory

Graph Theory originated rather recently

- L. Euler: Seven Bridges of Königsberg in 1736
- First monograph in 1936

Well known problem solved by Graph Theory:

- four color theorem
- shortest path in a graph
- maximum flow in a network

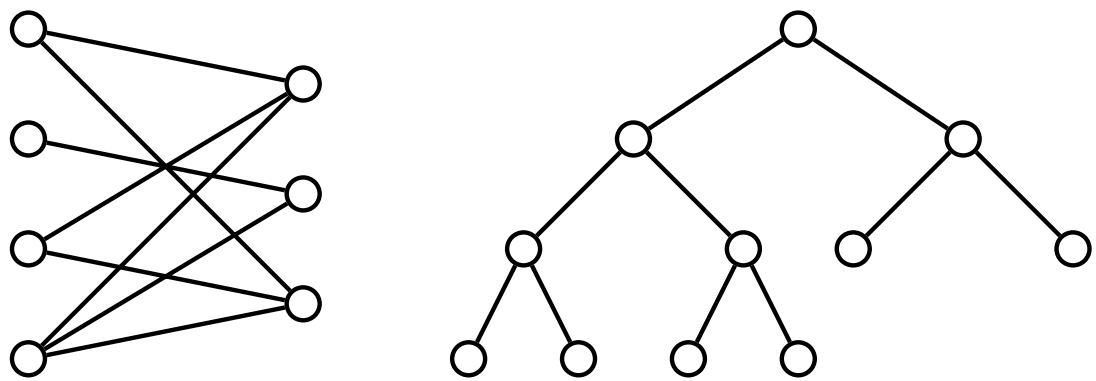
Motivation

Graph Theory is a very important discipline of Discrete Mathematics

- by graphs real life situations can be described easily
- intuitive interpretation
- easily implemented in computers

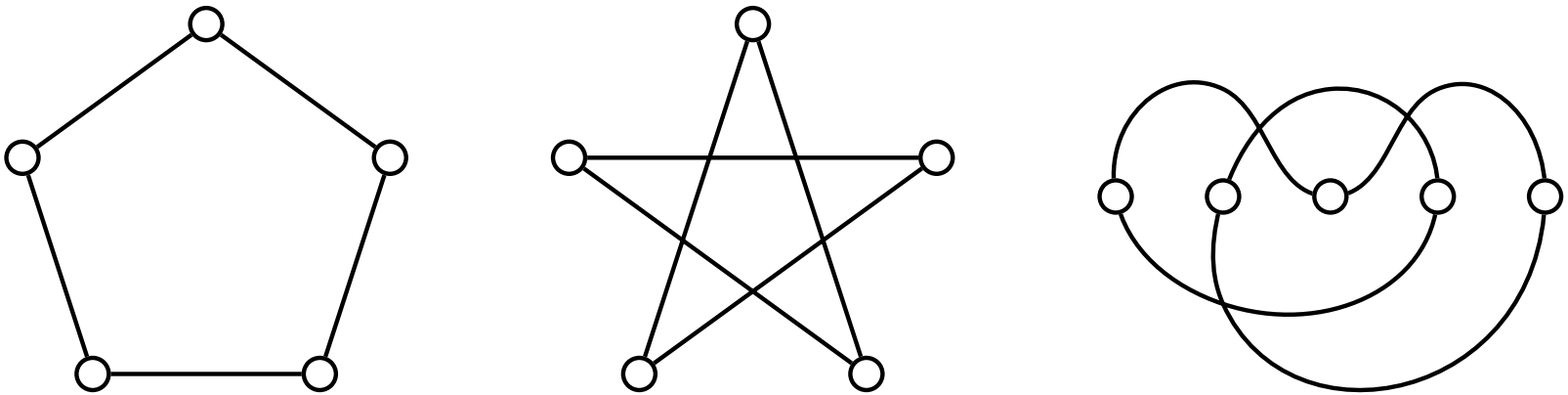
Informally: a graph contains

- vertices (nodes) – "dots" in the figure
- edges – "lines" joining two points in the drawing



Examples of graphs.

Figures of the same graph may be considerably different.



Different drawings of the same graph.

It may not be apparent if two different drawings represent the same graph (the same structure).

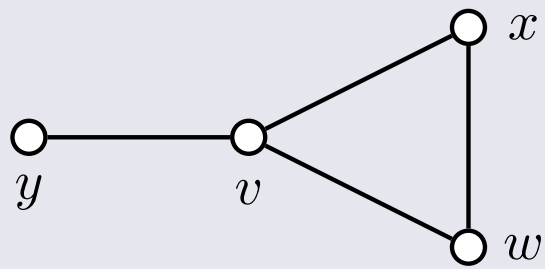
Definition of a graph

Definition

Graph G (*simple* graph) is an ordered pair $G = (V, E)$, where V is a nonempty set of *vertices* and E is the set of *edges* – the set of (some) two-element subsets of V.

Example

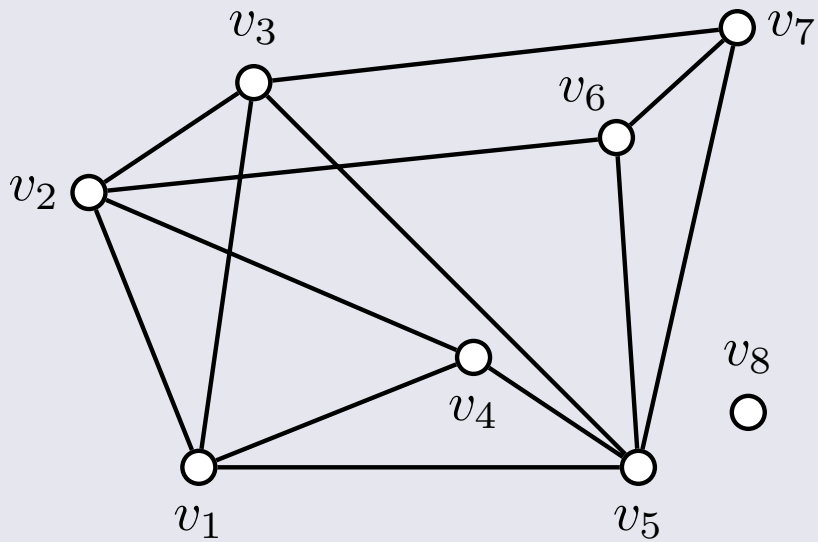
The graph $G = (V, E)$, where $V = \{v, w, x, y\}$ and $E = \big\{\{v, w\}, \{v, x\}, \{v, y\}, \{w, x\}\big\}$ we draw as follows:



The elements of V are called vertices, they are usually denoted by lower case letters u, v, ...
The elements of E are called edges. The edge between vertices u and v is the two-element subset $\{u, v\}$ of V, it is denoted by uv for short.

Example

Graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_6, v_3v_5, v_3v_7, v_4v_5, v_5v_6, v_5v_7, v_6v_7\}$.



If we are given a graph G, by $V(G)$ we understand the set of vertices of the graph G and by $E(G)$ the set of edges of G.

Note

Graph $G = (V, E)$ can be viewed as a special relation E on the set V, where E is *irreflexive* and *symmetric*.

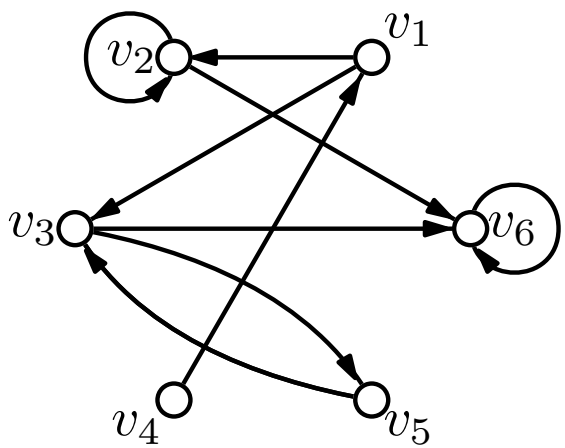
Oriented graphs and multigraphs

If a graph represents a transportation or road network, it is often necessary to impose orientation to edges, called arcs. We distinguish then the first (tail) and the end (head) vertex. The arc uv is not identical to the arc vu.

Definition

An oriented graph is the ordered pair $G = (V, E)$, where V is the set of *vertices* and a nonempty set of *arcs* is $E \subseteq V \times V$.

In a drawing we depict oriented edges by arrows.



An oriented graph.

Note

A simple oriented graph $G = (V, E)$ (without loops) can be considered as an irreflexive relation on a given set V.

Note

An oriented graph $G = (V, E)$ with loops can be considered as a (general) relation on a given set V.

Note

A multigraph is even more general than an oriented graph. Multiple edges, arcs and loops are allowed.

We focus on oriented graphs in the last chapter...

Common graph classes

Graph can be defined by giving

- sets V and E
- drawing
- name and parameter (parameters)

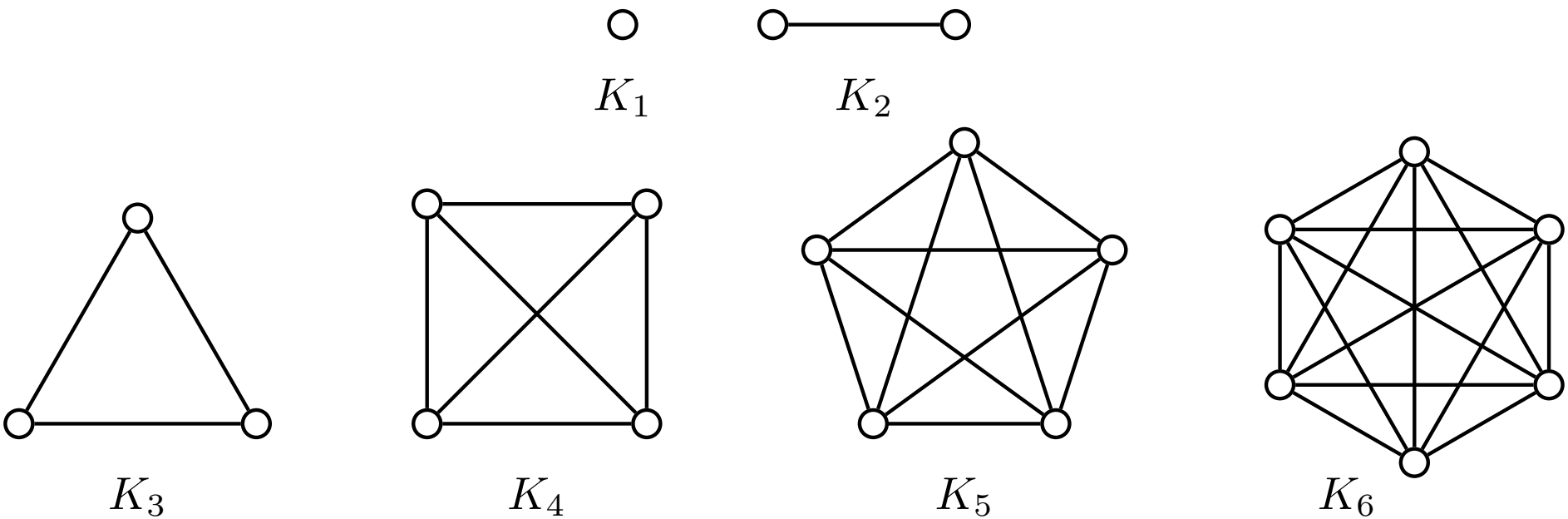
Some graph classes appear often and have their names

- paths
- trees
- caterpillars
- lollipops
- books
- ...

Complete graph K_n

The graph on n vertices ($n \geq 1$), which contains all $\binom{n}{2}$ edges is called complete graph and is denoted by K_n.

$$K_n = (V, E): \quad V = \{1, 2, \ldots, n\}, \quad E = \{ij : i, j = 1, 2, \ldots n \wedge i \neq j\}$$

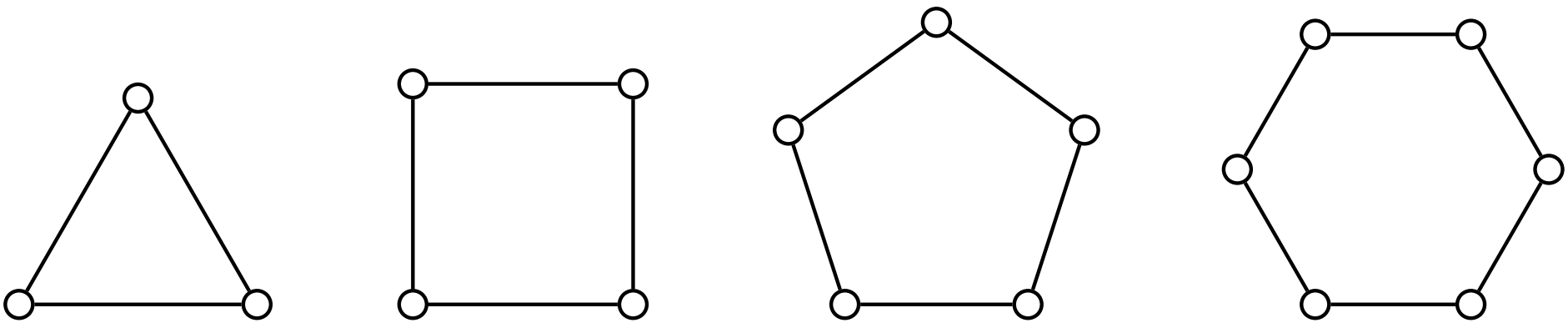


Trivial graph and complete graphs.

Cycle C_n

Graph on n vertices ($n \geq 3$), which are connected by edges into a single cycle is called a cycle on n vertices and denoted by C_n. The number n is the length of the cycle C_n.

$C_n = (V, E)$: $\quad V = \{1, 2, \ldots, n\}, \quad E = \{i(i+1) : i = 1, 2, \ldots n-1\} \cup \{1n\}$



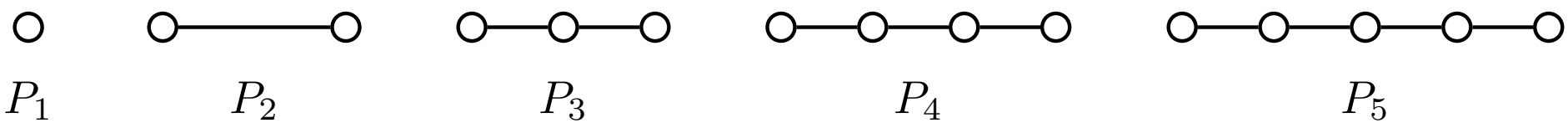
Cycles C_3, C_4, C_5 and C_6.

The cycle of length three is often called a triangle.

Path P_n

The graph on n vertices ($n > 0$), which are connected consecutively by $n-1$ edges is called a path and denoted by P_n.

$$P_n = (V, E): \quad V = \{1, 2, \ldots, n\}, \quad E = \{i(i+1): i = 1, 2, \ldots n-1\}$$



Paths P_1, P_2 P_3, P_4, and P_5.

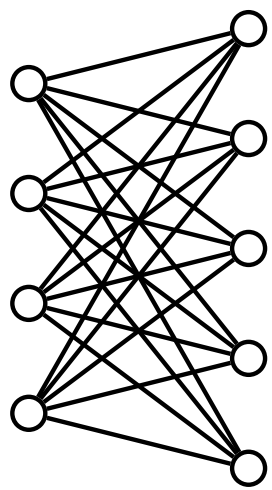
Notice, there is another notation (less common):
index = number of edges

Complete bipartite graph $K_{m,n}$

A graph, whose vertex set is partitioned into two disjoined nonempty subsets M and N ($|M| = m \geq 1$, $|N| = n \geq 1$) and which contains all $m \cdot n$ edges uv such that $u \in M$ and $v \in N$ is called complete bipartite graph and is denoted by $K_{m,n}$.

$$K_{m,n} = (M \cup N, E): \quad M = \{u_1, u_2, \ldots, u_m\}, \quad N = \{v_1, v_2, \ldots, v_n\},$$

$$M, N \neq \emptyset, \quad M \cap N = \emptyset, \quad E = \{u_i v_j : i = 1, 2, \ldots, m \wedge j = 1, 2, \ldots n\}.$$



Graph $K_{4,5}$.

Degree of a vertex in a graph

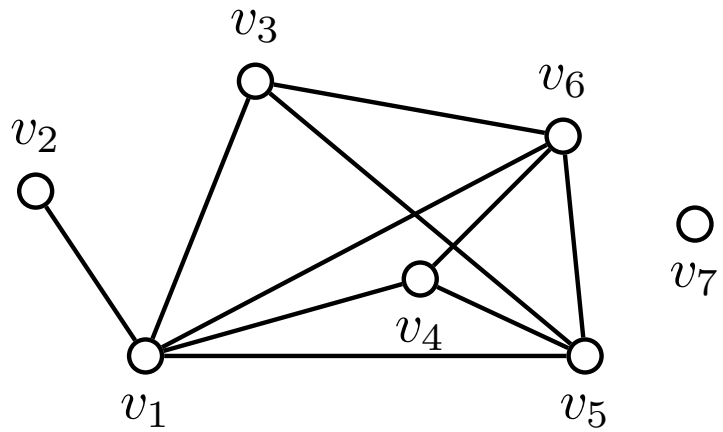
For a given edge uv, we call the vertices u and v the end-vertices of the edge uv.

We also say that u and v are incident with the edge uv ($u \in \{u, v\}$).

Two different end-vertices of the same edge we call neighbors. If such edge does not exist, we call the vertices independent.

Definition

Degree of a vertex in a graph G is the number of edges which are incident with the given vertex v.
The degree of a vertex v in a graph G is denoted by $\deg(v)$ (or $\deg_G(v)$).



Graph with degrees 5, 1, 3, 3, 4, 4, and 0.

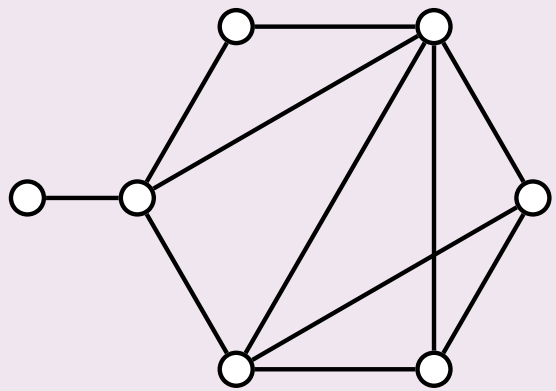
Theorem (Parity Principle)

The sum of all degrees in a graph is even and is equal to the double of edges.

$$\sum_{v \in V(G)} deg_G(v) = 2|E(G)|$$

Proof Both, the left and right side of the equation gives the count of end-vertices. Summing degrees every end-vertex contributes a 1 to a degree of some vertex. Moreover, every edge has two end-vertices, hence the sum of degrees is equal to the double of number of edges. □

Question



How many edges are in a graph with vertex degrees $5, 4, 4, 3, 3, 2, 1$*?*

Example

How many edges are in a graph G which has thirty vertices of degree 5 and five vertices of degree 4?

By Parity Principle the double of edges equals to the sum of vertex degrees.

$$\sum_{v \in V(G)} \deg_G(v) = 30 \cdot 5 + 5 \cdot 4 = 150 + 20 = 170$$

Thus, the graph G has $170/2 = 85$ edges. Does G exist? We show later.

Example

How many edges are in a graph G which has five vertices of degree 5 and thirty vertices of degree 4?

Similarly...

$$\sum_{v \in V(G)} \deg_G(v) = 5 \cdot 5 + 30 \cdot 4 = 145$$

By Parity Principle no such graph exists!

Example

Nine friends exchange Christmas presents. Everyone gave three presents to his friends. Show, that it is not possible that everyone receives presents from precisely those friends he gave his presents to. Hint: show that no graph model of such exchange can exist.

We set up a graph: vertices=friends, edges=pairs of exchanged presents.

$$\sum_{v \in V(G)} \deg_G(v) = 9 \cdot 3 = 27$$

By Parity Principle no such graph exists, thus there is no solution to the problem.

There is no exchange of presents possible such that each of the nine friends would receive presents from precisely those friends he/she gave his/her presents to.

Definition

Take a graph G with vertices $v_1, v_2, \ldots, v_n$. The sequence $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is called the degree sequence of the graph G.

Not every sequence is a degree sequence of a graph. How to tell?

Theorem (Havel–Hakimi)

Let $(d_1 \geq d_2 \geq \cdots \geq d_n)$ be a sequence of natural numbers. Then a non-trivial graph on n vertices with the degree sequence

$$D = (d_1, d_2, \ldots, d_n)$$

exists if and only if there exists a graph on $n-1$ vertices with the degree sequence

$$D' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n).$$

Since we deal only with finite graphs, we can by recursion decide about every finite sequence if it is a degree sequence of some graph. Moreover we can construct example of such a graph (needs not to be unique!).

Example

Does there exist a graph with the degree sequence $(3, 3, 3, 1, 1)$?

No, by Parity Principle.

Example

Does there exist a graph with the degree sequence $(3, 3, 3, 1)$?

The Parity Principle? No such graph exists, we use Theorem H-H to prove it.

Examining the degree sequence $(3, 3, 3, 1)$ we get

$$(3, 3, 3, 1) \overset{HH}{\sim} (2, 2, 0) \overset{HH}{\sim} (1, -1)$$

which obviously is not a degree sequence.

Example

Does there exist a graph with the degree sequence $(6, 4, 4, 1, 1)$?

Obviously, no such graph exists.

If you do not know why, try using Theorem H-H.

Example

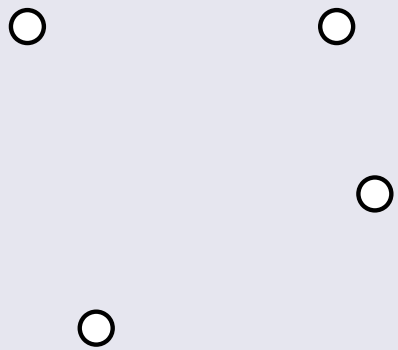
Does there exist a graph with the degree sequence $(6, 5, 4, 3, 3, 2, 1)$?

Yes, by Theorem H-H. Moreover, we construct such a graph.

Based on the degree sequence $(6, 5, 4, 3, 3, 2, 1)$ we get

$$(6, 5, 4, 3, 3, 2, 1) \overset{HH}{\sim} (4, 3, 2, 2, 1, 0) \overset{HH}{\sim} (2, 1, 1, 0, 0) \overset{HH}{\sim} (0, 0, 0, 0).$$

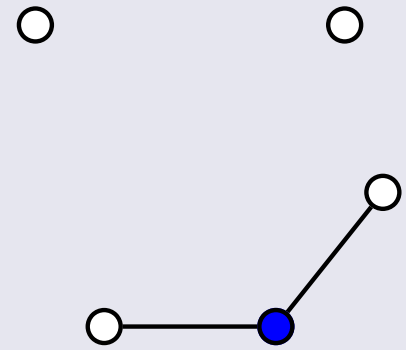
To construct the graph we add vertices.



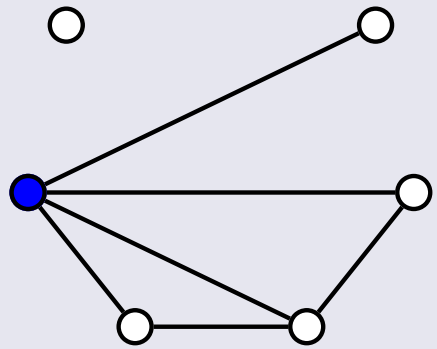
We start by drawing a graph with the degree sequence $(0, 0, 0, 0)$.

continued . . .

Then we add a vertex of degree 2 and connect it to two vertices of degree 0 (hereby we obtain vertices with degree 1).



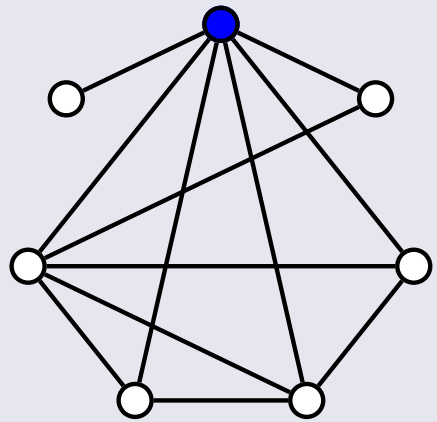
We obtain a graph with the degree sequence $(2, 1, 1, 0, 0)$.



To obtain a graph with the degree sequence $(4, 3, 2, 2, 1, 0)$, we add a vertex of degree 4 and connect it to vertices of degree 2, 1, 1 and 0.

continued . . .

Finally we add a vertex of degree 6 and connect it to all remaining vertices.



We get a graph with the degree sequence $(6, 5, 4, 3, 3, 2, 1)$.

Summary: using Havel-Hakimi Theorem we can

- decide, whether a given finite sequence of positive integers is a degree sequence of some graph,
- if yeas, the process yields an example of such a grah.

Definition

The highest degree of a vertex in a graph G we denote by $\Delta(G)$.
The smallest degree we denote by $\delta(G)$.

Questions

- Is every non-increasing sequence of natural numbers a degree sequence?
- How to tell if a particular sequence is a degree sequence of a graph?
- Is the graph uniquely determined by its degree sequence?
- How many graphs do exist with a particular degree sequence?

Subgraphs

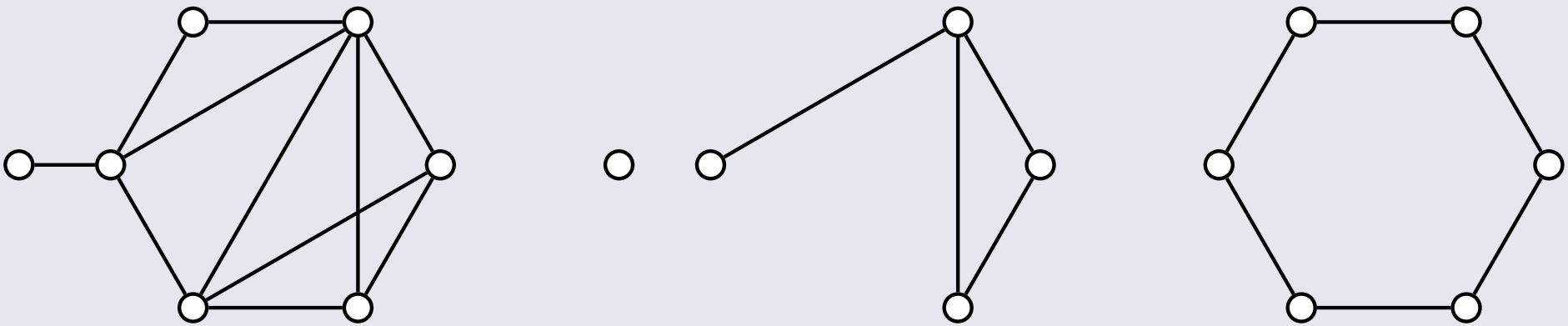
Often we consider graphs, that arose by deleting some vertices (and all incident edges) from a graph, or by deleting some edges or both. In this way we obtain a *subgraph*.

Definition

Graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$.

Notice: by deleting a vertex of G we have to delete all incident edges as well. The definition is correct, otherwise H would not be a graph.

Example

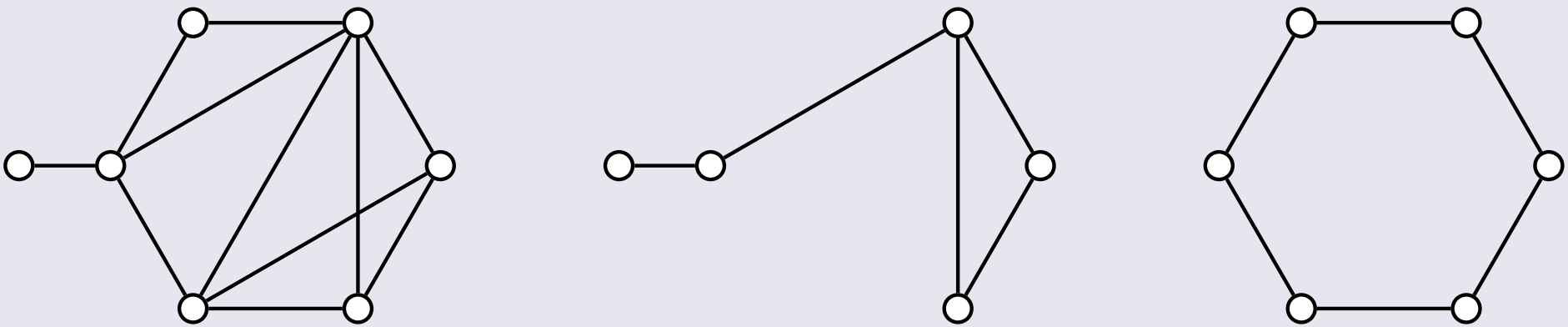


Graph G and its subgraphs H_1 and H_2.

Induced subgraph

A subgraph I of a graph G is called induced subgraph of G if $E(I)$ contains all edges of G, that are incident with vertices in $V(I)$ (we omit no other edges besides those incident with the deleted vertices).

Example



Graph G and subgraph H_1 (induced) and H_2 (not induced).

Factor of a graph

A subgraph F of a graph G is called a factor of G if $V(F) = V(G)$.

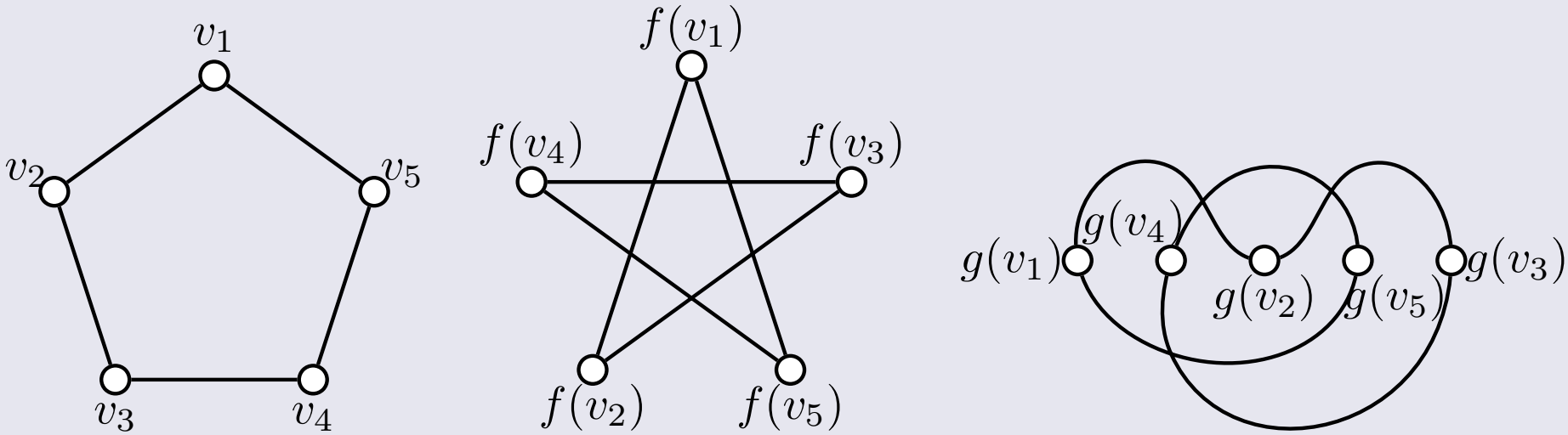
Graph isomorphisms

Definition

Isomorphism of graphs G and H is a bijective mapping $f : V(G) \to V(H)$, for which the following holds: every pair of vertices $u, v \in V(G)$ is joined by an edge in G if and only if $f(u), f(v)$ are joined by an edge in H.

For short $\forall u, v \in V(G) : uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$.

Example



Isomorphic graphs.

Fact

Isomorphic graphs G and H have

- the same number of vertices
- the same number of edges
- the same highest degree $\Delta(G) = \Delta(H)$
- the same lowest degree $\delta(G) = \delta(H)$
- the same degree sequence
- each subgraph of G has to be a subgraph of H and vice versa
- ...

If some bijection f is an isomorphism, it has to map vertices to vertices of the same degree, e.g.

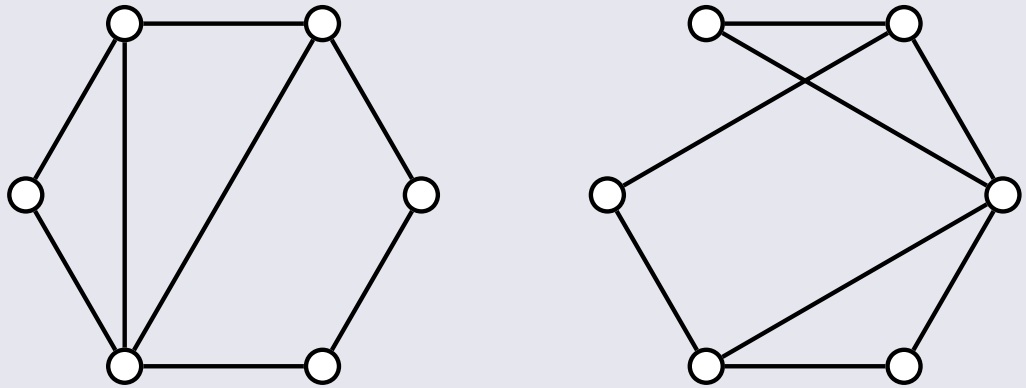
$$\deg_G(v) = \deg_H(f(v)).$$

This implication cannot be reversed!

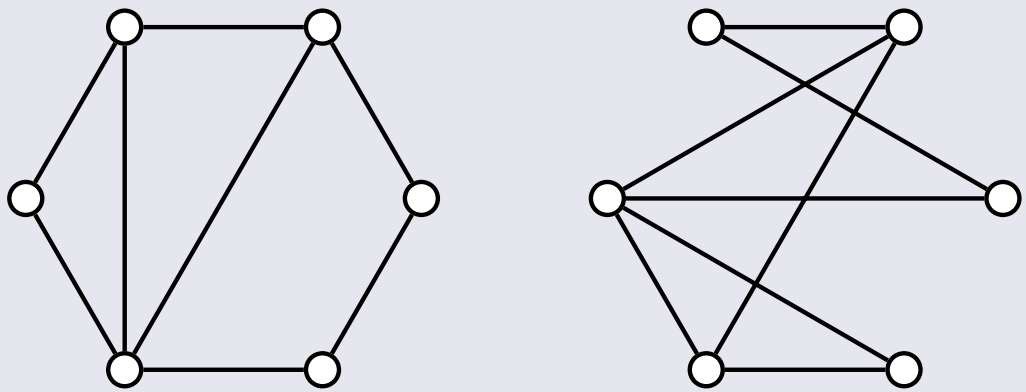
It is not enough to compare degree sequences!

Example

Are the following two graphs both with the degree sequence $(4, 3, 3, 2, 2, 2)$ isomorphic?

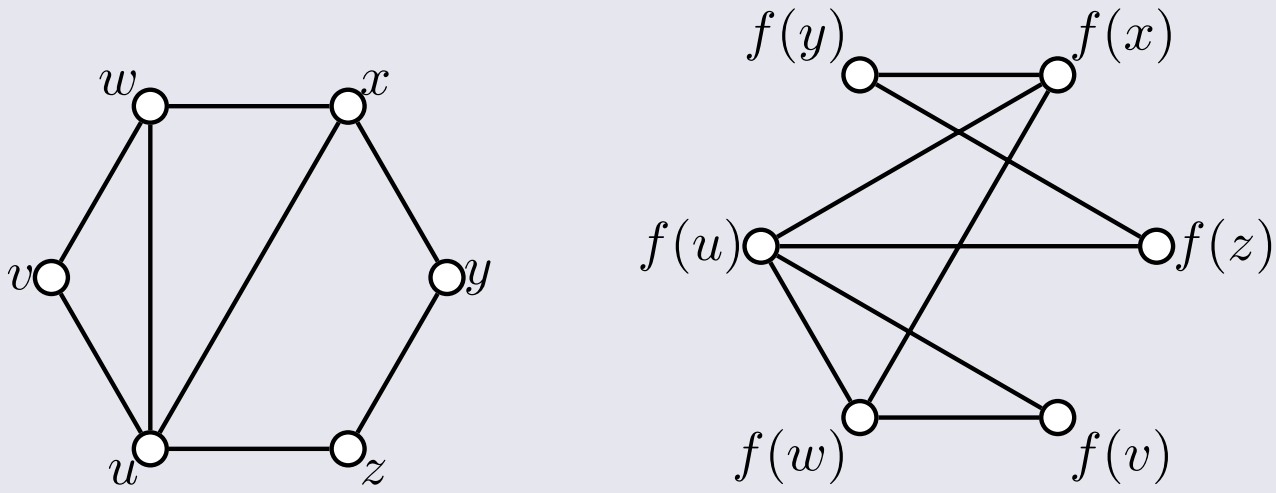


Are these two graphs isomorphic?



Are these two graphs isomorphic?

continued...



Isomorphism f between two given graphs.

Note

On the examples above we have practised some approaches used to determine, whether two given graphs are isomorphic or not.

No "fast and easy" universal algorithm for decision about being isomorphic is known!

The only universal algorithm for finding an isomorphism of two graphs or for deciding that such isomorphism does not exists is to *try all* feasible bijections between vertices (there can be up to $n!$ of them).

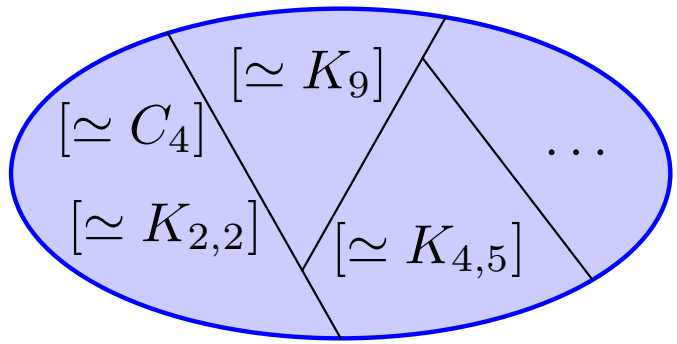
Theorem

Relation of "being isomorphic" $\simeq$ is an equivalence relation on the set of all graphs.

Proof Relation $\simeq$ is

- reflexive, since every graph is isomorphic to itself by identity,
- symmetric, since to every isomorphism (bijection) f there exists (a unique) f^{-1},
- transitive, since by composition of isomorphisms (mappings) we get again an isomorphism (mapping). □

When talking about a certain *graph*, we often address the entire isomorphism class, is does not depend on a particular representation, drawing or notation of a given graph.



Graph classes.

Implementation of graphs

We denote vertices of a graph G by natural numbers $0, 1, \ldots, n-1$.
Among the most common computer implementations of graphs are:

Adjacency matrix

Adjacency matrix of a graph G is a square matrix $A = (a_{ij})$ of order n, in which

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

It can be stored in a two-dimensional field `a[][]` where `a[i][j]=1` if the edge/arc ij is in the graph, otherwise `a[i][j]=0`.

The sum of the i-th row/column of the matrix equals the degree of vertex i.

Advantages:

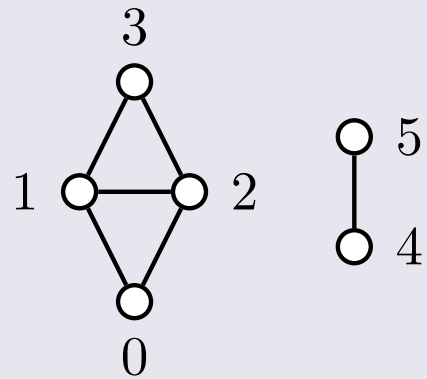
- quick to check the existence of and edge/arc uv,
- easy to add/remove an edge/arc ij.

Disadvantages:

- adjacency matrix is large, uses memory even for sparse graphs.

Example

Set up the adjacency matrix of the given graph.



The adjacency matrix is

$$A = \begin{array}{c} v\backslash v \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Incidence matrix

Incidence matrix B of a graph G is a rectangular matrix with n rows and $m = |E(G)|$ columns. Each row corresponds to a given vertex of the matrix B and each column corresponds to a column of B.

Element b_{ij} of the matrix B equals 1, if vertex i is incident to the edge e_j, otherwise we set $b_{ij} = 0$.

Sum of every column is 2 and the sum of row i equals the degree of vertex i.

Advantages:

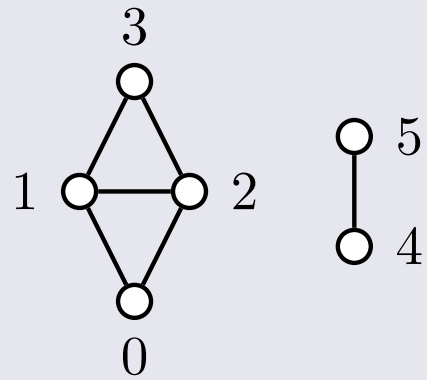
- easy to find end-vertices of a given edge,
- easy to "reattach" edge ij in a graph.

Disadvantages:

- expensive to add/remove edges to the incidence matrix,
- for large graphs is the incidence matrix large and sparse.

Example

Set up the incidence matrix of the given graph.



The incidence matrix is

$$B = \begin{array}{c} v\backslash e \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 01 & 02 & 12 & 13 & 23 & 45 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Neighbor lists

For every vertex $i = 0, 1, \ldots, n-1$ of a given graph G we create an array (list) of vertices, say `neig[i][]`, which contains vertices adjacent to i.

Each array will have $\deg(i)$ entries, where $\deg(i)$ is the degree of vertex i, stored in `deg[]`.

Elements `neig[i][0], neig[i][1], ..., neig[i][deg[i]-1]` contain vertices (or their indices) adjacent to vertex i.

Example

`deg[] = [2, 3, 3, 2, 1, 1],`

$$
\begin{aligned}
\texttt{neig[0][]} &= [1, 2], \\
\texttt{neig[1][]} &= [0, 2, 3], \\
\texttt{neig[2][]} &= [0, 1, 3], \\
\texttt{neig[3][]} &= [1, 2], \\
\texttt{neig[4][]} &= [5], \\
\texttt{neig[5][]} &= [4].
\end{aligned}
$$

We have to keep the symmetry of edges ij and ji in mind!

Advantages:

- suitable for graphs with few edges, easy to evaluate $\deg(v)$,

Disadvantages:

- costly to alter the structure of a given graph,
- costly to find a given edge.

All given representations can be used for oriented graphs as well.
In general, for multigraphs one can use the incidence matrix or perhaps neighbor list.

More complex structures are needed for labeled (weighted) graphs (we assign labels/weights to vertices and/or edges).

We can use additional fields or structured data types...

Next lecture

Chapter 2. Connectivity of graphs

- motivation
- connectivity and components of a graph
- searching through a graph
- higher degrees of connectivity
- Eulerian graphs traversable in "one trail"