Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/"(weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Chapter 2. Arrangements and selections

- selections: permutations, k-permutations and k-combinations
- two basic counting principles
  multiplication principle (of independent selections)
  method of double counting
- permutations, k-permutations and k-combinations with repetition

Arrangements and selections

We count the number of selections from a given set

- ordered arrangements / unordered selections,
- with repetition / without repeating elements.

Today:

- permutations (without repetition)
- k-combinations (without repetition)
- k-permutations (without repetition)
- $+$ problems leading to counting such selections
- independent and not independent selections
- permutations (with repetition)
- k-combinations (with repetition)
- k-permutations (with repetition)
- $+$ problems leading to counting such selections

Beware! While solving real life problems we usually need to split a complex problem into several sub-cases,

- complex selections/arrangements, during discussions

we have to distinguish common/different properties.

Definition

Permutation of an n-element set X is an *(ordered) arrangement of all n elements* from X (without repetition).

The total number of possible permutations of an n-element set is

$$P(n) = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

- the first element is chosen among n possibilities
- the second element is chosen among $n-1$ possibilities
- the third element is chosen among $n-2$ possibilities...

Problems, described by permutations (without repetition)

- number of orderings of elements from a set
- number of bijections from an n-element set onto another n-element set
- number of ways how to order the cards in a deck
- distribution of numbers at a start of a marathon
- distributing keys in a fully occupied hotel

Definition

Combination (or k-combination) from a set X is an *(unordered) selection of k distinct elements* from a given set X (a k-element subset of X.)

The number of k-combinations from an n-element set

$$C(n,k) = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$$

- $n!$ different orderings (permutations) of X
- we choose first k elements (not distinguishing their $k!$ orderings)
- we discard the last $n-k$ elements (not distinguishing their $(n-k)!$ orderings)

Problems, described by combinations (without repetition)

- number of k-element subsets of an n-element set
- binomial coefficients: coefficient at x^k in $(x+1)^n$

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Definition

k-permutation from a set X is an *ordered arrangement of k elements from an n-element set* X (without repetition)
(sequence of k-elements from X).

The number of k-permutations from an n-element set

$$V(n,k) = n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

- the first element is chosen among n possibilities
- the second element is chosen among $n-1$ possibilities
- ...
- the k-th element is chosen among $n-k+1$ possibilities.

or

- $n!$ possibilities how to order elements of X
- we take only first k elements
- we discard the last $n-k$ elements (not distinguishing their $(n-k)!$ orderings)

Problems, described by k-permutations (without repetition)

- setting up an k-element sequence from n elements
- number of injections (one-to-one mappings) from an k-element set to an n-element set
- number of different race outcomes (trio on a winner's podium)
- distributing keys in a partially occupied hotel

Examples

- team of four among ten employees
  calculation using *k-combinations* (we do not distinguish ordering)

$$C(10,4)=\binom{10}{4}=\frac{10!}{4!\cdot 6!}=\frac{10\cdot \cancel{9}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}}=\frac{10\cdot 3\cdot 7}{1}=210$$

- number of matches in a tennis tournament of seven players
  calculation using *k-combinations* (2-element subsets in a 7-element set)

$$C(7,2)=\binom{7}{2}=21$$

Examples

- number of possible orders after a tournament of seven players
  calculation based on *permutations*

$$P(7) = 7! = 5040$$

- number of triples on the winners podium in a tournament of seven
  calculation by *3-permutations*, because “the order does matter”

$$V(7,3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$$

Complex selections and arrangements

In some cases we add and in some cases we multiply the number of selections or arrangements to obtain the result. How to recognize which is correct?

Sum rule

Suppose there are n_1 selections (arrangements) obtained in one way and n_2 selections (arrangements) obtained in another way, where no selection (arrangement) can be obtained in both ways, then the total number of selections (arrangements) is $n_1 + n_2$.
"EITHER n_1 ways OR n_2 further ways."

Product rule

Suppose a selection (arrangement) can be broken into a sequence of two selections (arrangements). If the first stage can be obtained in n_1 ways and the second stage can be obtained in n_2 ways for each way (independently) of the first stage, then the total number of selections (arrangements) is $n_1 \cdot n_2$.
"First n_1 ways AND then n_2 ways."

If a selection is broken into two disjoint sets of selections, then we add the number of selections.

Example

In the game „člověče nezlob se“ we roll an ordinary dice and move a peg by the indicated number of fields. If we roll a 6 in the first roll, we roll an additional time. By how many fields can we move the peg in one round?

We distinguish two cases:

- if there is not a 6 in the first roll, we move by 1 up to 5 fields,
- if there is a 6 in the first roll, we move by $6+1$ up to $6+6$ fields.

There are 11 possibilities: 1, 2, 3, 4, 5, (no 6!) 7, 8, 9, 10, 11, 12.

If a selection can be broken into two stages (subselections), then we multiply the number of selections.

Example

The coach of a hockey team sets up a formation (three forwards, two full-backs and a goalkeeper). He has a team of 12 forwards, 8 full-backs, and two goalkeepers.
How many different formations can he set up?

Because there is no relation between the choice of full-backs, forwards, and goalkeepers we can count as follows

$$\binom{12}{3} \cdot \binom{8}{2} \cdot \binom{2}{1} = \frac{12 \cdot 11 \cdot 10}{6} \cdot \frac{8 \cdot 7}{2} \cdot 2 = 220 \cdot 28 \cdot 2 = 12320.$$

There are altogether 12 320 different formations.

When two selections are not independent...

we cannot just multiply the counts of each (sub)selection.

Example

The coach of a hockey team sets up a formation (three forwards, two full-backs and a goalkeeper). He has a team of 11 forwards, 8 full-backs, 1 universal player (either a full-bak or a forward), and two goalkeepers. How many different formations can he set up?

- choose 3 forwards: $\binom{12}{3}$
- choose 2 full-backs: $\binom{8}{2}$ or $\binom{9}{2}$?

It depends, whether the universal player was picked as forward or not.

... solution in the discussion

Question

We roll a dice three times. How many rolls are possible, such that every subsequent roll gives a higher number than the previous one?

Double counting

Suppose each arrangement can be further split into several *finer* ℓ *arrangements*. Moreover, suppose we know how to count the total number of the refined arrangements m. Then the total number of the original arrangements is *given by the ratio* m/ℓ.

Example

We have the characters T, Y, P, I, C. How many different (even meaningless) five-letter words can you construct? We do not distinguish Y and I letters.

If we distinguish all characters, we have $P(5) = 5! = 120$ words.
Not distinguishing Y, I: $\quad TYPIC = TIPYC.$

In the total of $m = 120$ we have every arrangement counted twice $\ell = 2$. The number of different words is

$$\frac{m}{\ell} = \frac{120}{2} = 60.$$

Arrangements with repetition

So far no repetition of selected elements was allowed
(people, subsets, ...)
In several problems repetition is expected (rolling dice, characters, ...)

Example

How many anagrams of the word MISSISSIPPI exist?
(*anagram* is a word obtained by rearranging all characters of a given word)

If no character in "MISSISSIPPI" would repeat, the calculation would rely on permutation. But they repeat: S 4times, I 4times, P 2times.

By double counting:

1. first we distinguish all characters (using colors, indices, etc.)
2. we count all arrangements: $(4+4+2+1)!$
3. divide by the number of indistinguishable arrangements: $4! \cdot 4! \cdot 2! \cdot 1!$

$$\frac{(4+4+2+1)!}{4! \cdot 4! \cdot 2! \cdot 1!} = \frac{11 \cdot 10 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{2}} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34650$$

Definition

Permutation with repetition from the set X is an *arrangement* of elements from X in a sequence such that every element from X occurs **a given number of times**. Denote the number of them by $P^*(m_1, m_2, \ldots, m_k)$.

(an arrangement with a given number of copies of elements from X)

The number of all permutations with repetition from a k-element set, where the i-th element is repeated in m_i identical copies $(i = 1, 2, \ldots, k)$:

$$P^*(m_1, m_2, \ldots, m_k) = \frac{(m_1 + m_2 + \cdots + m_k)!}{m_1! \cdot m_2! \cdots m_k!}.$$

Examples

- permutation with repetition of 2 elements, one element occurs in k copies the other in $(n-k)$ copies

$$\frac{(k+n-k)!}{k! \cdot (n-k)!} = \binom{n}{k} = C(n,k).$$

  first element = "is", the second element = "is not" an arrangement
- permutation of multisets (in multisets identical copies are allowed)

Example describing the idea of combination with repetition

Example

How man ways are there to select 6 balls of three colors, provided we have an unlimited supply of balls of each color?

We present a beautiful trick, how to count the total number of selections.
Suppose we pick $\bullet, \bullet, \bullet, \bullet, \bullet, \bullet$
This selection we *can order* (group) based on colors

$$\bullet\ \bullet\ \bullet | \bullet | \bullet\ \bullet$$

now we observe, that only the “bars”, not the colors are important

$$\bullet\ \bullet\ \bullet | \bullet | \bullet\ \bullet$$

The total number of selections is

$$C^*(3,6) = \binom{6+2}{2} = \binom{8}{2} = 28.$$

Definition

A *k-combination with repetition* from an n-element set X is a **selection** of k elements from X, while each element can occur in **an arbitrary number of identical copies**. The number of them we denote by $C^*(n,k)$.

The total number of all k-element selections with repetition from n possibilities is

$$C^*(n,k) = \binom{k+n-1}{n-1}.$$

- having n "colors", we need $n-1$ bars
- we can "select bars", or "select elements"

$$C^*(n,k) = \binom{k+n-1}{n-1} = \binom{k+n-1}{k}.$$

Problems solved using k-combinations with repetition

- number of ways how to write k using n nonnegative integer summands
- drawing k elements of n kinds provided after each draw we return the elements back to the polling urn

Example

How many ways are there to write k as the sum of n nonnegative integer summands? We distinguish the order of summands!

We have

$$k = x_1 + x_2 + \cdots + x_n.$$

We will select (draw) k ones and distribute them into n boxes (with the possibility of tossing more ones into each box).

- some boxes can remain empty ($0 \in \mathbb{N}_0$)
- we can toss all ones into one box
- we repeat boxes, not ones! (a different problem)

Questions

How many ways are there to write k as the sum of n positive summands?

How many ways are there to write k as the sum of at least n natural summands?

How many ways are there to write k as the sum of at most n natural summands?

Definition

A k-permutation with repetition from an n-element set X is an **arrangement** of k elements from X, while elements **can repeat** in an arbitrary number of identical copies. The number of them we denote by $V^*(n,k)$.

The arrangement is a sequence.

The number of all k-permutations with repetition from n possibilities is

$$V^*(n,k) = \underbrace{n \cdot n \cdots n}_{k} = n^k.$$

Problems solved by k-permutation with repetition

- number of mappings of an k-element set to an n-element set
- cardinality of the Cartesian power $|A^k|$

Question

How many odd-sized subsets has a given set of n elements?

Next lecture

Chapter 3. Discrete probability

- motivation
- sample space
- independent events