SILESIAN UNIVERSITY IN OPAVA
Faculty of Philosophy and Science

Master's degree thesis

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Historical notes on the inverse problem of the calculus of variations

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Opava 2002

Contents

1 Introduction

The inverse problem of the calculus of variations concerns a decision whether a system of differential equations is variational, i.e., if it can be expressed in the form of Euler–Lagrange equations for some Lagrangian. There are two questions connected to the inverse problem: first, how to construct a Lagrangian for variational differential equations and second, if it is possible to reduce the order of this Lagrangian.

In this thesis the history of deriving of the variationality conditions for ordinary differential equations of an arbitrary order (so-called higher-order Helmholtz conditions) is studied. The attention is turned mainly to the works originated at the end of the 19th century. Following advancement is left aside here; there are two nice review treatises available, the first by Santilli [21] and the second by Morandi et al. [19].

There are several motivations to this problem. First, even the authors studying deeply the inverse problem do not know how the problem was tackled by the "classics" one century ago. The quotations have been largely formal and the commentaries indistinct, authors confine themselves to a more or less general cliché. One reason is that there has been no comprehensive paper recounting the old history of the inverse problem and the historical remarks in the review papers have brought only a little light to the problem by then.

Second, the sources are not easily available to the broad scientific community; the digitalization of the sources has been going on very slowly, it is a very expensive and difficult objective. There has been some progress in the last years, however only in the field of the scientific papers. Still there hasn't been any attempt to digitalize other important sources such as correspondence.

Third, even if there will be all works digitalized, there would be another great difficulty, the language gap. Most of the papers written in the 19th century are in German, French, Italian or even Latin and the knowledge of these languages is not common at the time being and the situation is getting even worse.

Fourth, studying the history of mathematics is not common and popular among the mathematicians (e.g. in [13] there are only listed references with no commentary). However, there are some exceptions; the best account of the history of the inverse problem of the calculus of variations was written by Santilli in 1978, [21]. He fixed the attention mainly on the works since Douglas (1941) but in his book there were the most important remarks on the considered problem so far. Twelve years later, in the comprehensive paper [19], Morandi, Ferrario, Lo Vecchio, Marmo and Rubano discussed the progress of the research in the seventies and eighties of the last century with no respect to the previous development.

Last but not least, though there is a more or less accurate literature dealing with the history of the inverse problem in the 20th century, inaccurate commentaries on *old history* of the inverse problem and consequent inaccurate quoting of these commentaries have brought about a broad confusion. That is why I posed myself for one of the main goals to correct the inappropriate opinions originated from the authors during the 20th century.

These are the main reasons why I restrict myself to the old history of the inverse problem in this thesis.

However, there is another and a more general setting of the inverse problem of the calculus of variations, a so-called multiplier problem. This case is not systematically considered in this thesis, but there are some remarks to be made; these are collected in the Appendix.

In this thesis it is shown that exploring the history of mathematics is not waste of time. For example, the equivalent form of the Helmholtz conditions, which was cited many times as result of Mayer, Davis or others, was originally derived by Helmholtz himself. However, in my opinion the most interesting result is that the so-called higher-order Helmholtz conditions, which were believed to be a new result of the late 1970s, were already known in 1899! First they appeared in the work of Karl Böhm.

1.1 Sources and literature

The main sources, the scientific papers of the "classics" have not been published in English translation yet. However, some of them are easily available from the electronic libraries. But the rest of them is the more difficult to be obtained. However, the situation is worse in the Czech Republic than, e.g., in Germany. In the enclosures the reader can find either the copy of an article or a link to a full-text transcription.

In the last year there has been a considerable progress in digitalizing the sources for the history of Mathematics in the 19th century. The European Mathematical Society project of digitalization of *Jahrbücher über die Fortschritte der Mathematik*, hereafter JFM in short, see `www.emis.de/projects/JFM`, is a searchable database of all reviews published in JFM from 1868 till 1942. This is the most useful source for the first searching for an article and also for further research. In June 2002 the digitalization was complete for the years 1868–1911. It contains also links to 12 196 full-text article and book facsimiles.

However, in the other EMS project called *Classical Works, Selecta, and Opera Omnia* only the works of Riemann and Hamilton have been digitalized so far.

From the other electronic sources let me make a note on *The MacTutor History of Mathematics archive*, see `www-history.mcs.st-andrews.ac.uk/history/`, by John O'Connor and Edmund Robertson from the University of St. Andrews, the biographical index of Mathematicians. The profiles of the more than 1 500 famous mathematicians are relatively short but there is a list of references for each personality. Unfortunately, there are many important things missing, such as a list of papers which the person authored, links to electronic transcriptions of the papers, etc.

A considerable part of mathematicians responsible for the rise of the inverse problem are connected with Heidelberg (Helmholtz, Königsberger, Mayer and Böhm). In the virtual library of the *Ruprecht–Karls University in Heidelberg* (for the link see [35]) there are biographies of these personalities with a lot of excellent additional information.

There is one rule holding: the less famous the person is, the more difficult it is to find any information about him. For example, there are several biographies of Helmholtz in various languages, but there is no paper on the life of Hirsch or Böhm.

It is very difficult to find any details about a personality which is not listed in St. Andrews archive. One way is to find a obituary dedicated to him; e.g. fill-in his surname as title by searching JFM. But only the famous mathematicians obituaries are reviewed. The other possibility, searching the journals or archives, brings similar problems, e.g. when the mathematician died, where their inheritance files are archived and how to get to them?

The most progressive way is to combine more sources. For the first time it is effective to search library and archive catalogues accessible by Internet searching engines such as `google.com`. Here one can find out the basic information after some difficulties; the most useful are these: where the person was engaged, when and where he died or where his inheritance is stored. One can proceed with ordering the papers in libraries and visiting the archives personally.

The composing style of the scientific papers was completely different in the late 19th century, a more belletristic, less binded by the strict structure of definitions, propositions and proofs. That is why it is a more difficult to follow the proofs and check their correctness.

1.2 A brief survey of the used terminology

There is a close connection between the inverse problem of the calculus of variations and the Law of Least Action, das Princip der kleinsten Wirkung (Action) in German. The law can be stated in this form:

"The mean value of a Lagrangian computed for the same time elements is minimal on the actual trajectory of the system (for longer paths a boundary value) in the comparison with all admissible trajectories which runs in the same time from the initial point to the end point."

There is an extensive literature on the Law of Least Action; this formulation, which is cited due to Helmholtz [8, p. 139], originally descends from Hamilton, see [6, 7]. Let me remark here only that there is also second, alternative formulation, for details see the papers of Mayer [17] and Helmholtz [9] on the history of the Law of Least Action. The reader can also consult the paper by Mayerhofer [32] named "The Law of Least Action by Hermann von Helmholtz"; however, no remarks on connection to the inverse problem are stated there.

In the mechanical system corresponding to the Law of Least Action the external forces are expressible in the form of Euler–Lagrange equations

$$\sum_{k=0}^{n}(-1)^k \frac{d^k}{dx^k}\left(\frac{\partial L}{\partial y_i^{(k)}}\right) = 0$$

for some Lagrangian $L(x, y_i, \ldots, y_i^{(n)})$, where x is an independent variable, $y_i(x)$ are dependent variables and $y_i^{(j)} = d^j y_i/dx^j$. These forces are called *potential*. Hence, these forces can be identified with the differential expressions which are variational.

For one of the central terms in the field, a Lagrangian, Helmholtz proposed the term *a kinetic potential*, das kinetische Potential in the original, see [8, p. 138].

In [8] Helmholtz proposed an important generalization for the research of the systems which obeys the Law of Least Action: Former, the velocities could be contained only in the kinetic energy (living force, die lebendige Kraft in the original) and, moreover, in the form of the homogeneous function with respect to $\dot{q}^2$ only. Helmholtz wrote that it is necessary to make a research of the whole matter with a Lagrangian in the form of an *arbitrary function of coordinates and velocities*.

2 Helmholtz conditions

The impulse for the rise of the inverse problem is of a physical kind. In the nineteenth century many physicists were searching for the unifying principle of the Nature. One admissible principle was the Law of Least Action.

In 1870s Hermann Helmholz was working on Electrodynamics Theory. By the investigation of determining an admissible Lagrangian for Maxwell Electrodynamics Theory he was brought into the research on the Law of Least Action — he tried to generalize the variational functionals from Mechanics to Electrodynamics and then to some other fields of Physics.

2.1 Hermann von Helmholtz — the necessary conditions for the existence of a Lagrangian

Let me make some notes on the life of Helmholtz first; for detailed biography see [30]. Hermann Ludwig Ferdinand Helmholtz was born on August 31, 1821 in Potsdam. He studied medicine and in 1850s he made a research in the Physiology (science that deals with the functions and activities of life or of living matter and of the physical and chemical phenomena involved); in 1858–1871 he was a professor of Physiology in Heidelberg. From 1871, when he was appointed the professor of Physics in Berlin he turned his interest to Physics. In 1888 he was appointed the first president of he Imperial Physico-Technical Institute in Charlottenburg (near Berlin, today a part of Berlin). He died on September 8, 1894 in Berlin.

In the famous treatise *On the physical significance of the Law of Least Action*, published in Journal für die reine und angewandte Mathematik (after its editors often called Crelle's or Borchardt or Kronecker Journal) in 1886, see [8] he posed his famous conditions (Helmholtz conditions) for external forces depending on coordinates, velocities and accelerations, i.e., of the form $P_j(p_i, p_i', p_i'')$ to be *potential*.

Proceeding from the Euler–Lagrange equations in the form

$$P_i = -\frac{\partial H}{\partial p_i} + \frac{\partial^2 H}{\partial p_i' \partial p_\kappa} p_\kappa' + \frac{\partial^2 H}{\partial p_i' \partial p_\kappa'} p_\kappa'', \tag{1}$$

he derived the relation between the forces and accelerations first.

He pointed at the fact that (1) are linear in the accelerations. Then, utilizing the interchangeability of the partial derivatives of the second order he expressed the value of p_κ'' as follows:

$$\frac{\partial P_i}{\partial p_\kappa''} = \frac{\partial^2 H}{\partial p_i' \partial p_\kappa'} = \frac{\partial P_\kappa}{\partial p_i''},$$

i.e, *"When the acceleration p_κ'' increases the force P_i of a definite value, then the same acceleration p_i'' increases the force P_κ of the same value,"* [8, p. 161].

He found out the relationship between the forces and velocities by differentiating the P_i with respect to p'_λ,

$$\frac{P_i}{\partial p'_\lambda} = -\frac{\partial^2 H}{\partial p'_\lambda \partial p_i} + \frac{\partial^2 H}{\partial p'_i \partial p_\lambda} + \frac{\partial^3 H}{\partial p'_\lambda \partial p'_i \partial p_\kappa} p'_\kappa + \frac{\partial^3 H}{\partial p'_\lambda \partial p'_i \partial p'_\kappa} p''_\kappa,$$

$$= -\frac{\partial^2 H}{\partial p'_\lambda \partial p_i} + \frac{\partial^2 H}{\partial p'_i \partial p_\lambda} + \frac{d}{dt}\left(\frac{\partial^2 H}{\partial p'_i \partial p'_\lambda}\right)$$

and then adding this expression to $\partial P_\lambda / \partial p'_i$ he arrived to:

$$\frac{P_i}{\partial p'_\lambda} + \frac{P_\lambda}{\partial p'_i} = 2.\frac{d}{dt}\left(\frac{\partial^2 H}{\partial p'_i \partial p'_\lambda}\right) = 2.\frac{d}{dt}\left(\frac{\partial P_i}{\partial p''_\lambda}\right) \overset{(2)}{=} 2.\frac{d}{dt}\left(\frac{\partial P_\lambda}{\partial p''_i}\right)$$

He also noticed that in the many cases when $\partial^2 H / \partial p'_i \partial p'_\kappa = \text{const}$, this condition take the more simple form:

$$\frac{P_i}{\partial p'_\lambda} + \frac{P_\lambda}{\partial p'_i} = 0,$$

i.e. *"When the escalation of the velocity p'_κ by the permanently same position and accelerations increases the force P_i, then the corresponding escalation of p''_i decreases the force P_κ,"* [8, p. 163].

Analogously, for the finding the relationship between forces and coordinates Helmholtz differentiated P_i with respect to p_λ,

$$\frac{P_i}{\partial p_\lambda} = -\frac{\partial^2 H}{\partial p_\lambda \partial p_i} + \frac{d}{dt}\left(\frac{\partial^2 H}{\partial p'_i \partial p_\lambda}\right)$$

and took it away $P_\lambda / \partial p_i$ he computed the third condition

$$\frac{P_i}{\partial p_\lambda} - \frac{P_\lambda}{\partial p_i} = \frac{d}{dt}\left(\frac{\partial^2 H}{\partial p'_i \partial p_\lambda} - \frac{\partial^2 H}{\partial p'_\lambda \partial p_i}\right) = \frac{1}{2}\frac{d}{dt}\left(\frac{P_i}{\partial p'_\lambda} - \frac{P_\lambda}{\partial p_i}\right)$$

Helmholtz summed up the conditions at the end of the first part of [8] in this form (p. 165–166):

$$\frac{\partial P_i}{\partial p''_\kappa} = \frac{\partial P_\kappa}{\partial p''_i}, \tag{2}$$

$$\frac{\partial P_i}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_i} = \frac{d}{dt}\left(\frac{\partial P_i}{\partial p''_\kappa} + \frac{\partial P_\kappa}{\partial p''_i}\right). \tag{3}$$

$$(4) \qquad \frac{\partial P_i}{\partial p_\kappa} - \frac{\partial P_\kappa}{\partial p_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_i}\right).$$

These three sets of conditions are necessary and sufficient for the existence of a Lagrangian H such that

$$(5) \qquad P_i = -\frac{\partial H}{\partial p_i} + \frac{d}{dt}\left(\frac{\partial H}{\partial p'_i}\right).$$

However, in [8] Helmholtz proved only the necessity of these conditions and moreover under the following restrictive assumption:

$$(6) \qquad \frac{\partial^2 P_i}{\partial p''_j \partial p''_k} = 0.$$

It is noteworthy that Helmholtz did not notice that (6) follows directly from his conditions. The latter condition means that forces P_i are affine in the accelerations, i.e., they are of the form

$$(7) \qquad B^{ij}(p_k, p'_k)p''_j + A^i(p_k, p'_k) = 0.$$

For the complete derivation see [8, § 4].

Remark 1 There are two different opinions concerning the year when the paper [8] was originally published. On the cover of the Journal the year 1887 is printed and some authors follow this. But there are many good reasons that the paper was published in 1886. First, Helmholtz himself mentioned at the end of the paper that it was finished already in April 1886; second, this year is quoted in all contemporary papers, especially in the third volume of his posthumously edited *Scientific papers*, see [8]; third, the supplement to this article, lecture *On the history of the Law of Least Action*, was delivered on the session of the Academy of Sciences on January 27, 1887 and the participants of the lecture should have been familiar with this paper. Last but not least, the paper was reviewed in JFM in 1886, see JFM 18.0941.01.

Nowadays, the Helmholtz conditions for system (7) are usually considered in this equivalent form, see e.g. [16, p. 5]:

$$(8) \qquad \begin{aligned} &B_{ij} = B_{ji}, \qquad \frac{\partial B_{ij}}{\partial p'_k} = \frac{\partial B_{ik}}{\partial p'_j} \\ &\frac{\partial A_i}{\partial p'_j} + \frac{\partial A_j}{\partial p'_i} = 2\left(\frac{\partial B_{ij}}{\partial t} + p'_k \frac{\partial B_{ij}}{\partial p_k}\right) \\ &\frac{\partial A_i}{\partial p_j} - \frac{\partial A_j}{\partial p_i} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial p'_j} - \frac{\partial A_j}{\partial p'_i}\right) + \frac{1}{2} p'_k \frac{\partial}{\partial p_k}\left(\frac{\partial A_i}{\partial p'_j} - \frac{\partial A_j}{\partial p'_i}\right) \end{aligned}$$

Santilli [21] wrote that this equivalent form was *"apparently derived for the first time by Mayer (1896) and then worked out in more details by Davis (1928 and 1929),"* see p. 65. Looking through the Supplement 6 we elicit that his statement is not true. Mayer derived the Helmholtz conditions in the original form, cf. also Problem 2.

In this connection it is interesting that the conditions (8) were first derived by Helmholtz himself; though he did not publish it and it was found in his inheritance by Leo Königsberger and published in 1905, see [10].

However, I cannot state when the manuscript was written. It is possible that it was finished already in 1886 by composing [8] but it could have been rewritten later. The question may be impossible to answer even after the study of the original which was not accessible for me. Unfortunately from the Königsberger's quotation[1] it is not evident where the manuscript is stored, because his vast inheritance is stored by several archives all over Germany. Fortunately, Mayerhofer [32] noticed that this part of the Helmholtz's inheritance is stored in the Academy-archive of the Berlin-Brandenburgische Akademie der Wissenschaften.

In [30], the Helmholtz's letter to Kronecker was published, which was sent on April 25, 1886. Helmholtz wrote on the problem which he is still involved with and which makes him nervous because of the delay of the printing. He wrote: *"In the attempt to reverse my propositions, I have been led to the theory of polydimensional functions, where one has to walk very warily, and I have not decided whether to make this discussion a digression in the main essay, or to treat it separately. Even in the second case, however, I must first get my excursus worked out ..."* Thus, it seems that at least the first part of the manuscript was written already in 1886.

Definitely, the manuscript was written between 1886 and 1894.

The most important aspect on the Helmholtz's work is that it made a considerable impulse into the research of questions related to the inverse problem of the calculus of variations.

Helmholtz's friends and colleagues were trying to prove the sufficiency of Helmholtz conditions and to generalize it to the higher-order field theory. Although some of these were reached already at the end of the 19th century and at the beginning of the 20th century (see below), the complete solution in the case of the field theory remained to my best knowledge unsolved until the early 1980s when Ian Anderson with T. Duchamp [1] and Demeter Krupka [15] published the solution which is nowadays known as the Anderson–Duchamp–Krupka test.

Helmholtz did not give up the attempts to prove the sufficiency at the end of his life, but he did not manage to finish the computations to a publishable form. His attempts were found in his inheritance and published with a commentary by Königsberger in 1905, see [10].

[1] "In the inheritance of Helmholtz there is a note which was originally § 5 in the above quoted paper and which holds the title 'reverse of the problem'...", see [10, p. 865]

2.2 Leo Königsberger — the first published attempt to prove the sufficiency

A collaborator and a good friend of Helmholtz,[2] Leo Königsberger (1837–1921) wrote a series of papers during the nineties of the 19th century which was collected into a whole monograph *Über die Prinzipien der Mechanik* in 1901. In one of these papers from 1986, [14], he provided a proof of the sufficiency of the Helmholtz conditions for the Lagrangians depending on the coordinates and velocities (first-order Lagrangians) in the case of two coordinates.

The proof is pure analytical and that is why he could construct the Lagrangian explicitly.

Problem 1 *Let P_1, P_2 be linear functions with respect to both p_1'' and p_2'' such that the conditions* (2)–(4) *hold.*

Then there exists a Lagrangian $H(p_1, p_2, p_1', p_2')$ such that both Euler–Lagrange equations

$$P_1 = -\frac{\partial H}{\partial p_1} + \frac{d}{dt}\left(\frac{\partial H}{\partial p_1'}\right), \qquad P_2 = -\frac{\partial H}{\partial p_2} + \frac{d}{dt}\left(\frac{\partial H}{\partial p_2'}\right). \tag{9}$$

hold at once.

According to the assumptions of the above problem P_i is of the form

$$P_i = f_{0i}(p_1, p_2, p_1', p_2') + f_{1i}(p_1, p_2, p_1', p_2').p_1'' + f_{2i}(p_1, p_2, p_1', p_2').p_2''. \tag{10}$$

Hence, from the Helmholtz conditions (2), (3) and (4) Königsberger obtained the conditions on the coefficients of P_i, see [14, eq. 99–104]:

$$f_{21} = f_{12} \tag{11}$$

$$\frac{\partial f_{11}}{\partial p_2'} = \frac{\partial f_{12}}{\partial p_1'}, \qquad \frac{\partial f_{12}}{\partial p_2'} = \frac{\partial f_{22}}{\partial p_1'} \tag{12}$$

$$\frac{\partial f_{01}}{\partial p_2'} + \frac{\partial f_{02}}{\partial p_1'} = 2\left(p_1'\frac{\partial f_{12}}{\partial p_1} + p_2'\frac{\partial f_{12}}{\partial p_2'}\right) \tag{13}$$

$$\frac{\partial f_{11}}{\partial p_2} - \frac{\partial f_{12}}{\partial p_1} = \frac{1}{2}\left(\frac{\partial^2 f_{01}}{\partial p_2'\partial p_1'} - \frac{\partial^2 f_{02}}{\partial p_1'\partial p_1'}\right) \tag{14}$$

$$\frac{\partial f_{21}}{\partial p_2} - \frac{\partial f_{22}}{\partial p_1} = \frac{1}{2}\left(\frac{\partial^2 f_{01}}{\partial p_2'\partial p_2'} - \frac{\partial^2 f_{02}}{\partial p_1'\partial p_2'}\right) \tag{15}$$

[2] Also the author of the most comprehensive biography of Helmholtz so far, [30]. See biography at [34].

$$\frac{\partial f_{01}}{\partial p_2} - \frac{\partial f_{02}}{\partial p_1} = \frac{1}{2}\left(p_1'\frac{\partial^2 f_{01}}{\partial p_2'\partial p_1} + p_2'\frac{\partial^2 f_{01}}{\partial p_2'\partial p_2} - p_1'\frac{\partial^2 f_{02}}{\partial p_1'\partial p_1} - p_2'\frac{\partial^2 f_{02}}{\partial p_1'\partial p_2}\right). \tag{16}$$

However, there are two other conditions which are not mentioned in the paper. These conditions are as follows:

$$\frac{\partial f_{01}}{\partial p_1'} = \frac{\partial f_{11}}{\partial p_1}p_1' + \frac{\partial f_{11}}{\partial p_2}p_2', \tag{17}$$

$$\frac{\partial f_{02}}{\partial p_2'} = \frac{\partial f_{22}}{\partial p_1}p_1' + \frac{\partial f_{22}}{\partial p_2}p_2'. \tag{18}$$

They follow from (3) for $i = \kappa = 1$, respectively $i = \kappa = 2$.

Then he utilized (9) to obtain conditions on f_{ij}. He computed the right-hand sides of (9) and by the comparison with the right-hand side of (10) he arrived to:

$$\frac{\partial^2 H}{\partial p_1'\partial p_1'} = f_{11}, \qquad \frac{\partial^2 H}{\partial p_1'\partial p_2'} = f_{12} = f_{21}, \qquad \frac{\partial^2 H}{\partial p_2'\partial p_2'} = f_{22}, \tag{19}$$

$$\begin{aligned} -\frac{\partial H}{\partial p_1} + \frac{\partial^2 H}{\partial p_1'\partial p_1}p_1' + \frac{\partial^2 H}{\partial p_1'\partial p_2}p_2' &= f_{01}, \\ -\frac{\partial H}{\partial p_2} + \frac{\partial^2 H}{\partial p_2'\partial p_1}p_1' + \frac{\partial^2 H}{\partial p_2'\partial p_2}p_2' &= f_{02}. \end{aligned} \tag{20}$$

Then he set

$$H_1 = \int f_{11}\, dp_1' + \int\left[f_{21} - \int \frac{\partial f_{11}}{\partial p_2'}dp_1'\right]dp_2' \tag{21}$$

$$H_2 = \int f_{22}\, dp_2' + \int\left[f_{21} - \int \frac{\partial f_{22}}{\partial p_1'}dp_2'\right]dp_1' \tag{22}$$

and then from (12) and (19) it implies that

$$H = \int H_1\, dp_1' + \int\left[H_2 - \int \frac{\partial H_1}{\partial p_2'}dp_1'\right]dp_2' + \omega_1 p_1' + \omega_2 p_2' + \omega, \tag{23}$$

where ω_1, ω_2 and ω are functions of p_1 and p_2 such that equations (20) hold.

Putting

$$\frac{\partial \omega_1}{\partial p_2} - \frac{\partial \omega_2}{\partial p_1} = \Omega,$$

he re-wrote (20) to the form

$$\text{(24)}\quad \begin{aligned} &-\int \frac{\partial H_1}{\partial p_1} dp_1' - \int \left[\frac{\partial^2 H}{\partial p_1} - \int \frac{\partial^2 H_1}{\partial p_1 \partial p_2'} dp_1' \right] dp_2' + p_1' \frac{\partial H_1}{\partial p_1} + p_2' \frac{\partial H_1}{\partial p_2} + p_2' \Omega - \frac{\partial \omega}{\partial p_1} = f_{01} \\ &-\int \frac{\partial H_1}{\partial p_2} dp_1' - \int \left[\frac{\partial^2 H}{\partial p_2} - \int \frac{\partial^2 H_1}{\partial p_2 \partial p_2'} dp_1' \right] dp_2' + p_1' \frac{\partial H_2}{\partial p_1} + p_2' \frac{\partial H_2}{\partial p_2} + p_1' \Omega - \frac{\partial \omega}{\partial p_2} = f_{02} \end{aligned}$$

In the end he determined Ω and ω from (24). He considered Ω and ω as functions of p_1, p_2, p_1', p_2' and introduced the short cut

$$p_2' \Omega - \frac{\partial \omega}{\partial p_1} = M, \qquad p_1' \Omega + \frac{\partial \omega}{\partial p_2} = N.$$

from (24). He proved that if they obey the conditions (13)–(16), then they must be of the form:

$$p_2' \Omega - \frac{\partial \omega}{\partial p_1} = p_2' \phi(p_1, p_2) + \int R(p_1, p_2) dp_2 + R_1(p_1, p_1')$$

$$p_1' \Omega + \frac{\partial \omega}{\partial p_2} = p_1' \phi(p_1, p_2) + \int R(p_1, p_2) dp_1 + R_2(p_2, p_2').$$

where ϕ, R, R_i are arbitrary functions.

Let me remark that from (17), (18) it follows that R_1 (respectively R_2) is independent on p_1' (respectively p_2').

Then, in a series of papers from 1901–1905, he tried to generalize the results to the higher-order Lagrangians and field theory.

In [10] he wrote down (without any quotation): *"By the last occasion I have finished the determination of independent necessary and sufficient conditions for existence of a kinetic potential of an arbitrary order depending on unlimited number of both independent and dependent variables. It was done by means of analytical computations of identic solutions of the principal equations of the variation of simple and multiple integrals."*

Unfortunately, I have no opportunity to look over these papers, but from the reviews it seems to me that Königsberger did not solve the problem completely, cf. JFM 32.0691.02, 33.0714.01, 36.0761.01 and 36.0836.04.

2.3 Adolph Mayer — proof of sufficiency

Christian Gustav Adolph Mayer (1839–1908) studied in Heidelberg and his professional career is linked also with Leipzig.

In 1896 Mayer wrote a paper for the Proceedings of the Academy of Sciences in Leipzig, entitled *The conditions for existence of a kinetic potential*, [18]. His paper is written purely from the mathematical point of view, without any physical aspects. At first he deduced the Helmholtz conditions with the help of the Jacobi principle

$$\delta \frac{dV}{dt} = \frac{d\delta V}{dt}$$

for any function $V(t, q_i, q_i')$. He applied it by means of the following conditions:

$$\text{(J1)} \qquad \frac{\partial}{\partial p_\lambda}\frac{dV}{dt} \equiv \frac{d}{dt}\frac{\partial V}{\partial p_\lambda},$$

$$\text{(J2)} \qquad \frac{\partial}{\partial p_\lambda'}\frac{dV}{dt} \equiv \frac{\partial V}{\partial p_\lambda} + \frac{d}{dt}\frac{\partial V}{\partial p_\lambda'},$$

$$\text{(J3)} \qquad \frac{\partial}{\partial p_\lambda''}\frac{dV}{dt} \equiv \frac{\partial V}{\partial p_\lambda'}.$$

The problem is set as follows:

Problem 2 (Mayer, 1896) *Let $P_1, \dots, P_n$ be given functions depending on the n variables q_i, its first and second derivatives with respect to t and possibly also on the independent variable t.*

Find the conditions on P_i such that if they are fulfilled then there exists a function $H(t, p_i, p_i')$ that identically obeys n following (Euler–Lagrange) equations:

$$(25) \qquad -\frac{\partial H}{\partial p_i} + \frac{d}{dt}\frac{\partial H}{\partial p_i'} = P_i.$$

By the substitution

$$\frac{\partial H}{\partial p_i'} = \psi_i,$$

Mayer put the Euler–Lagrange equations into the form

$$(26) \qquad \frac{\partial H}{\partial p_i'} = \psi_i, \quad \frac{\partial H}{\partial p_i} = \frac{d\psi_i}{dt} - P_i$$

Differentiating the second relation with respect to p''_k, he found out with help of (J3) that

$$\frac{\partial \psi_i}{\partial p'_k} - \frac{\partial P_i}{\partial p''_k} = 0. \tag{27}$$

This condition means that P_i are linear with respect to p''.

From the interchangeability of the partial derivatives of the second order and equations (J1) and (J2) he stated that the necessary and sufficient conditions for the existence of a function H, which fulfils all $2n$ equations (26); together with (27) are as follows:

$$\frac{\partial \psi_i}{\partial p'_\kappa} = \frac{\partial \psi_\kappa}{\partial p'_i}, \tag{28}$$

$$\frac{\partial \psi_\kappa}{\partial p_i} = \frac{\partial \psi_i}{\partial p_\kappa} + \frac{d}{dt}\frac{\partial \psi_i}{\partial p'_\kappa} - \frac{\partial P_i}{\partial p'_\kappa}, \tag{29}$$

$$\frac{d}{dt}\frac{\partial \psi_i}{\partial p_\kappa} - \frac{\partial P_i}{\partial p_\kappa} = \frac{d}{dt}\frac{\partial \psi_\kappa}{\partial p_i} - \frac{\partial P_\kappa}{\partial p_i}. \tag{30}$$

But from (27) and (28) he obtained the first Helmholtz condition (2). Hence, from (29) he deduced:

$$\frac{\partial P_i}{\partial p'_k} = \frac{\partial \psi_i}{\partial p_k} - \frac{\partial \psi_k}{\partial p_i} + \frac{d}{dt}\frac{\partial \psi_i}{\partial p'_k}$$

$$\left(\frac{1}{2}\frac{\partial P_i}{\partial p'_k} + \frac{1}{2}\frac{\partial P_i}{\partial p'_k}\right) + \left(\frac{1}{2}\frac{\partial P_i}{\partial p'_k} - \frac{1}{2}\frac{\partial P_i}{\partial p'_k}\right) = \frac{\partial \psi_i}{\partial p_k} - \frac{\partial \psi_k}{\partial p_i} + \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \psi_i}{\partial p'_k} + \frac{\partial \psi_k}{\partial p'_i}\right)$$

and

$$\frac{\partial P_i}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_i} = \frac{d}{dt}\left(\frac{\partial \psi_i}{\partial p'_\kappa} + \frac{\partial \psi_\kappa}{\partial p'_i}\right), \tag{31}$$

$$\frac{\partial P_i}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_i} = 2\left(\frac{\partial \psi_i}{\partial p_\kappa} - \frac{\partial \psi_\kappa}{\partial p_i}\right), \tag{32}$$

These systems are equivalent to (29). The system (31) is by virtue of (27) the second Helmholtz condition

$$\frac{\partial P_i}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_i} = \frac{d}{dt}\left(\frac{\partial P_i}{\partial p''_\kappa} + \frac{\partial P_\kappa}{\partial p''_i}\right).$$

Finally, from (30) and (32) Mayer obtained the third Helmholtz condition

$$\frac{\partial P_i}{\partial p_\kappa} - \frac{\partial P_\kappa}{\partial p_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_i}\right).$$

Solution 1 *Let* $P_i(t, q_j, q'_j, q''_j)$ *be functions linear in* p'' *such that the conditions* (2), (3), (4) *hold.*

Then there exists a function H *fulfilling the equations* (25).

The Mayer's paper follows on with the proof of sufficiency of the above conditions.

Problem 3 (Mayer, 1896) *Let* $P_i(t, q_j, q'_j, q''_j)$ *be functions linear in* p'' *such that the conditions* (2), (3), (4) *hold.*

There exist n functions $\psi_i(t, q_j, q'_j)$ *such that the equations*

$$\frac{\partial \psi_i}{\partial p'_k} = \frac{\partial P_k}{\partial p''_i}, \tag{33}$$

$$\frac{\partial \psi_i}{\partial p_k} - \frac{\partial \psi_k}{\partial p'_i} = \frac{1}{2}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) \tag{34}$$

are satisfied identically.

The Helmholtz conditions imply, with help of (27) and (32), the original integrability conditions (28), (29) and (30). Mayer asserted that (27) and (32) are equivalent to (33), (34); the equivalence is obvious. Hence, the sufficiency was proved for the conditions as it was stated above.

The condition (33) implies that the function ψ must be of the form

$$\psi_\lambda = \chi_\lambda(t, p_i, p'_i) + \omega_\lambda(t, p_i) \tag{35}$$

where χ_λ can be evaluated by the quadrature and ω_λ is an arbitrary function.

Substituting (35) into (34) he deduced:

$$\frac{1}{2}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) - \left(\frac{\partial \chi_i}{\partial p_k} - \frac{\partial \chi_k}{\partial p_i}\right) = \frac{\partial \omega_i}{\partial p_k} - \frac{\partial \omega_k}{\partial p_i} = \Omega_{ik},$$

where $\Omega_{ki} = -\Omega_{ik}$. Moreover, these symbols are independent on q'' and q'. The first statement follows from

$$\frac{\partial}{\partial p''_\lambda}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) = 0,$$

which is a consequence of the independence of (4) on the third derivatives. The second statement follows from the following calculation

$$
\begin{aligned}
\frac{\partial \Omega_{ik}}{\partial p'_\lambda} &= \frac{1}{2}\,\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) - \frac{\partial}{\partial p'_\lambda}\left(\frac{\partial \chi_i}{\partial p_k} - \frac{\partial \chi_k}{\partial p_i}\right) \\
&\overset{(J3)}{=} \frac{1}{2}\,\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) - \frac{\partial}{\partial p''_\lambda}\left[\frac{d}{dt}\left(\frac{\partial \chi_i}{\partial p_k} - \frac{\partial \chi_k}{\partial p_i}\right)\right] \\
&\overset{(4)}{=} \frac{1}{2}\,\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_i}{\partial p'_k} - \frac{\partial P_k}{\partial p'_i}\right) - \frac{\partial}{\partial p''_\lambda}\left(\frac{\partial P_i}{\partial p_k} - \frac{\partial P_k}{\partial p_i}\right) \\
&= 0
\end{aligned}
$$

Differentiating (4) and utilizing the formula (J3) the latter identity follows. Hence, Ω are functions of t and $p_1, \ldots, p_n$ only.

Since the formula

$$
\frac{\partial \Omega_{ik}}{\partial p_\lambda} + \frac{\partial \Omega_{k\lambda}}{\partial p_i} + \frac{\partial \Omega_{\lambda i}}{\partial p_k} = 0
$$

holds, for the details see [18, p. 526], we can define ω_i recurrently. More precisely, if we choose the arbitrary $\omega_n(t, p_1, \ldots, p_n)$, then the remaining ones are defined by

$$
\begin{aligned}
\frac{\partial \omega_\lambda}{\partial p_{\lambda+1}} &= \frac{\partial \omega_{\lambda+1}}{\partial p_\lambda} + \Omega_{\lambda,\lambda+1}, \\
\frac{\partial \omega_\lambda}{\partial p_{\lambda+2}} &= \frac{\partial \omega_{\lambda+2}}{\partial p_\lambda} + \Omega_{\lambda,\lambda+2}, \\
&\vdots \\
\frac{\partial \omega_\lambda}{\partial p_n} &= \frac{\partial \omega_n}{\partial p_\lambda} + \Omega_{\lambda,n}.
\end{aligned}
$$

Substituting these ω_i into the equations (35) yields n functions ψ_i which identically obeys the relations (33), (34). Hence, the proof was finished.

In the conclusive remarks Mayer determined the most general form of functions ψ as follows:

$$
\psi_i = \chi_i(t, p_1, \ldots, p_n, p'_1, \ldots, p'_n) + u_i(t, p_1, \ldots, p_n) + \frac{\partial \Phi(t, p_1, \ldots, p_n)}{\partial p_i}.
$$

3 The Generalization of the Helmholtz conditions to the higher-order Lagrangians

Soon after proving the sufficiency of the Helmholtz conditions, the attention was paid to the generalization of the Helmholtz conditions to the higher-order Lagrangians.

Adolph Mayer supposed that his method, see [18], is easily generalizable to the higher-order case. In the same year 1896, Leo Königsberger posed precisely the problem.

Already in 1897 and 1898, two papers by Arthur Hirsch were published, in which the problem was solved in general. In the first paper Hirsch tackled the problem for the differential equation of the form $F(x, y(x), y'(x), \ldots, y^{(m)}(x)) = 0$, in the second one he generalized the question for the n equations of the form

$$F_k(x, y_1, \ldots, y_n, y_1', \ldots, y_n', \ldots, y_1^{(r_1)}, \ldots, y_n^{(r_n)}) = 0.$$

The generalizations were nearly complete, but Hirsch did not succeed in writing the conditions of variationality of F in an explicit manner.

Hence in summer 1897 Karl Böhm wrote a paper which was published not until two years in Journal für die reine und angewandte Mathematik; in the note behind the paper he explicitly expressed the so-called higher-order Helmholtz conditions.

The paper was written independently on the research of Hirsch, Böhm completely solved the problem with another method for the Lagrangians depending on the velocities and accelerations, i.e., of the form $H(t, p_1, \ldots, p_n, p_1', \ldots, p_n', p_1'', \ldots, p_n'')$. However, he did not expect any serious problems in generalizing the results to the arbitrary order.

3.1 Arthur Hirsch – a successful generalization

The Zürich professor Arthur Hirsch (1866–1948) had no connections to the group around Helmholtz. He also used different methods to solve the inverse problem — the self-adjoint theory.

He defined the property of self-adjointness for differential expression F by this formula:

$$v \cdot \delta_u F = u \cdot \delta_v F. \tag{36}$$

It utilizes the properties of the second variation. By means of this theory he proved that for a self-adjoint differential equation there exist a Lagrangian.

The first paper *On the characteristic property of the differential equations in the calculus of variations* from 1897, see [11], is a voluminous treatise on more problems connected to the inverse problem. He began with the problem for one ODE with one dependent variable:

Problem 4 *If the function $F(x, y, y', \ldots, y^{(2n)})$ of the even order $2n$ has the property that the derived linear differential expression*

$$\delta F = \sum_{k=0}^{n} F_k \cdot u^{(k)}$$

is self-adjoint, then F can be computed by means of quadrature of $f(x, y, y', \ldots, y^{(n)})$ such that

$$F = V(f) = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial f}{\partial y^k}\right).$$

The differential equation $F = 0$ is then equivalent to the problem of the calculus of variations to extremize the integral

$$J = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx.$$

In the [11, § 3] the Problem 4 is proved for $n = 1$. Hirsch started from $F(x, y, y', y'')$ such that

$$\delta F = F_0 u + F_1 u' + F_2 u''.$$

Then it holds

$$\frac{dF_2}{dx} - F_1 = 0, \tag{37}$$

from this it implies that sole y'' is not contained in F_2 and that F must be of the form

$$F = M(x, y, y').y'' + N(x, y, y')$$

Integrating M with respect to y', setting

$$\int M(x, y, y')\, dy' = P(x, y, y')$$

and utilizing the formula

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} y' + M.y'' \tag{38}$$

he expressed F in the form

$$F = \frac{d}{dx} P(x, y, y') + Q(x, y, y')$$

for which he could rewrite (37) to the form

$$\frac{d}{dx}\left(\frac{\partial P}{\partial y'}\right) - \frac{\partial}{\partial y'}\left(\frac{dP}{dx}\right) - \frac{\partial Q}{\partial y'} = 0.$$

Differentiating (38) with respect to y' and comparing with the latter formula he finally arrived to (37) in this form:

$$\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y'} = 0.$$

By means of quadrature he obtained a function $f(x, y, y')$ which fulfils the following relations:

$$\frac{\partial f}{\partial y'} = -P, \qquad \frac{\partial f}{\partial y} = Q.$$

Then it holds

$$F = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right).$$

By means of an induction the proof is generalized to arbitrary n in § 4, § 5 and § 6. Similarly as in § 3, it it shown that F must be affine in the highest derivatives, i.e. of the form

$$F = M(x, y, y', \dots, y^{(2n-1)}).y^{2n} + N(x, y, y', \dots, y^{(2n-1)})$$

and moreover it is shown that M is independent with respect to $y^{(n+1)}, \dots, y^{(2n-1)}$.

In § 7 a so-called multiplier problem is studied. For a closer exposition see the Appendix below. Then, in § 9 the problem is generalized to partial differential equations of the form $F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy})$ and to the second-order PDEs with three independent variables in § 10, but only necessary conditions determining the form of the equations were obtained.

One year later (in 1898) Hirsch published the paper *The conditions for the existence of the generalized kinetic potential*, see [12]. Here he generalized the previous results to the case of a system of ODEs with an arbitrary number of dependent variables of an arbitrary order, i.e. the equations of the form

$$F_k(x, y_1, \dots, y_n, y_1', \dots, y_n', \dots, y_1^{(r_1)}, \dots, y_n^{(r_n)}) = 0.$$

The results are of the existence character, there are stated neither necessary nor sufficient conditions on F to be self-adjoint (and thus variational).

3.2 Karl Böhm – an alternative approach

Karl Böhm (1873–1958) studied in Heidelberg in 1890s, in 1900 he presented in Heidelberg his habilitation thesis, in 1904 he was appointed extraordinary professor and he taught in Heidelberg university until 1914.

He posed his objective in the following form:

Problem 5 *For the existence of a Lagrangian depending on the first and second derivatives defined by the equations*

$$P_i = -\left\{\frac{\partial H}{\partial p_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial p_i'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial p_i''}\right)\right\} \tag{39}$$

it is necessary and sufficient that P_i is such function of p, p', p'', $p^{(3)}$, $p^{(4)}$ that the conditions

(40) $$\frac{\partial P_i}{\partial p_k^{(4)}} = \frac{\partial P_k}{\partial p_i^{(4)}},$$

(41) $$\frac{\partial P_i}{\partial p_k^{(3)}} + \frac{\partial P_k}{\partial p_i^{(3)}} = 2\frac{d}{dt}\left(\frac{\partial P_i}{\partial p_k^{(4)}} + \frac{\partial P_k}{\partial p_i^{(4)}}\right),$$

(42) $$\frac{\partial P_i}{\partial p_k^{(2)}} - \frac{\partial P_k}{\partial p_i^{(2)}} = \frac{3}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial p_k^{(3)}} - \frac{\partial P_k}{\partial p_i^{(3)}}\right),$$

(43) $$\frac{\partial P_i}{\partial p_k'} + \frac{\partial P_k}{\partial p_i'} = \frac{d}{dt}\left(\frac{\partial P_i}{\partial p_k^{(2)}} + \frac{\partial P_k}{\partial p_i^{(2)}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial P_i}{\partial p_k^{(4)}} + \frac{\partial P_k}{\partial p_i^{(4)}}\right),$$

(44) $$\frac{\partial P_i}{\partial p_k} - \frac{\partial P_k}{\partial p_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_i}{\partial p_k'} - \frac{\partial P_k}{\partial p_i'}\right) - \frac{1}{4}\frac{d^3}{dt^3}\left(\frac{\partial P_i}{\partial p_k^{(3)}} - \frac{\partial P_k}{\partial p_i^{(3)}}\right),$$

hold.

Böhm uses Mayer's methods directly to derive the Helmholtz conditions for the second-order potentials.

At first he generalized the formulae for the variations (J1)–(J3) to the case of the functions depending on the higher derivatives, say up to the order ν:

(J5) $$\frac{\partial}{\partial p_\lambda^{(\sigma)}}\left(\frac{d^\rho V}{dt^\rho}\right) = \binom{\rho}{\sigma}\frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\left(\frac{\partial V}{\partial p_\lambda}\right) + \binom{\rho}{\sigma-1}\frac{d^{\rho-\sigma+1}}{dt^{\rho-\sigma+1}}\left(\frac{\partial V}{\partial p_\lambda'}\right) + \cdots + \frac{d^\rho}{dt^\rho}\left(\frac{\partial V}{\partial p_\lambda^{(\sigma)}}\right),$$

for $\sigma \leq \rho$

(J6) $$\frac{\partial}{\partial p_\lambda^{(\sigma)}}\left(\frac{d^\rho V}{dt^\rho}\right) = \frac{\partial V}{\partial p_\lambda^{(\sigma-\rho)}} + \binom{\rho}{\rho-1}\frac{d}{dt}\left(\frac{\partial V}{\partial p_\lambda^{(\sigma-\rho+1)}}\right) + \cdots + \frac{d^\rho}{dt^\rho}\left(\frac{\partial V}{\partial p_\lambda^{(\sigma)}}\right),$$

for $\sigma \geq \rho$

On these two formulae all subsequent calculations are grounded.

Similarly as Mayer, he proceeded with introducing the substitutions:

(45) $$\frac{\partial H}{\partial p_i^{(2)}} = \phi_i, \qquad \frac{\partial H}{\partial p_i'} - \frac{d\phi_i}{dt} = \psi_i.$$

Then the equations (39) take the form

$$-\frac{\partial H}{\partial p_i} + \frac{d\psi_i}{dt} = P_i. \tag{46}$$

Hence, he studied the problem on what conditions the functions ϕ_i, ψ_i and H obey the relations (45), (46). The proof is analogous to the Mayer's, see my interpretation above or the original proof in [3].

Let me translate the Böhm's note mentioned above.

Solution 2 *Let H be a function of the first ν derivatives of the coordinates p_i defined by the equations*

$$-\left\{\frac{\partial H}{\partial p_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial p_i'}\right) + \cdots + (-1)^\nu \frac{d^\nu}{dt^\nu}\left(\frac{\partial H}{\partial p_i^{(\nu)}}\right)\right\} = P_i$$

Then $P_j(p_i, p_i', \ldots, p_i^{(2\nu)})$ must identically satisfy the $(2\nu + 1)$ equations

$$\begin{aligned}\frac{\partial P_i}{\partial p_\kappa^{(\tau)}} &- \binom{\tau+1}{1}\frac{d}{dt}\left(\frac{\partial P_i}{\partial p_\kappa^{(\tau+1)}}\right) + \binom{\tau+2}{2}\frac{d^2}{dt^2}\left(\frac{\partial P_i}{\partial p_\kappa^{(\tau+2)}}\right) - \cdots \\ &+ (-1)^{2\nu-\tau}\binom{2\nu}{2\nu-\tau}\frac{d^{2\nu-\tau}}{dt^{2\nu-\tau}}\left(\frac{\partial P_i}{\partial p_\kappa^{(2\nu)}}\right) = (-1)^\tau \frac{\partial P_\kappa}{\partial p_i^{(\tau)}}\end{aligned} \tag{47}$$

$$\tau = 0, 1, 2, \ldots, 2\nu.$$

These relations coincide for $\nu = 2$ with the conditions (40)–(44) [of this thesis].

It is easy to see that for $\nu = 1$ the equations (47) coincide with the Helmholtz conditions (2)–(4). There is no proof of this statement. To my best knowledge, the proof was first realized by A.L. Vanderbauwhede in [27].

It is possible that Böhm simply guessed the form of (47); it is not difficult to understand the structure of the Helmholtz conditions and thus he had to precise the combinatorial numbers by the individual terms only.

Appendix Multiplier problem

It is essential to mention again that there is another and a more general level of the inverse problem. In this generalized formulation, the problem is to find everywhere regular matrix B^i_j called *variational multiplier* or *integrating factor* such that the equations $B^i_j F_i = 0$ are variational.

This generalized formulation was precisely stated by Hirsch ([11]) and its complete solution is still unknown. But there are some well-known results concerning mainly second-order ODEs; for one second-order ODE a multiplier always exist, but even for the system of two second-order equations a multiplier need not exist.

Let me make some remarks here, because the first attempt to solve the multiplier problem is surprisingly older than the Helmholtz conditions. This attempt is due to N.Y. Sonin.

Forgotten writing of Nikolay Sonin

This first contribution concerns the multiplier problem for one second-order ODE. It was achieved in voluminous paper *On the determination of maximal and minimal properties of the plane curves* by Russian mathematician Nikolay Yakovlevich Sonin (1849–1915).

The paper was finished in November 1885 (only five months before the famous paper by Helmholtz) and published in *Reports of the Warsaw University* next year. Let me remark that Sonin was engaged on the University of Warsaw (this part of Poland was occupied by Russian dominion during the 19th century) in the years 1872–1894.

Sonin considered the equations which are solved with respect to the second derivatives, i.e. of the form

$$y'' = \varphi(x, y, y'). \tag{48}$$

The maximal and minimal properties are defined by extremizing the integral

$$\int_a^b f(x, y, y')\, dx,$$

which is equivalent to the following condition:

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0. \tag{49}$$

Then he was looking for the unknown function f.

Differentiating (49) with respect to y' and setting $\partial^2 f/\partial p^2 = z$ he obtained an equivalent first-order partial differential equation

$$\frac{\partial z}{\partial x} + p\frac{\partial z}{\partial y} + \varphi\frac{\partial z}{\partial y'} + \frac{\partial \varphi}{\partial y'} = 0.$$

After integration of this equation he utilized the substitution of z and then expressed f in the form:

$$f(x, y, p) = p \int_A^p \Phi(\psi, \sigma).e^{-\int((\partial \log \varphi)/\partial p)\,dp}\,dp - \int_A^p p\Phi(\psi, \sigma).e^{-\int((\partial \log \varphi)/\partial p)\,dp}\,dp + Bp + C, \tag{50}$$

where the arbitrary functions $A,\ B,\ C$ depend on x and y.

He proceeded by searching the relations between the arbitrary functions and his final solution is as follows:

Solution 3 *A general solution of the equation* (49) *has the following expression:*

$$\begin{aligned} f(x, y, p) = p &\int_A^p \Phi(\psi, \sigma).e^{-\int((\partial \log \varphi)/\partial p)\,dp}\,dp \\ &- \int_A^p p\Phi(\psi, \sigma).e^{-\int((\partial \log \varphi)/\partial p)\,dp}\,dp \\ &+ \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} p, \end{aligned} \tag{51}$$

where A is the root of the equation

$$\Psi[\phi(x, y, A), \sigma(x, y, A)] = 0, \tag{52}$$

or

$$[\Phi(\psi, \sigma)e^{-\int((\partial \log \varphi)/\partial p)\,dp}]_{p=A} = 0. \tag{53}$$

The equation (51) is the explicit formula for all (infinite) equivalent Lagrangians. The ϕ, σ in the condition (52) are first integrals of the equation (48).

Even though there appeared a review in Jahrbuch über die Fortschritte der Mathematik in 1886 (JFM 19.0359.01) and scientist in the western Europe could familiarize themselves with this work, this did not turn up. This work laid nearly forgotten for the whole century. Actually, Sonin's biographer A.I. Kropotov also passed this question over, cf. [31].

At the beginning of 1980s the work of Sonin was quoted (according to the Science Citation Index, see `http://wos.cesnet.cz/CIW.cgi`) four times quoted and especially in the review paper [5] there are some good historical remarks. Finally, in 1990s the Helmholtz–Sonin mapping and form was introduced, but there is no connection between this form and Sonin's research.

The Famous book of Gaston Darboux

In the considerable part of the papers on the inverse problem one can read that the beginning of the multiplier problem is connected with the work [4] by Jean Gaston Darboux. It is interesting that on the page 53, which is often quoted, Darboux wrote that he *"only reproduces the results of Beltrami and Dini."*

However, there is no exact quotation and it is not evident which work he meant. E. Beltrami wrote more than 100 of papers in 1868–1894 and U. Dini about 30 paper in the same time. The most of these papers are written in Italian and they are inaccesible in the Czech republic. That is why this question remains unsolved.

Commentary of Arthur Hirsch

Hirsch discussed the multiplier problem in [11, § 7]. First he posed the problem for one even order ordinary differential equation solved with respect to the highest derivative.

For the one second-order equation he reviewed the solution due to Darboux [4], for the higher-order equations he stated the following solution:

Solution 4 *Let* $F = M \cdot (y^{(2n)} + \Phi(x, y, y', \ldots, y^{(2n-1)})) = 0$ *be a differential equation such that*

$$\sum (-1)^k \frac{d^k}{dx^k} \left(\frac{\partial \Phi_{2n-1}}{\partial y^{(k)}} \right) = 0.$$

If δF *is self-adjoint, then is* F *equivalent to some variational problem.*

Some remarks to the following advancement

The interest in the self-adjoint theory grew up in the USA in the late 1920s and 1930. It led to the composing of one of the most famous works on the inverse problem, the one written by Jesse Douglas.

Douglas solved the problem for two second-order ODEs. By a sophisticated method he proved that the multiplier does not exist in general.

Since 1970s the interest on the inverse problem has been renewed and intensified. This is depicted in the cited works [21, 19]. For a more contemporary works on the multiplier problem, see [2].

There were developed two main approaches. First, classical, *operator approach* utilized property of adjointness mentioned in this thesis. A.L. Vanderbauwhede worked up the theory in [26] and then, in [27], he derived a test for $F_j(x, y_i, y_i' \cdots, y_i^{(n)})$ to be variational from (36) .

By the second approach, based on modern methods of *differential geometry*, the closedness conditions for certain differential forms is studied. Among other important works let me quote [2, 20, 23, 24, 25].

It is not usual to provide the quotings, especially names of journals in the complete form. In the 19th century the journals were often quoted according to the name of the managing editor and that is why the full names are of special importance.

References

[1] I.M. Anderson and T. Duchamp, On the existence of global variational principles, *American Journal of Mathematics* 102 (1980) 781–868.

[2] I.Anderson and G. Thompson, The Inverse Problem of the Calculus of Variations for Ordinary Differential Equations, *Memoirs of the American Mathematical Society* 98 (1992) (473) 1–110.

[3] K. Böhm, Die Existenzbedingungen eines von den ersten und zweiten Differentialquotienten der Coordinaten abhängigen kinetischen Potentials, *Journal für die reine und angewandte Mathematik* 121 (1899) (2) 124–140.

[4] G. Darboux, *Leçons sur la Théorie Générale des Surfaces* III. (Gauthier–Villars, Paris, 1894).

[5] V.V. Dodonov, V.I. Man′ko and V.D. Skarzhinsky, The inverse problem of the variational calculus and the nonuniqueness of the quantization of classical systems, *Hadronic Journal* 4 (1981) (5) 1734–1804.

[6] W.R. Hamilton, On a general method in dynamics, *Philosophical Transactions of the Royal Society* 1834 (2) 247–308.

[7] W.R. Hamilton, Second Essay on a general method in dynamics, *Philosophical Transactions of the Royal Society* 1835 (1) 95–144.

[8] H. Helmholtz, Über die physikalische Bedeutung des Princips der kleinsten Wirkung, *Journal für die reine und angewandte Mathematik* 100 (1886) 137–166, 213–222. Also published in: *Wissenschaftliche Abhandlungen* III. volume (J.A. Barth, Leipzig, 1895) 203–248.

[9] H. Helmholtz, Zur Geschichte des Princips der kleinsten Action, *Sitzungsberichte der kgl. Preuss. Akademie der Wissenschaften zu Berlin* 1887 (10. März 1887) 225–236. Also published in: *Wissenschaftliche Abhandlungen* III. volume (J.A. Barth, Leipzig, 1895) 249–263.

[10] H. Helmholtz, Über die physikalische Bedeutung des Princips der kleinsten Wirkung (Aus den hinterlassenen Papieren bearbeitet von Leo Königsberger), *Sitzungsberichte der kgl. Preuss. Akademie der Wissenschaften zu Berlin* 1905 (26. October 1905) 863–883.

[11] A. Hirsch, Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung, *Mathematische Annalen* 49 (1897) 49–72.

[12] A. Hirsch, Die Existenzbedingungen des verallgemeinerten kinetischen Potentials, *Mathematische Annalen* 50 (1898) 429–441.

[13] J.A. Kobussen, Some comments on the Lagrangian formalism for systems with general velocity-dependent forces, *Acta Physica Austriaca* 51 (1979) 293–309.

[14] L. Königsberger, Über die Prinzipen der Mechanik, *Sitzungsberichte der kgl. Preuss. Akademie der Wissenschaften zu Berlin* 1886 (30. Juli 1896) 899–944.

[15] D. Krupka, On the local structure of the Euler–Lagrange mapping of the calculus of variations, in: *Proceedings of the Conference (CSSR-GDR-Poland) on Differential Geometry and Its Applications*, September 1980; O. Kowalski ed. (Charles University in Prague, Czechoslovakia, 1981) 181–188.

[16] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture notes in Mathematics 1678 (Springer, Berlin, 1997).

[17] A. Mayer, *Geschichte des Princips der kleinsten Action,* Akademische Antrittsvorlesung (Veit & Cie, Leipzig, 1877).

[18] A. Mayer, Die Existenzbedingungen eines kinetischen Potentiales, *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig*, Mathematisch-Physische Classe 48 (1896) 519–529.

[19] G. Morandi, C. Ferrario, G. Lo Vechio, G. Marmo and C. Rubano, The Inverse Problem in the Calculus of Variations and the Geometry of the tangent Bundle, *Phys. Rep.* 188 (1990) 147–284.

[20] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1993).

[21] R.M. Santilli, *Foundations of Theoretical Mechanics I.* The Inverse Problem in Newtonian Mechanics (Springer, New York, 1978).

[22] N.Y. Sonin, Ob opredlenij maximalnych i minimalnych svojstv ploskich krivich. *Warsawskye Universitetskye Izvestiya* (1886) (1–2) 1–68.

[23] F. Takens, A global version of the inverse problem of the calculus of variations, *J. Diff. Geom.* 14 (1979) 543–562.

[24] E. Tonti, Variational formulation of nonlinear differential equations I, II, *Bulletin de la Classe des Sciences, Académie royale de Belgique* 55 (1969) (4) 137–165, 262–278.

[25] M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators* (Holden-Day, San Francisco, 1964). Transl. from Russian (GITL, Moscow, 1959).

[26] A.L. Vanderbauwhede, Potential operators and the inverse problem of classical mechanics. *Hadronic Journal* 1 (1978) 1177–1197.

[27] A.L. Vanderbauwhede, Potential operators and variational principles. *Hadronic Journal* 2 (1979) 620–641.

[28] V. Volterra, *Leçons sur les Fonctions de Lignes* (Gauthier–Villars, Paris, 1913).

[29] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 2nd edition (Cambridge University Press, Cambridge, 1917)

— — — — — — — — — —

[30] L. Königsberger, *Hermann von Helmholtz* (Dover Publications, New York, 1965) 440 pp.

[31] A.I. Kropotov, *Nikolay Yakovlevich Sonin, 1849–1915* (Nauka, Leningrad, 1967) 135 pp.

[32] B. Mayerhofer, Das Prinzip der kleinsten Wirkung bei Hermann von Helmholtz, *Centaurus* 37 (1994) 304–320.

— — — — — — — — — —

[33] *Jahrbücher über die Fortschritte der Mathematik*,
`http://www.emis.de/MATH/JFM/JFM.html`.

[34] *The MacTutor History of Mathematics archive*,
`http://www-history.mcs.st-andrews.ac.uk/history/`

[35] *Homo Heidelbergensis mathematicus*,
`http://www.ub.uni-heidelberg.de/helios/fachinfo/www/math/homoheid.htm`

List of Supplements

Supplement 1 *Karl Böhm*

– Die Existenzbedingungen eines von den ersten und zweiten Differentialquotienten der Coordinaten abhängigen kinetischen Potentials, *Journal für die reine und angewandte Mathematik* 121 (1899) (2) 124–140.

Supplement 2 *Gaston Darboux*

– *Leçons sur la Théorie Générale des Surfaces* III. (Gauthier–Villars, Paris, 1894).

Supplement 3 *Hermann von Helmholtz:*

– Über die physikalische Bedeutung des Princips der kleinsten Wirkung, in: *Wissenschaftliche Abhandlungen* III. volume (J.A. Barth, Leipzig, 1895) 203–248.

– Über die physikalische Bedeutung des Princips der kleinsten Wirkung (Aus den hinterlassenen Papieren bearbeitet von Leo Königsberger), *Sitzungsberichte der königliche Preussische Akademie der Wissenschaften zu Berlin* 1905 (26. October 1905) 863–883.

Supplement 4 *Arthur Hirsch*

– Über eine charakteristische Eigenschaft der Differentialgleichungen der Variationsrechnung, *Mathematische Annalen* 49 (1897) 49–72.

– Die Existenzbedingungen des verallgemeinerten kinetischen Potentials, *Mathematische Annalen* 50 (1898) 429–441.

Supplement 5 *Leo Königsberger*

– Über die Prinzipen der Mechanik, *Sitzungsberichte der kgl. Preuss. Akademie der Wissenschaften zu Berlin* 1886 (30. Juli 1896) 899–944. (the pages 932–935 only)

Supplement 6 *Adolph Mayer*

– Die Existenzbedingungen eines kinetischen Potentiales, *Berichte über die Verhandlungen der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig,* Mathematisch-Physische Classe 48 (1896) 519–529.

Supplement 7 *Nikolay Sonin*

– Ob opredlenij maximalnych i minimalnych svojstv ploskich krivich. *Warsawskye Universitetskye Izvestiya* (1886) (1–2) 1–68.