

Kočka leze dírou, pes oknem - Fourierova transformace a její využití

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... aneb jak jsem získal srdce své ženy 😊

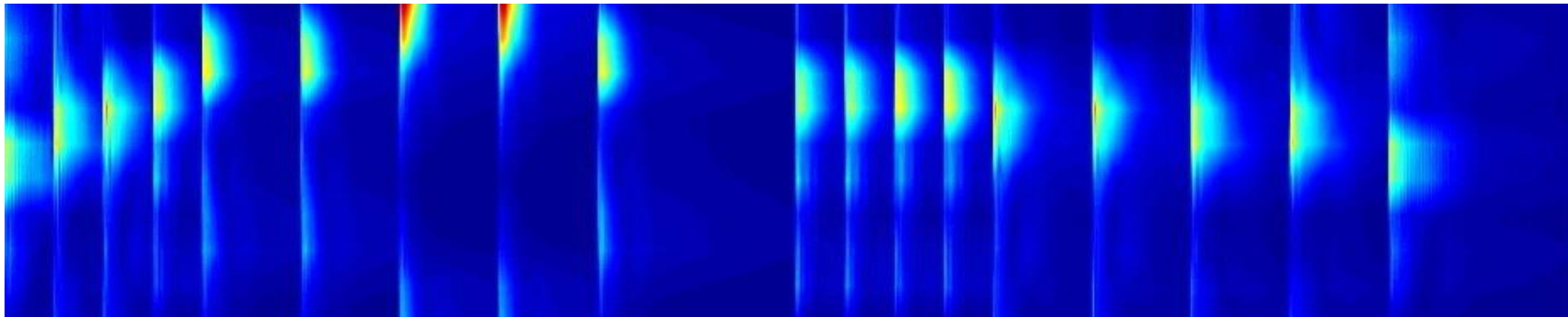
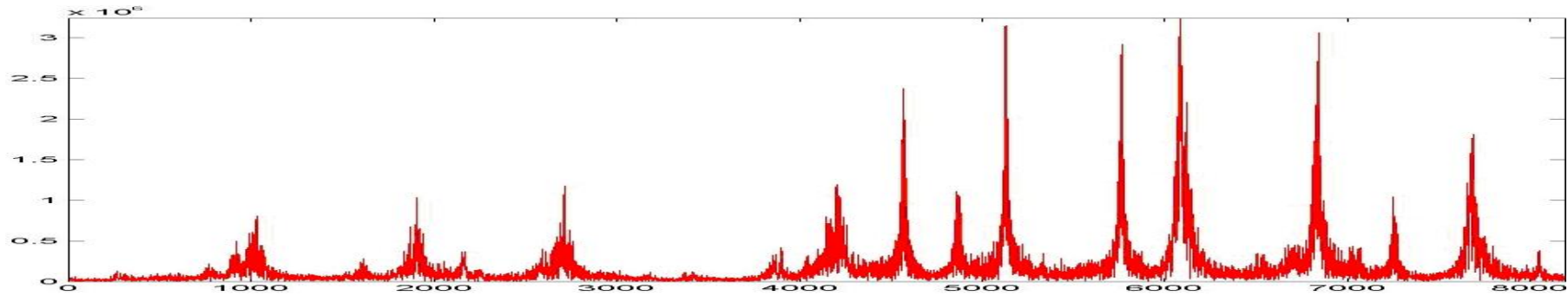
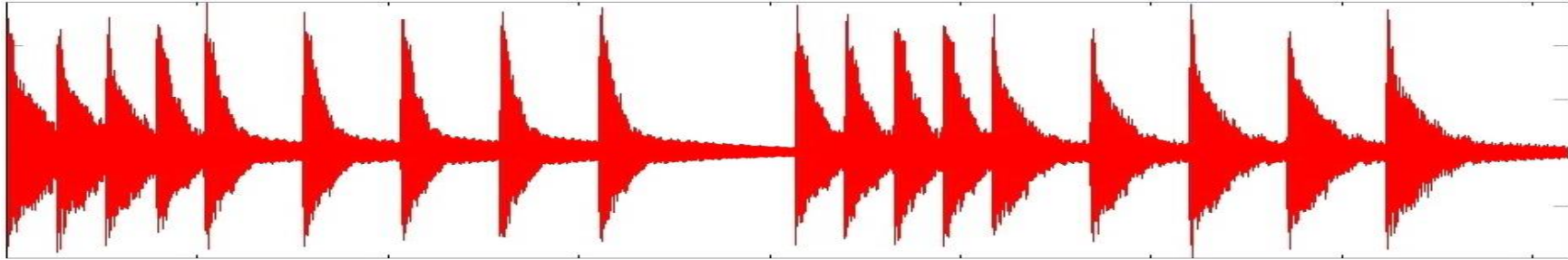
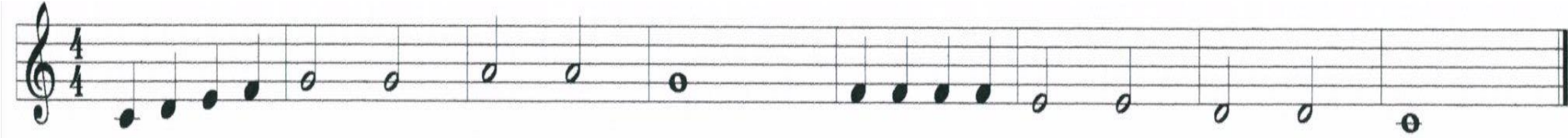


Díky paní doc. Ing. Nině Častové, CSc.!!!



Motivace

Ko-čka le-ze dí - rou, pes ok - nem, ne-bu-de-li pr - šet, ne - zmo - knem



Integrální, diskretizované a diskrétní transformace

Integrální transformace, $L^2(I)$: $F(q) = \int_I f(t) \bar{\chi}(t, q) dt = \langle f(t), \chi(t, q) \rangle$
 $f(t) \leftrightarrow F(q)$

$$f(t) = \int_{\hat{I}} F(q) \chi(t, q) dq = \langle F(q), \bar{\chi}(t, q) \rangle$$

Diskrétní časová (DIT), l^2 : $F(q) = \sum_{n=0}^{\infty} f_n \bar{\chi}_n(q) = \langle \{f_n\}_{n=0}^{\infty}, \chi_n(q) \rangle$

$$\{f_n\}_{n=0}^{\infty} \leftrightarrow F(q)$$

$$\{f_n\}_{n=0}^{\infty}, f_n = \int_{\hat{I}} F(q) \chi_n(q) dq = \langle F(q), \bar{\chi}_n(q) \rangle$$

Diskretizovaná IT, l^2 : $\{F_n\}_{n=0}^{\infty} = \sum_{n=0}^{\infty} f_n \{\bar{\chi}_{n,k}\}_{k=0}^{\infty}, F_n = \sum_{k=0}^{\infty} f_k \bar{\chi}_{n,k} = \langle \{f_k\}_{k=0}^{\infty}, \{\chi_{n,k}\}_{k=0}^{\infty} \rangle$

(Diskrétní FŘ v l^2)

$$\{f_n\}_{n=0}^{\infty} \leftrightarrow \{F_n\}_{n=0}^{\infty}$$

$$\{f_n\}_{n=0}^{\infty} = \sum_{n=0}^{\infty} F_n \{\chi_{n,k}\}_{k=0}^{\infty}, f_n = \sum_{k=0}^{\infty} F_k \chi_{n,k} = \langle \{F_k\}_{k=0}^{\infty}, \{\bar{\chi}_{n,k}\}_{k=0}^{\infty} \rangle$$

Diskrétní transformace, $l^2(N)=\mathbf{C}^N$: $\mathbf{F} = (F_0, F_1, \dots, F_{N-1})^T = \sum_{n=0}^{N-1} f_n \bar{\chi}_n, F_n = \sum_{k=0}^{N-1} f_k \bar{\chi}_{n,k} = \langle \mathbf{f}, \chi_n \rangle$

$$\mathbf{F} = \mathbf{X} \mathbf{f}$$

(Diskrétní FŘ v $l^2(N)$)

$$\mathbf{f} \leftrightarrow \mathbf{F}$$

$$\mathbf{f} = (f_0, f_1, \dots, f_{N-1})^T = \sum_{n=0}^{N-1} F_n \chi_n, f_n = \sum_{k=0}^{N-1} F_k \chi_{n,k} = \langle \mathbf{F}, \bar{\chi}_n \rangle$$

$$\mathbf{f} = \mathbf{X}^{-1} \mathbf{F} = \bar{\mathbf{X}}^T \mathbf{F}$$

Jean-Baptiste Joseph Fourier (1768-1830)

- francouzský matematik a fyzik, který se nejvíce proslavil zkoumáním Fourierových řad (FŘ) a jejich aplikací k problémům toků tepla (1807, *Treatise on the propagation of heat in solid bodies*).
- Na jeho počest byla také nazvána jeho jménem Fourierova transformace (FT).



Fourierova řada (FŘ)

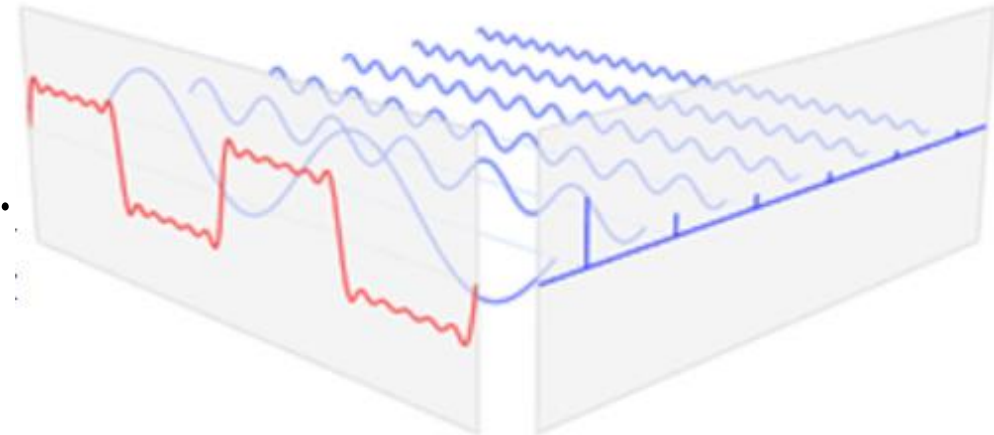
Věta: Necht' $\{\chi_0(t), \chi_1(t), \dots, \chi_n(t), \dots\}$ je ortonormální systém $(\langle \chi_m(t), \chi_n(t) \rangle = \int_0^T \chi_m(t) \bar{\chi}_n(t) dt = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases})$ v prostoru $L^2(0, T)$. Pak platí: pro $\forall f(t) \in L^2(0, T)$ existuje:

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N \langle f(t), \chi_n(t) \rangle \chi_n(t) \stackrel{\text{ozn.}}{=} \sum_{n=0}^{\infty} \langle f(t), \chi_n(t) \rangle \chi_n(t) \quad \dots \text{tzv. (zobecněná) Fourierova řada } f(t).$$

Definice: Platí-li $\forall f(t) \in L^2(0, T)$: $f(t) = \sum_{n=0}^{\infty} \langle f(t), \chi_n(t) \rangle \chi_n(t)$, nazveme $\{\chi_0(t), \chi_1(t), \dots, \chi_n(t), \dots\}$ ortonormální bází $L^2(0, T)$.

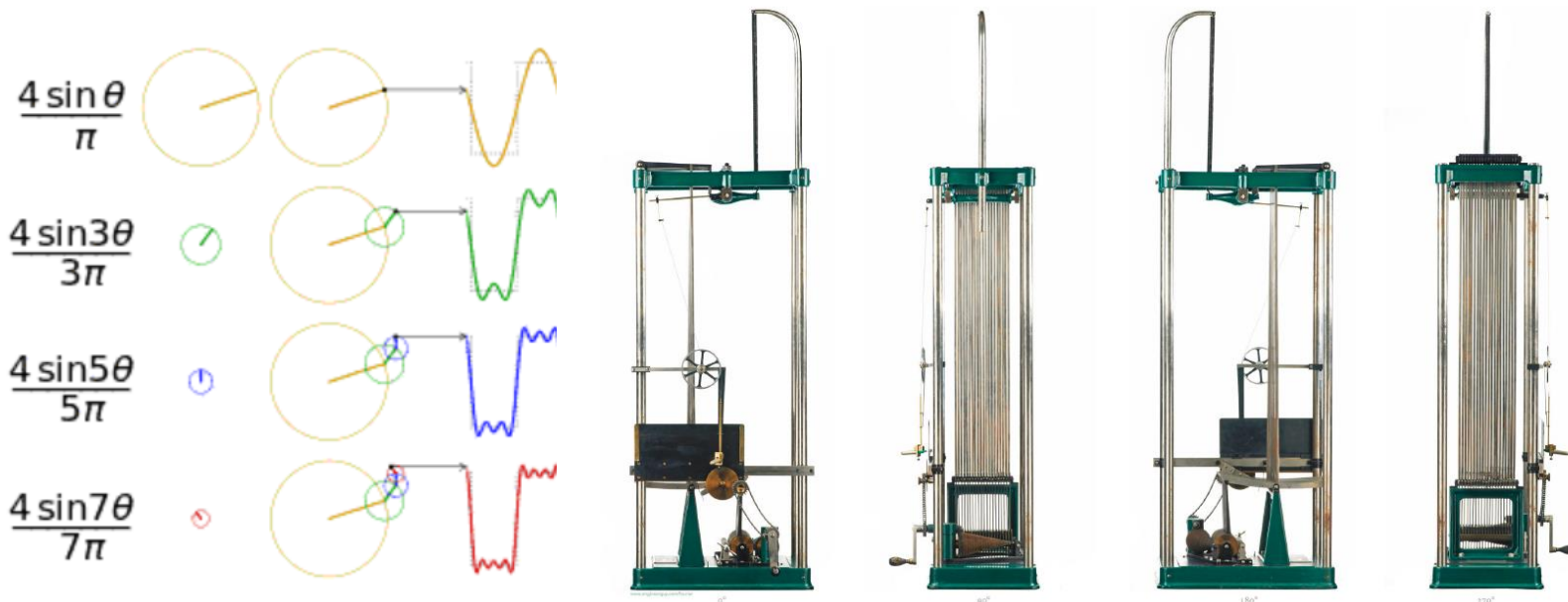
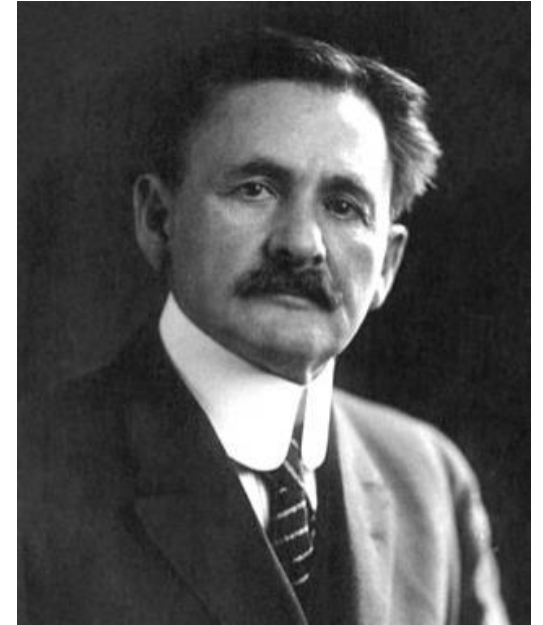
Pozn.: Lze dokázat, že systém $\left\{ \frac{e^{i\omega t}}{\sqrt{T}} \right\}_{n=0}^{\infty}$ je ortonormální bází $L^2(0, T)$ a danou řadu nazveme trigonometrickou FŘ v komplexním tvaru:

$$f(t) = \sum_{-\infty}^{+\infty} c_n e^{i\omega t}, \quad c_n = \frac{1}{T} \langle f(t), e^{i\omega t} \rangle = \frac{1}{T} \int_0^T f(t) e^{-i\omega t} dt, \quad \omega = \frac{2\pi}{T}.$$

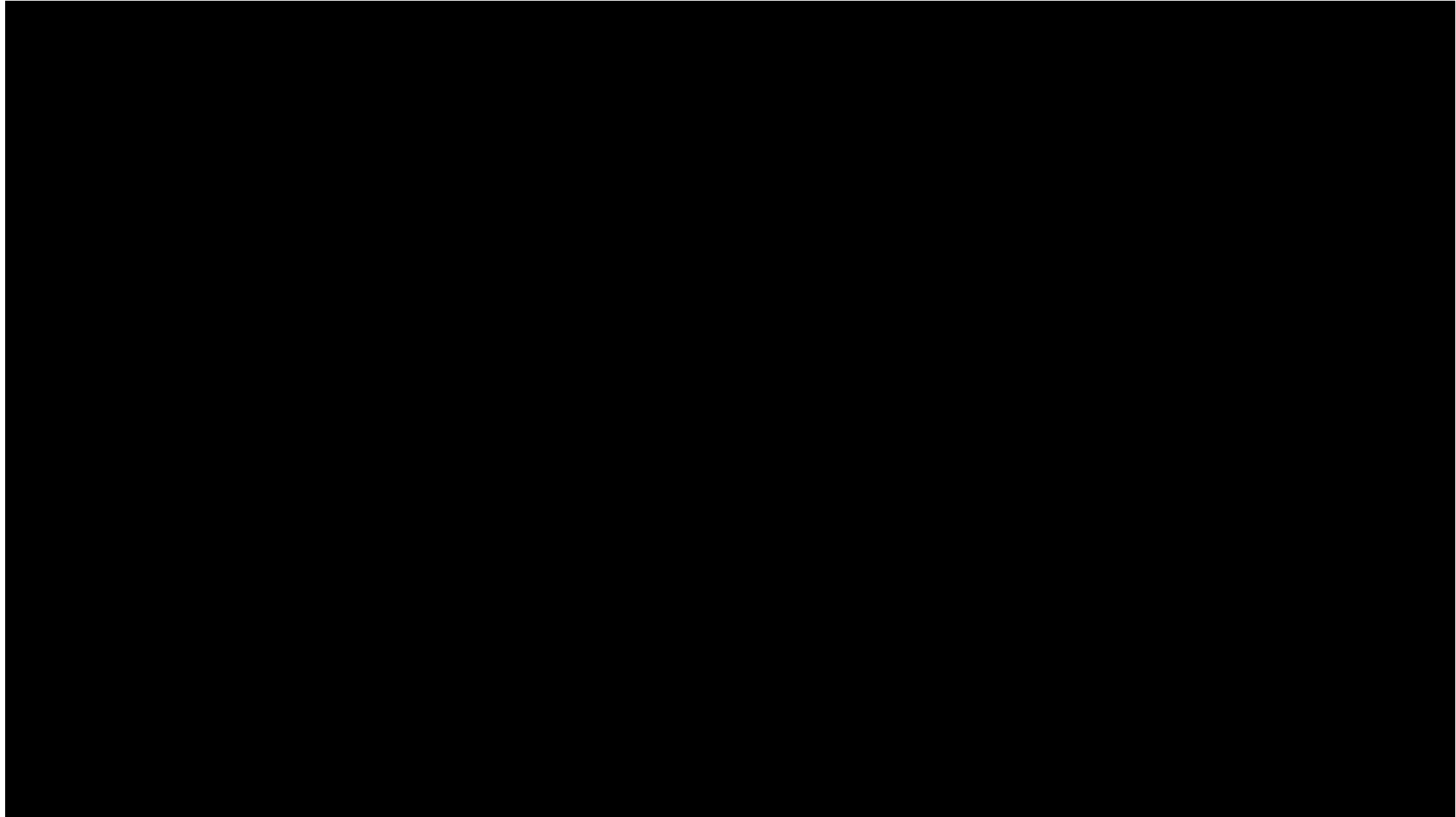


Albert Michelson (1852-1931) - Harmonický analyzátor

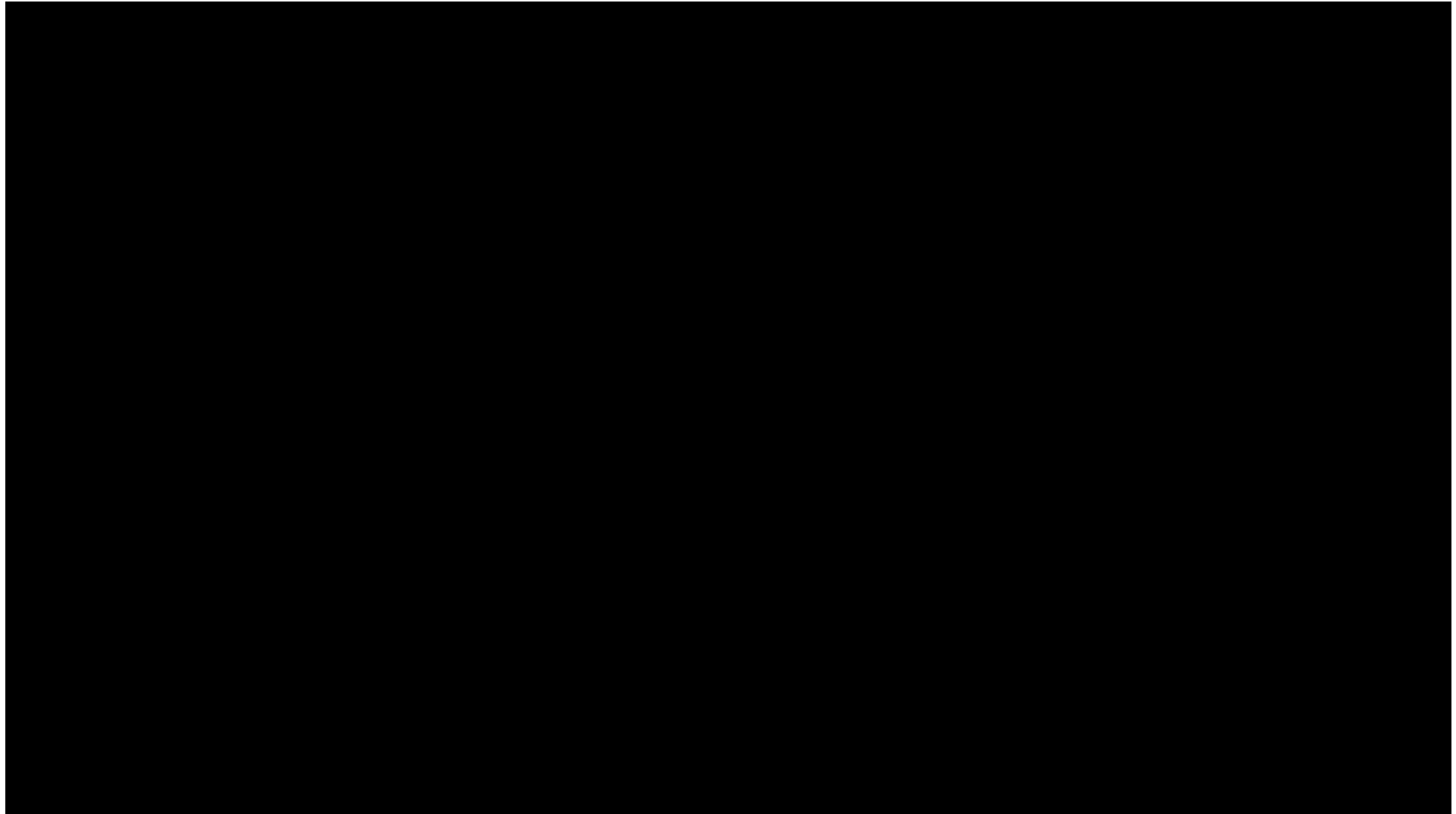
- americký fyzik narozený v Polsku do židovské rodiny, 1907 Nobelova cena za fyziku za své přesné optické přístroje, světlo se šíří všemi směry stejnou, neměnnou rychlostí
- 1897 navrhl harmonický analyzátor, vyrobeny min. 4 stroje
- „ This 19th century mechanical marvel does Fourier analysis: it can find the frequency components of a signal using only gears, springs and levers.“, engineerguy.com/fourier



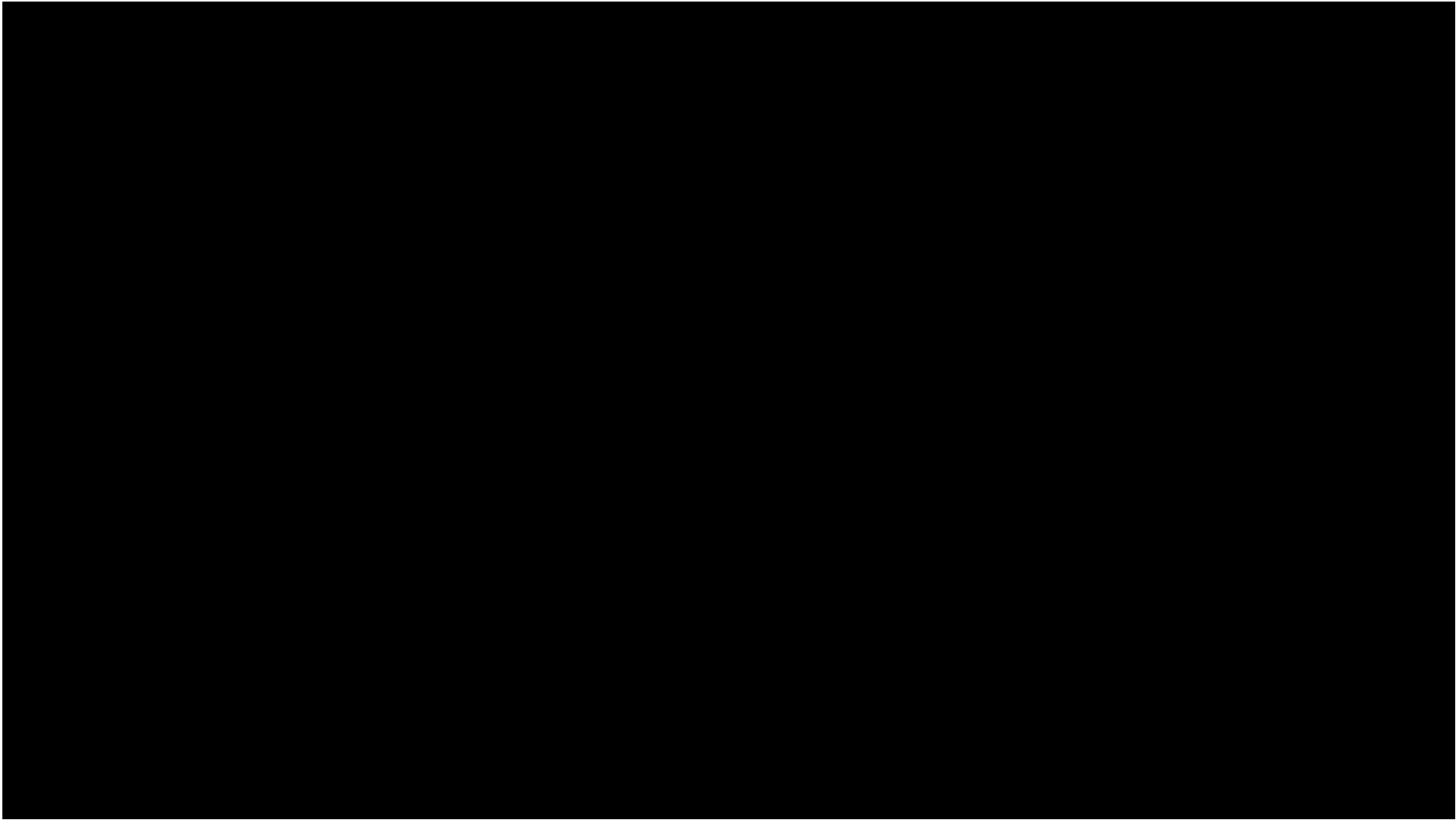
Harmonický analyzátor - Historie



Harmonický analyzátor - Syntéza



Harmonický analyzátor - Analýza



Od Fourierovy řady k Fourierově transformaci (FT)

FT je rozšíření FŘ skrze prodlužování periody $T=2L$ limitně až do nekonečna

$$f(t) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(\frac{1}{2L} \int_{-L}^{+L} f(x) e^{-i\omega_n x} dx \right)}_{c_n} e^{i\omega_n t} = \sum_{n=-\infty}^{+\infty} \frac{1}{2L} \int_{-L}^{+L} f(x) e^{i\omega_n(t-x)} dx$$

$T \rightarrow \infty \Rightarrow L \rightarrow \infty \Rightarrow \Delta\omega = \omega(n+1) - \omega n = \pi/L \rightarrow 0$, sumace přejde v integrál:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\left(\int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx \right)}_{F(\omega) = FT\{f\}} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega.$$

Diskrétní Fourierova transformace (DFT)

C_n DFŘ je totožný s F_n DFT: $\mathbf{F} = (F_0, F_1, \dots, F_{N-1})^T = \sum_{n=0}^{N-1} f_n \bar{\chi}_n, \quad F_n = \sum_{k=0}^{N-1} f_k \bar{\chi}_{n,k} = \langle \mathbf{f}, \chi_n \rangle$

Označme $w = e^{-i\frac{2\pi}{N}}$, pak $e^{-i\frac{2\pi}{N}nk} = w^{nk}$ a $\bar{\chi}_n = (w^{n0}, w^{n1}, \dots, w^{n(N-1)})^T$

$w^k, k=0,1,2,\dots,N-1$ jsou kořeny rovnice $z^N - 1 = 0$ ležící v komplexní rovině na kružnici o poloměru 1

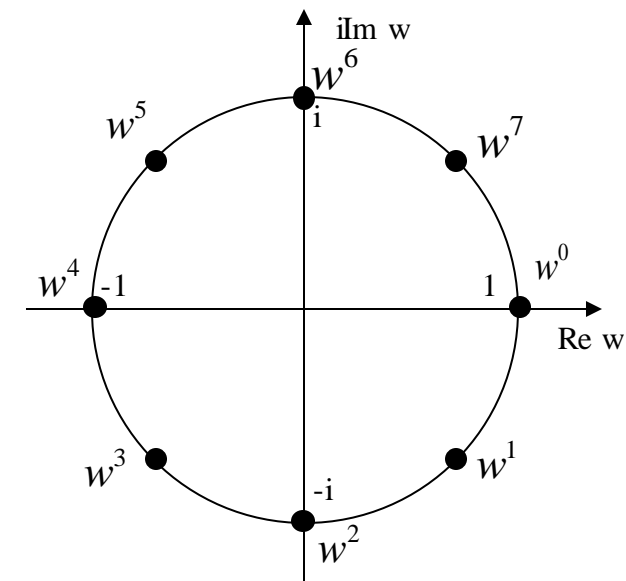
Přímá DFT{ \mathbf{f} }: $\mathbf{F} = \mathbf{W}\mathbf{f}$, zpětná DFT{ \mathbf{F} }: $\mathbf{f} = \frac{1}{N} \mathbf{W}^T \mathbf{F}$

$$\mathbf{X} = \mathbf{W} = \begin{pmatrix} \bar{\chi}_0^T \\ \bar{\chi}_1^T \\ \bar{\chi}_2^T \\ \vdots \\ \bar{\chi}_{N-1}^T \end{pmatrix} = \begin{pmatrix} w^0 & w^0 & w^0 & \dots & w^0 \\ w^0 & w^1 & w^2 & \dots & w^{N-1} \\ w^0 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)(N-1)} \end{pmatrix}$$

Pro $N=8$: $w^0 = 1 = -w^4$ $w^1 = \frac{\sqrt{2}}{2}(1-i) = -w^5$ $w^2 = -i = -w^6$ $w^3 = -\frac{\sqrt{2}}{2}(1+i) = -w^7$

$$(w^0, w^1, w^2, w^3, w^4, w^5, w^6, w^7) = (1, w^1, w^2, w^3, -1, -w^1, -w^2, -w^3)$$

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^1 & w^2 & w^3 & -1 & -w^1 & -w^2 & -w^3 \\ 1 & w^2 & -1 & -w^2 & 1 & w^2 & -1 & -w^2 \\ 1 & w^3 & -w^2 & w^1 & -1 & -w^3 & w^2 & -w^1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -w^1 & w^2 & -w^3 & -1 & w^1 & -w^2 & w^3 \\ 1 & -w^2 & -1 & w^2 & 1 & -w^2 & -1 & w^2 \\ 1 & -w^3 & -w^2 & -w^1 & -1 & w^3 & w^2 & w^1 \end{pmatrix}$$



Rychlá Fourierova transformace (FFT): $O(N^2) \rightarrow O(N \log N)$

C. F. Gauss vs. J. W. Cooley & J. W. Tuckey (1965)

Algoritmus včetně jeho rekurzivní aplikace navržen Carl Friedrichem Gaussem pro trigonometrickou interpolaci trajektorie asteroidů Pallas a Juno, publikován v 1805 v *Theoria Interpolationis Methodo Nova Tractata* (až 1807 J. Fourier publikuje FŘ)

Mezi Top 10 algoritmy s největším vlivem na rozvoj vědy a inženýrství v 20. století



An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N -vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2 . These methods are applied here to the calculation of complex Fourier series. They are useful in situations where the number of data points is, or can be chosen to be, a highly composite number. The algorithm is here derived and presented in a rather different form. Attention is given to the choice of N . It is also shown how special advantage can be obtained in the use of a binary computer with $N = 2^m$ and how the entire calculation can be performed within the array of N data storage locations used for the given Fourier coefficients.

Consider the problem of calculating the complex Fourier series

$$(1) \quad X(j) = \sum_{k=0}^{N-1} A(k) \cdot W^{jk}, \quad j = 0, 1, \dots, N-1,$$

where the given Fourier coefficients $A(k)$ are complex and W is the principal N th root of unity,

$$(2) \quad W = e^{2\pi i/N}.$$

A straightforward calculation using (1) would require N^2 operations where "operation" means, as it will throughout this note, a complex multiplication followed by a complex addition.

The algorithm described here iterates on the array of given complex Fourier amplitudes and yields the result in less than $2N \log_2 N$ operations without requiring more data storage than is required for the given array A . To derive the algorithm, suppose N is a composite, i.e., $N = r_1 \cdot r_2$. Then let the indices in (1) be expressed

$$(3) \quad \begin{aligned} j &= j_1 r_1 + j_0, & j_0 &= 0, 1, \dots, r_1 - 1, & j_1 &= 0, 1, \dots, r_2 - 1, \\ k &= k_1 r_2 + k_0, & k_0 &= 0, 1, \dots, r_2 - 1, & k_1 &= 0, 1, \dots, r_1 - 1. \end{aligned}$$

Then, one can write

$$(4) \quad X(j_1, j_0) = \sum_{k_0} \sum_{k_1} A(k_1, k_0) \cdot W^{j_1 r_2 k_1 + j_0 k_0}.$$

Received August 17, 1964. Research in part at Princeton University under the sponsorship of the Army Research Office (Durham). The authors wish to thank Richard Garwin for his essential role in communication and encouragement.

Jiné: Prime-factor FFT, Bruun's FFT, Rader's FFT, Bluestein's FFT, Hexagonal FFT

Algoritmus FFT – Radix-2

Splitting method - alg. publikován 1965 J.W.Cooleyem a J.W. Tukeyem po názvem Butterfly (motýlek) pro $N=2^m$, $m \in \mathbb{N}$.

Rozdělením vektoru $\mathbf{f} = (f_0, f_1, \dots, f_{N-1})^T$ majícího N složek na 2 vektory o $N/2$ složkách se sudými indexy $y_k = f_{2k}$: $\mathbf{y} = (f_0, f_2, \dots, f_{N-2})^T$ a lichými indexy $z_k = f_{2k+1}$: $\mathbf{z} = (f_1, f_3, \dots, f_{N-1})^T$, pak pro $n = 0, 1, 2, \dots, N-1$, $w = w_N = e^{-i2\pi/N}$

$$F_n = \sum_{k=0}^{N-1} f_k e^{-ink2\pi/N} = \sum_{k=0}^{N-1} f_k w^{nk} = f_0 w^0 + f_1 w^n + \dots + f_{N-1} w^{n(N-1)} =$$

$$= (f_0 w^0 + f_2 w^{2n} + \dots + f_{N-2} w^{(N-2)n}) + w^n (f_1 w^0 + f_3 w^{2n} + \dots + f_{N-1} w^{n(N-2)}) = \sum_{k=0}^{\frac{N}{2}-1} [y_k w_N^{2nk} + z_k w_N^{(2k+1)n}]$$

V důsledku symetrie komplex. čísel (koeficientů na kružnici) :

$$w^{2nk} = w_N^{2nk} = w_{N/2}^{nk} = e^{-i4\pi nk/N} = e^{-i2\pi nk/(N/2)} \quad (\text{obecně } w_N^{pq} = w_{N/q}^p)$$

platí:

$$F_n = \underbrace{\sum_{k=0}^{\frac{N}{2}-1} y_k w_{N/2}^{nk}}_{Y_n} + w_N^n \underbrace{\sum_{k=0}^{\frac{N}{2}-1} z_k w_{N/2}^{nk}}_{Z_n} \quad n = 0, 1, 2, \dots, N-1,$$

$$\mathbf{Y} = DFT \{ \mathbf{y} \} \quad \mathbf{Z} = DFT \{ \mathbf{z} \}$$

1. polovina koeficientů: $F_n = Y_n + w_N^n Z_n \quad n = 0, 1, 2, \dots, \left(\frac{N}{2} - 1\right)$

2. polovina koeficientů: $F_{n+\frac{N}{2}} = Y_{n+\frac{N}{2}} + w_N^{(n+\frac{N}{2})} Z_{n+\frac{N}{2}} = Y_n - w_N^n Z_n \quad \text{neboť: } w_N^{\frac{N}{2}} = e^{-i\frac{2\pi}{N} \cdot \frac{N}{2}} = e^{-i\pi} = -1$

Maticový zápis FFT

$$F_n = Y_n + w_N^n Z_n$$

$$F_{n+\frac{N}{2}} = Y_n - w_N^n Z_n$$

$$DFT(y_0) = Y_0 = \sum_{k=0}^0 y_0 w_1^0 = y_0$$

$$DFT(z_0) = Z_0 = \sum_{k=0}^0 z_0 w_1^0 = z_0$$

Princip dělení posl. se opakuje dokud nedostaneme jeden prvek, pak:

$$\mathbf{F} = \mathbf{W}_N \cdot \mathbf{f} = \begin{pmatrix} \mathbf{I}_{\frac{N}{2}} & \mathbf{D}_{\frac{N}{2}} \\ \mathbf{I}_{\frac{N}{2}} & -\mathbf{D}_{\frac{N}{2}} \end{pmatrix} \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \mathbf{B}_N \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$$

$$\mathbf{I}_r, \mathbf{D}_r = \text{diag}(1, w_{2r}^1, w_{2r}^2, \dots, w_{2r}^{r-1}) \quad \text{matice řádu } r \quad w_N = e^{-i2\pi/N}$$

Další krok:

$$\mathbf{Y} = \begin{pmatrix} \mathbf{I}_{\frac{N}{4}} & \mathbf{D}_{\frac{N}{4}} \\ \mathbf{I}_{\frac{N}{4}} & -\mathbf{D}_{\frac{N}{4}} \end{pmatrix} \begin{pmatrix} \mathbf{Y}' \\ \mathbf{Z}' \end{pmatrix} = \mathbf{B}_{N/2} \begin{pmatrix} \mathbf{Y}' \\ \mathbf{Z}' \end{pmatrix}$$

$$\mathbf{Z} = \begin{pmatrix} \mathbf{I}_{\frac{N}{4}} & \mathbf{D}_{\frac{N}{4}} \\ \mathbf{I}_{\frac{N}{4}} & -\mathbf{D}_{\frac{N}{4}} \end{pmatrix} \begin{pmatrix} \mathbf{Y}'' \\ \mathbf{Z}'' \end{pmatrix} = \mathbf{B}_{N/2} \begin{pmatrix} \mathbf{Y}'' \\ \mathbf{Z}'' \end{pmatrix}$$

...

Pak:

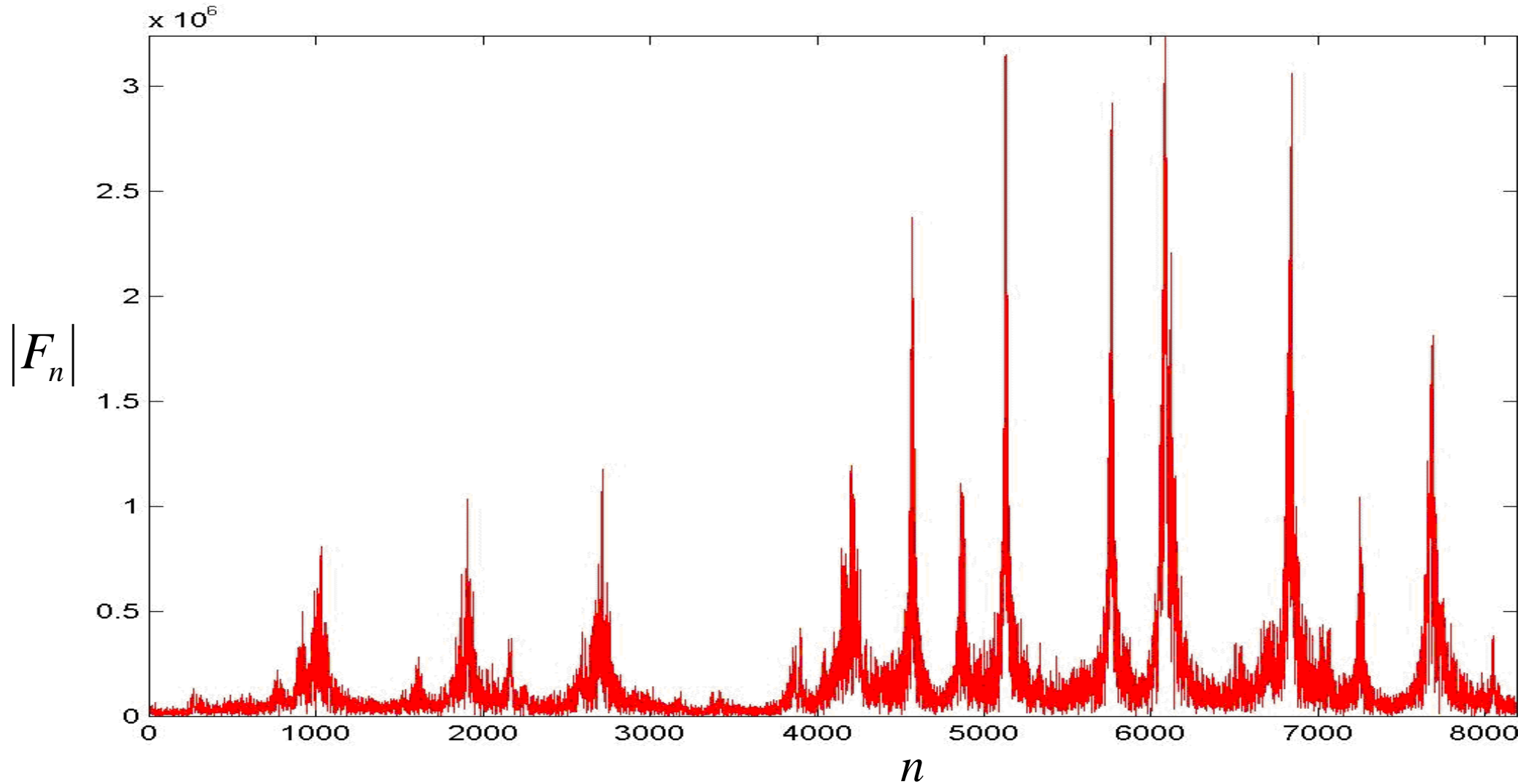
$$\mathbf{F} = \mathbf{B}_N \begin{pmatrix} \mathbf{B}_{N/4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{N/4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{N/4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{N/4} \end{pmatrix} \dots$$

$$\begin{pmatrix} \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{B}_2 \end{pmatrix} \mathbf{P}^T \cdot \mathbf{f}$$

Např. $\mathbf{W}_{16} = \mathbf{B}_{16} \mathbf{B}_8 \mathbf{B}_4 \mathbf{B}_2 \mathbf{P}^T$

```
function F=myfft(f);
N=length(f); n=log2(N);
% Vytvoreni permutacniho vektoru pt
% Prenasobovani dilycimi maticemi
F=f(pt);
for i=1:n
    K=2^i; m=N/K; % m pocet bloku B, K rad bloku B
    w=exp(2i*pi/K);
    D=diag(w.(0:-1:-(K/2-1)));
    I=eye(K/2);
    B=[[I,D];[I,-D]];
    V=diags(B,m); % Matice sestavena z m bloku B
    F=V*F;
end;
```


Fourier. jednostranné amplitudové spektrum $|F|$



Okenní Fourierova transformace (WFT)

Definice: Necht' $f(t), g(t) \in L^2(\mathbb{R})$, pak pro τ definujeme WFT jako FT funkce $f_\tau(t) = f(t)\bar{g}(t-\tau) \quad \forall \tau \in \mathbb{R}$:

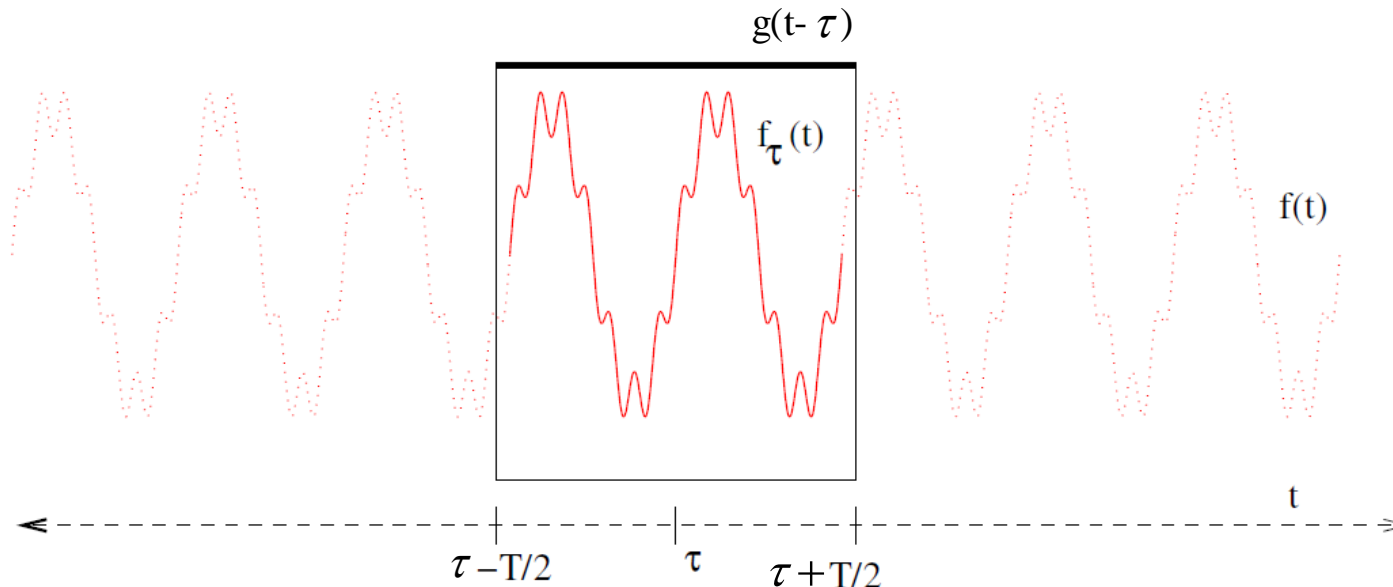
$$WFT\{f\} = F(\omega, \tau) = FT\{f_\tau\} = \int_{-\infty}^{+\infty} f_\tau(t) e^{-i\omega t} dt = \int_{-\infty}^{+\infty} f(t) \bar{g}(t-\tau) e^{-i\omega t} dt$$

a zpětnou WFT:

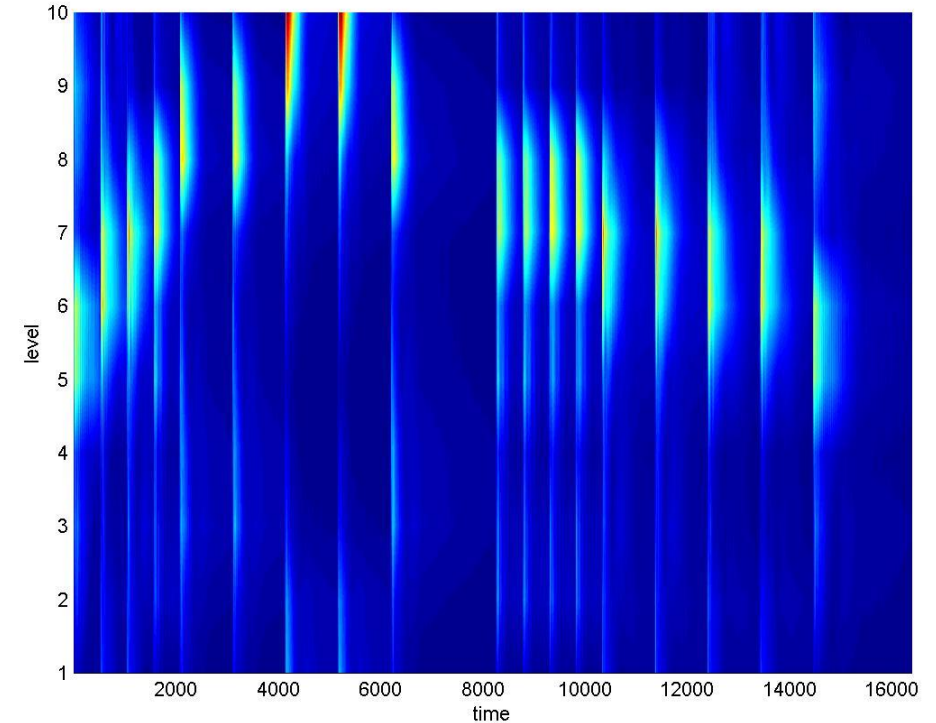
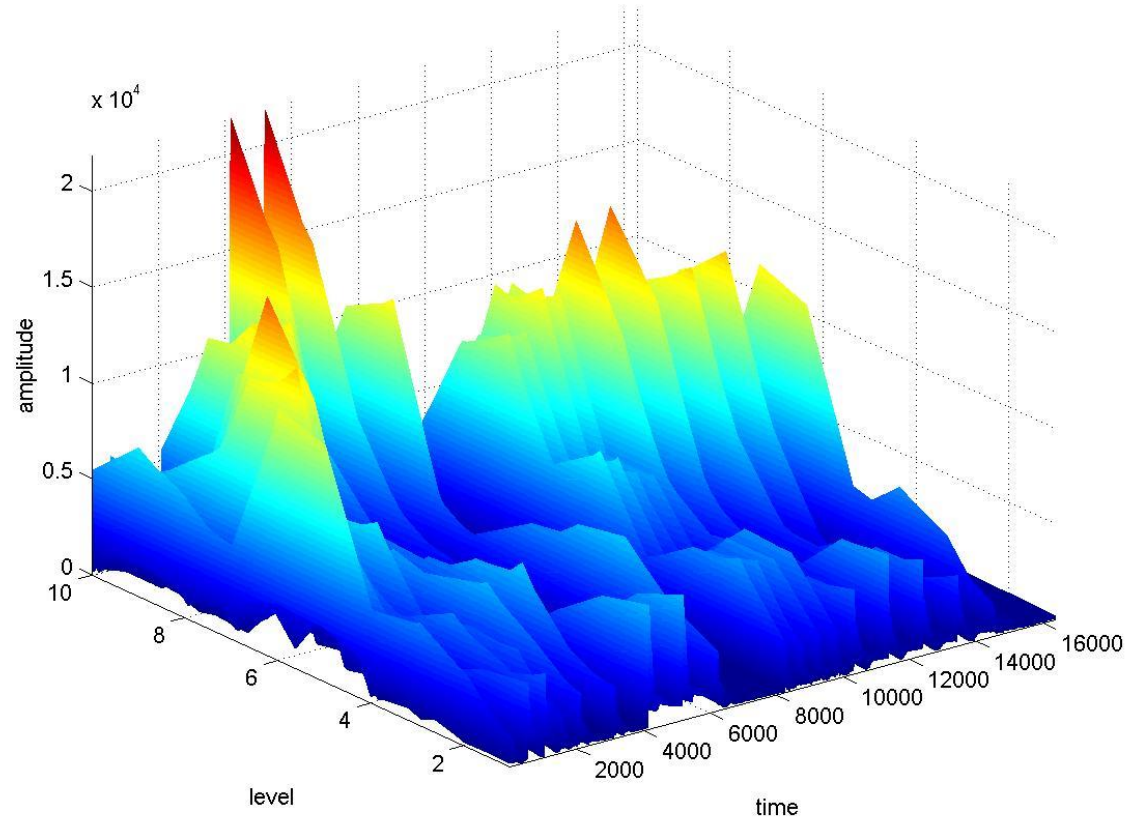
$$WFT^{-1}\{f\} = F^{-1}(\omega, \tau) = f(t) = \frac{1}{2\pi \|g(t)\|^2} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} F(\omega, \tau) g(t-\tau) e^{i\omega t} d\omega \right) d\tau.$$

Generování $g(t)$ je založeno např. na filtračních vlastnostech jádra v konvolučním integrálu, lokálních kosinových řadách, atd.

Vzor v modelu slyšení



WFT – příklad CFS pomocí Hammingova okna



Ukázka přepočtu frekvenčních rozsahů (lidský sluch 20 Hz – 20 kHz):

777492 bodů/17.63 sec -> 44100 bodů/1sec (44,1 kHz) -> max. frekvence (Nyquistova) 22050 Hz.

Okno s 20 body -> 10 frekv. hladin -> 2205 Hz na 1 hladinu.

Další použití FT

- Komprese, filtrace
- Násobení obrovských čísel (šifrování)
- Waveletová transformace
- Kepstrální analýza signálu:

komplexní kepstrum

power kepstrum

snadnější separace hlásek

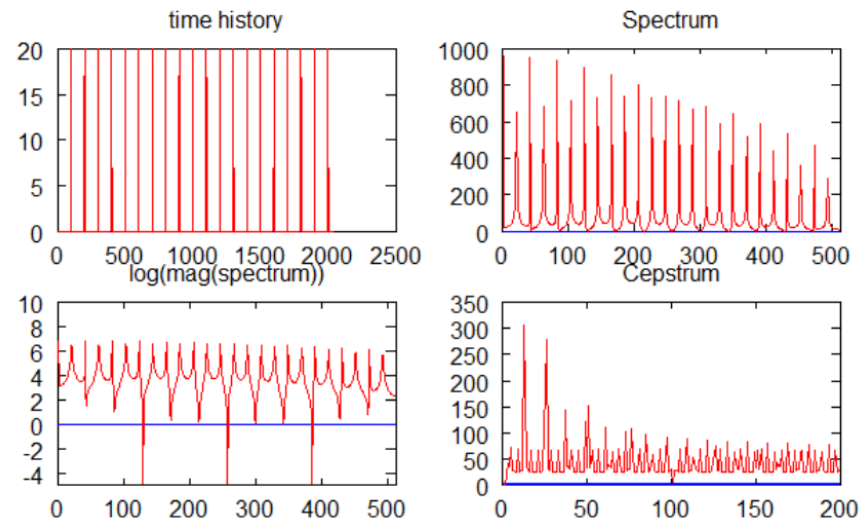
...

$$\mathbf{a}, \mathbf{b}; \mathbf{c} = FFT^{-1} \{ FFT \{ [\mathbf{a} \ \mathbf{o}] \} \cdot FFT \{ [\mathbf{b} \ \mathbf{o}] \} \}$$

$$WT\{f\} = F(a, b) = \int_{-\infty}^{+\infty} f(t) \bar{\psi}(a(t-b)) dt$$

$$CC\{f\} = FT^{-1} \{ \log(FT\{f(t)\}) + 2\pi mi \}$$

$$PC\{f\} = \left| FT^{-1} \{ \log(|FT\{f(t)\}|^2) \} \right|^2$$



Děkuji za pozornost!!!

