Taylorův mnohočlen

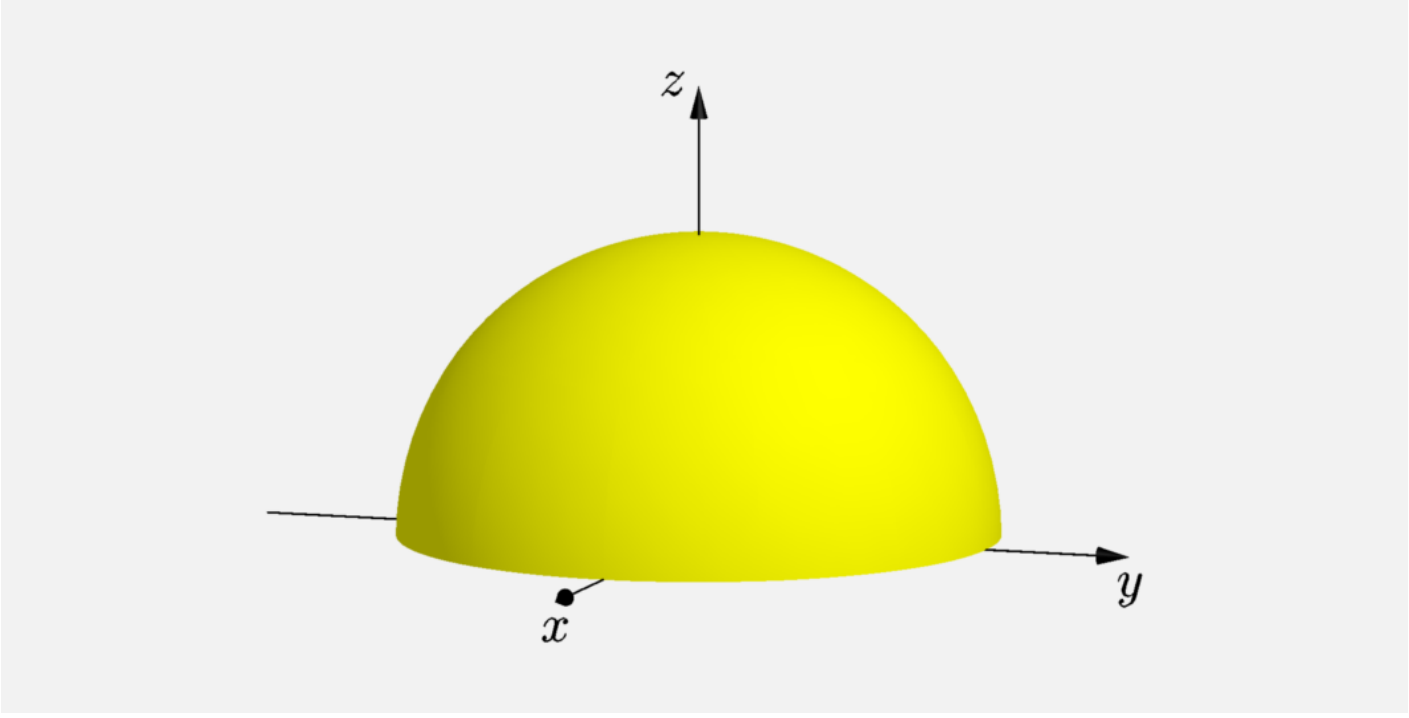
Příklad 2.

Zadání:

Najděte Taylorův mnohočlen prvního, druhého, třetího a čtvrtého řádu funkce

$$f(x, y) = \sqrt{1 - x^2 - y^2}$$

se středem v bodě $S = \left(\frac{1}{2}, \frac{1}{2}\right)$.



Obrázek 1: Graf funkce $f(x, y)$

Řešení:

Pro Taylorův mnohočlen čtvrtého řádu platí

$$\begin{aligned} T_4(x,y) = f\left(\frac{1}{2},\frac{1}{2}\right) + \mathrm{d}f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\ + \frac{1}{2}\mathrm{d}^2 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\ + \frac{1}{6}\mathrm{d}^3 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\ + \frac{1}{24}\mathrm{d}^4 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) \ . \end{aligned}$$

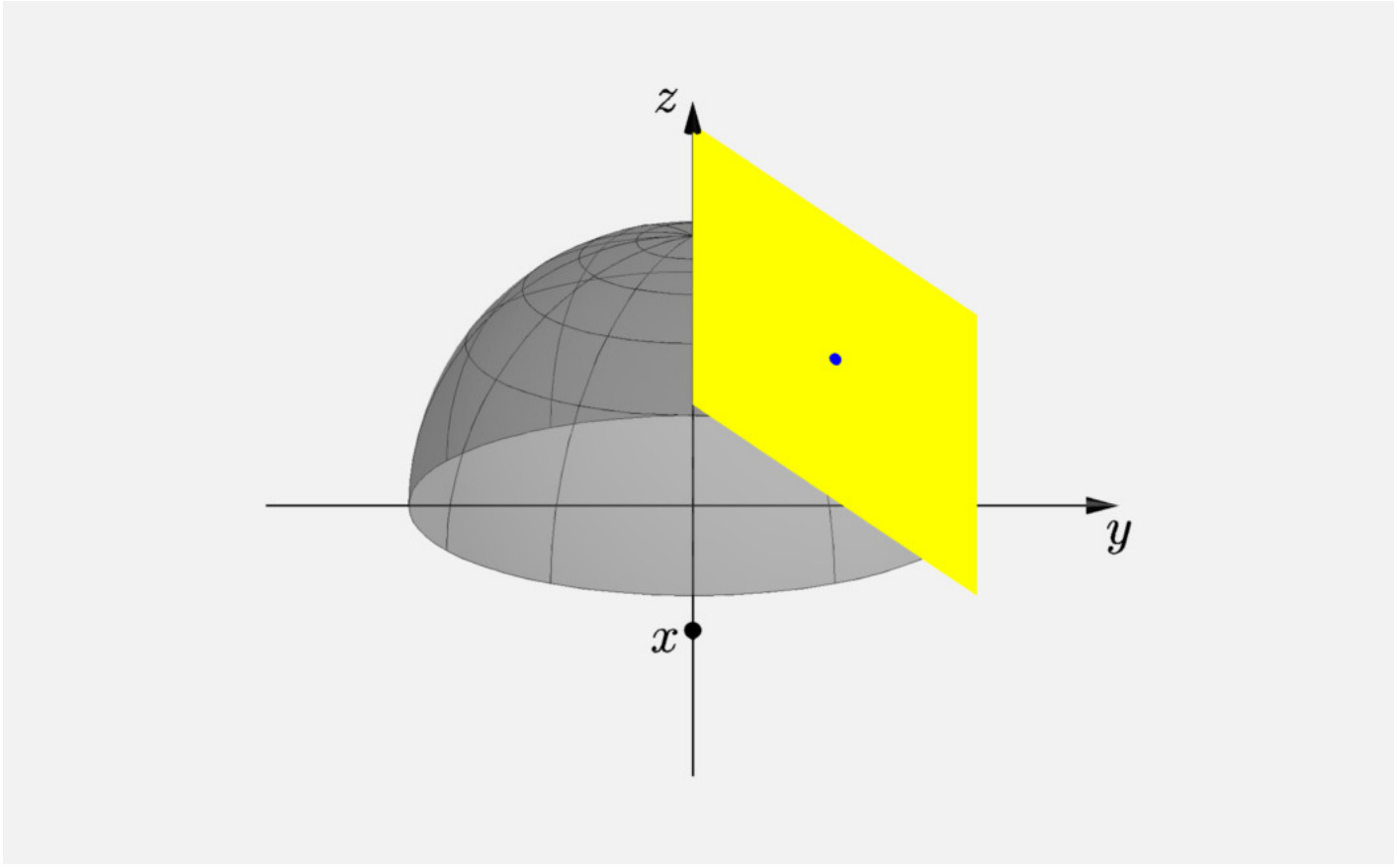
Parciální derivace prvního řádu

$$\frac{\partial f}{\partial x}(x,y) = -\frac{x}{\sqrt{1-x^2-y^2}}\,,$$
$$\frac{\partial f}{\partial y}(x,y) = -\frac{y}{\sqrt{1-x^2-y^2}}\,.$$

Taylorův mnohočlen prvního řádu

$$T_1(x,y) = f\left(\frac{1}{2},\frac{1}{2}\right) + \mathrm{d}f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right)$$

$$T_1(x,y) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(y-\frac{1}{2}\right)\,.$$



Obrázek 2: Taylorův mnohočlen 1. řádu

Parciální derivace druhého řádu

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{-1+y^2}{\sqrt{(1-x^2-y^2)^3}},$$

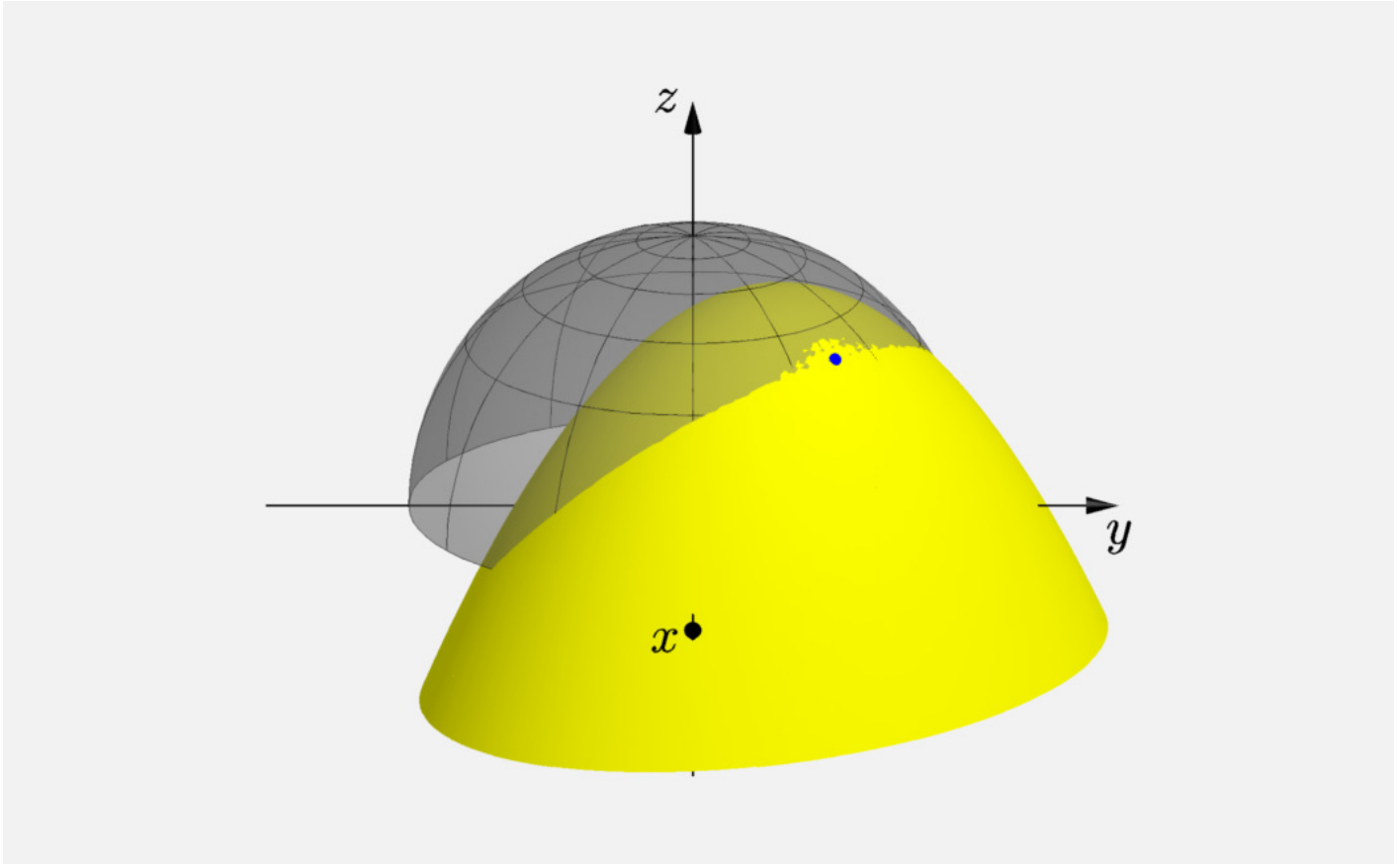
$$\frac{\partial^2 f}{\partial xy}(x,y) = -\frac{xy}{\sqrt{(1-x^2-y^2)^3}},$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{-1+x^2}{\sqrt{(1-x^2-y^2)^3}}.$$

Taylorův mnohočlen druhého řádu

$$T_2(x,y) = f\left(\frac{1}{2},\frac{1}{2}\right) + \mathrm{d}f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\ + \frac{1}{2}\mathrm{d}^2 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right)$$

$$T_2(x,y) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^2 - \\ - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(y-\frac{1}{2}\right)^2 .$$



Obrázek 3: Taylorův mnohočlen 2. řádu

Parciální derivace třetího řádu

$$\frac{\partial^3 f}{\partial x^3}(x,y) = \frac{3x\left(-1+y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^5}},$$

$$\frac{\partial^3 f}{\partial x^2 y}(x,y) = -\frac{y\left(1+2x^2-y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^5}},$$

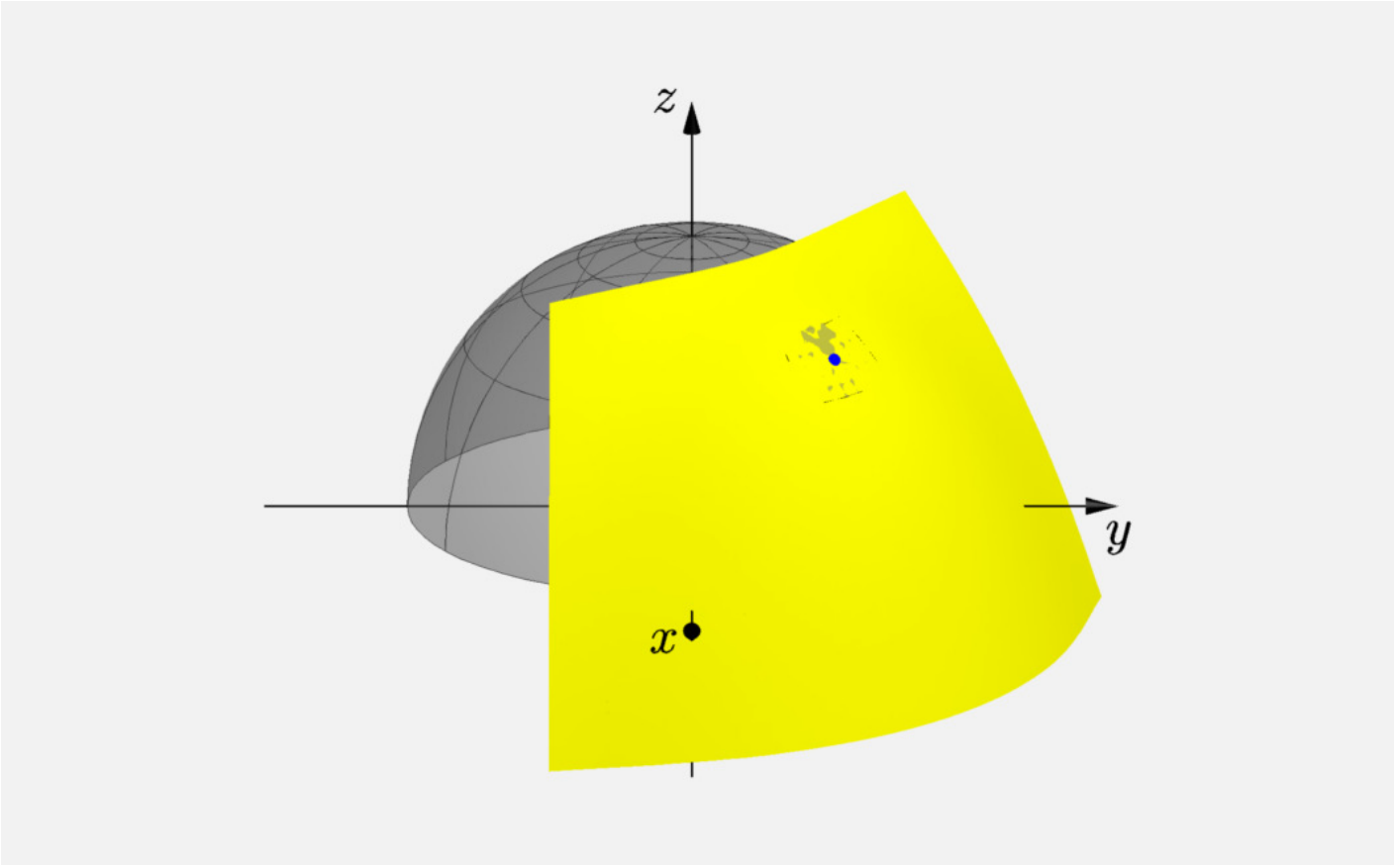
$$\frac{\partial^3 f}{\partial x y^2}(x,y) = \frac{x\left(-1+x^2-2y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^5}},$$

$$\frac{\partial^3 f}{\partial y^3}(x,y) = \frac{3y\left(-1+x^2\right)}{\sqrt{\left(1-x^2-y^2\right)^5}}.$$

Taylorův mnohočlen třetího řádu

$$T_3(x,y) = f\left(\frac{1}{2},\frac{1}{2}\right) + \mathrm{d}f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\ + \frac{1}{2}\mathrm{d}^2 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \frac{1}{6}\mathrm{d}^3 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right)$$

$$T_3(x,y) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^2 - \\ - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(y-\frac{1}{2}\right)^2 - \\ + \frac{3\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^3 - \frac{5\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^2\left(y-\frac{1}{2}\right) - \\ - \frac{5\sqrt{2}}{4}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)^2 - \frac{3\sqrt{2}}{4}\left(y-\frac{1}{2}\right)^3 .$$



Obrázek 4: Taylorův mnohočlen 3. řádu

Parciální derivace čtvrtého řádu

$$\frac{\partial^4 f}{\partial x^4}(x,y) = \frac{3\left(-4x^2+4x^2y^2-1+2y^2-y^4\right)}{\sqrt{\left(1-x^2-y^2\right)^7}},$$

$$\frac{\partial^4 f}{\partial x^3 y}(x,y) = -\frac{3xy\left(2x^2+3-3y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^7}},$$

$$\frac{\partial^4 f}{\partial x^2 y^2}(x,y) = \frac{-x^2+2x^4-11x^2y^2-1-y^2+2y^4}{\sqrt{\left(1-x^2-y^2\right)^7}},$$

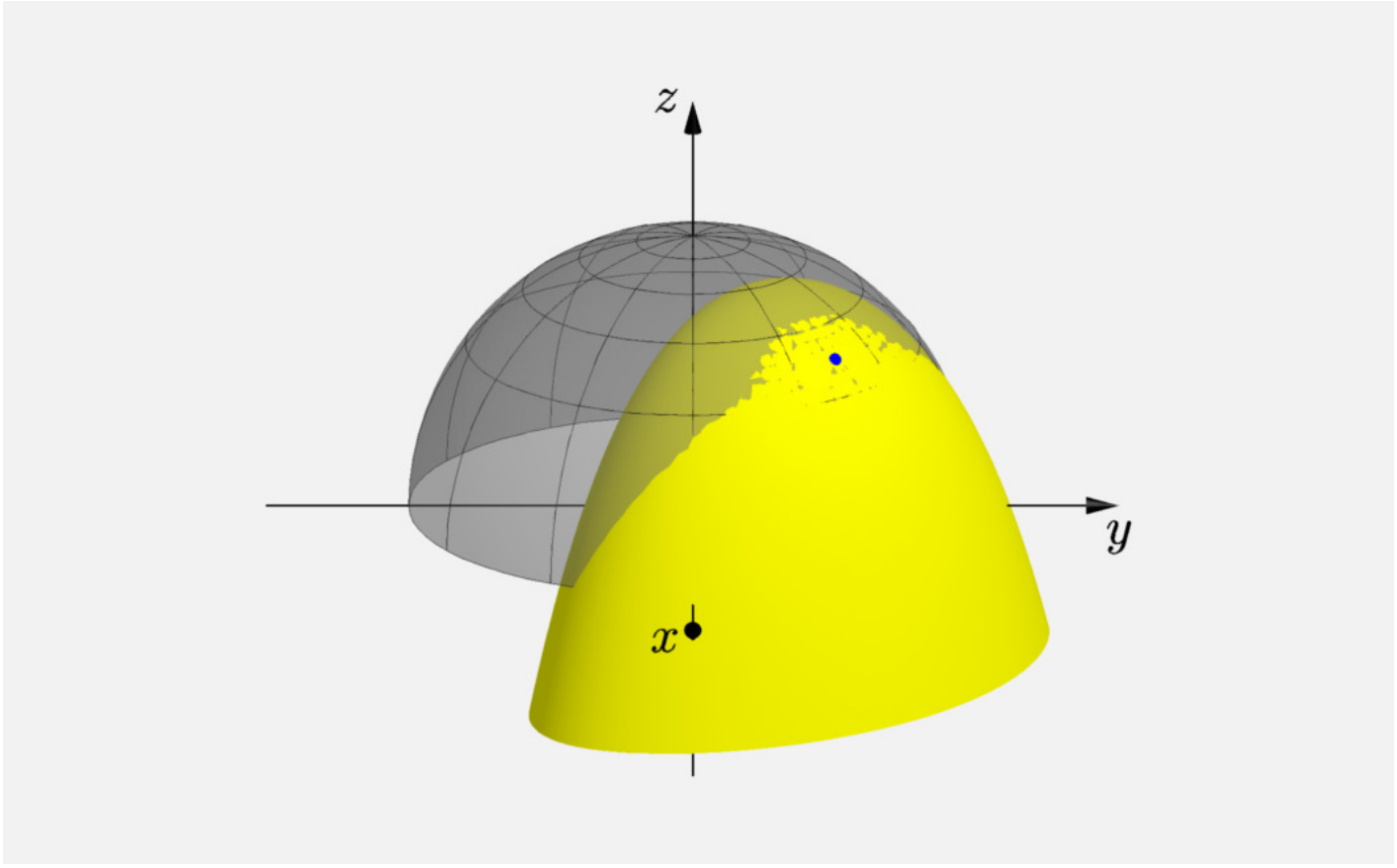
$$\frac{\partial^4 f}{\partial x y^3}(x,y) = \frac{3xy\left(-3+3x^2-2y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^7}},$$

$$\frac{\partial^4 f}{\partial y^4}(x,\ y) = \frac{3\left(1-2x^2+4y^2+x^4-4x^2y^2\right)}{\sqrt{\left(1-x^2-y^2\right)^7}}.$$

Taylorův mnohočlen čtvrtého řádu

$$
\begin{aligned}
T_4(x,y) = f\left(\frac{1}{2},\frac{1}{2}\right) + \mathrm{d}f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\
+ \frac{1}{2}\mathrm{d}^2 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\
+ \frac{1}{6}\mathrm{d}^3 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right) + \\
+ \frac{1}{24}\mathrm{d}^4 f_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(x-\frac{1}{2},y-\frac{1}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
T_4(x,y) = {} & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^2 - \\
& - \frac{\sqrt{2}}{2}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right) - \frac{3\sqrt{2}}{4}\left(y-\frac{1}{2}\right)^2 - \\
& - \frac{3\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^3 - \frac{5\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^2\left(y-\frac{1}{2}\right) - \\
& - \frac{5\sqrt{2}}{4}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)^2 - \frac{3\sqrt{2}}{4}\left(y-\frac{1}{2}\right)^3 - \\
& - \frac{21\sqrt{2}}{16}\left(x-\frac{1}{2}\right)^4 - \frac{11\sqrt{2}}{4}\left(x-\frac{1}{2}\right)^3\left(y-\frac{1}{2}\right) - \\
& - \frac{31\sqrt{2}}{8}\left(x-\frac{1}{2}\right)^2\left(y-\frac{1}{2}\right)^2 - \frac{11\sqrt{2}}{4}\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)^3 - \\
& - \frac{21\sqrt{2}}{16}\left(y-\frac{1}{2}\right)^4 \quad .
\end{aligned}
$$



Obrázek 5: Taylorův mnohočlen 4. řádu

	$(x,y)=\left(\frac{7}{16},\frac{7}{16}\right)$	$(x,y)=\left(\frac{1}{4},\frac{1}{4}\right)$	$(x,y)=\left(\frac{1}{8},\frac{1}{8}\right)$
$f(x,y)$	0,7856128181	0,9354143468	0,9842509842
$T_1(x,y)$	0,7954951286	1,060660172	1,237436867
chyba	0,0098823105	0,1252458252	0,2531858828
$T_2(x,y)$	0,7844465852	0,8838834762	0,8396893024
chyba	0,0011662329	0,0515308706	0,1445616818
$T_3(x,y)$	0,7858276531	0,9722718239	1,137999976
chyba	0,0002148350	0,0368574771	0,1537489918
$T_4(x,y)$	0,7855687028	0,9059805632	0,8024004683
chyba	0,0000441153	0,0294337836	0,1818505159

Tabulka 1: Tabulka funkčních hodnot