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Series

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Contents

1 Series (of real numbers)

1.1 Sum and convergence of a series

Definition 1.1. The expression

$$a_1 + a_2 + \cdots + a_n + \ldots \ = \sum_{n=1}^{\infty} a_n \tag{1.1}$$

(i.e. a formal ordered sum) with $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$ is called a series (of real numbers).[1]

The number a_n is the n-th element of the series (1.1), the sequence (s_n) defined by the expression

$$s_n := a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of (1.1).

If the limit

$$s := \lim s_n \in \mathbb{R}^*$$

exists, it is called the sum of the series (1.1) and we write [2]

$$\sum_{n=1}^{\infty} a_n = s;$$

moreover, if $s \in \mathbb{R}$, then the series (1.1) is convergent (summable). If the series $\sum\limits_{n=1}^{\infty} a_n$ has no sum,[3] or if $\sum\limits_{n=1}^{\infty} a_n \in \{+\infty, -\infty\}$, then the series (1.1) is divergent.

Examples 1.2.

a)

$$1 + 2 + 3 + \ldots \ = \sum_{n=1}^{\infty} n = +\infty \quad \text{... } \texttt{divergent} \text{ (arithmetic) series.}$$

$$\Big(s_n = \frac{n(n+1)}{2} \to +\infty\,.\Big)$$

[1] I.e. (a_n) is a sequence of real numbers.

[2] Notice that by the symbol $\sum\limits_{n=1}^{\infty} a_n$ we denote both the series and its sum, i.e. a number! There is no need to worry though, it will always be possible to distinguish between these two from context.

[3] Meaning that $\lim s_n$ does not exist.

b)

$$1 + (-1) + 1 + (-1) + \dots \ = \sum_{n=1}^{\infty} (-1)^{n+1} \quad \dots \ \texttt{divergent} \text{ series.}$$

$$\left(s_n = \begin{cases} 0, & \text{for } n \text{ even,} \\ 1, & \text{for } n \text{ odd.} \end{cases} \right)$$

Be careful about `the placement of parentheses`. It holds that

$$(1-1) + (1-1) + (1-1) + \dots \ = 0 + 0 + 0 + \dots \ = 0,$$

$$1 + (-1+1) + (-1+1) + \dots \ = 1 + 0 + 0 + \dots \ = 1.$$

c) The sum of (<u>the geometric</u>) series

$$1 + q + q^2 + \dots \ = \sum_{n=1}^{\infty} q^{n-1}$$

with $q \in \mathbb{R}$ exists if and only if $q > -1$. In particular, we have

$$\sum_{n=1}^{\infty} q^{n-1} = \begin{cases} +\infty, & \text{for } q \geq 1, \\ \frac{1}{1-q}, & \text{for } -1 < q < 1. \end{cases}$$

$$\left(s_n = \begin{cases} n, & \text{for } q = 1, \\ \frac{1-q^n}{1-q}, & \text{for } q \neq 1. \end{cases} \right)$$

d)

$$1 + \frac{1}{2} + \frac{1}{3} + \dots \ = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad \dots \ \texttt{divergent} \text{ (the so-called \underline{harmonic}) series.}$$

Try to prove the assertion above by the (obvious) inequality

$$\forall k \in \mathbb{N} : \quad \frac{1}{2^k+1} + \frac{1}{2^k+2} + \frac{1}{2^k+3} + \dots + \frac{1}{2^{k+1}} \geq \frac{1}{2^{k+1}} \left(2^{k+1} - 2^k \right) = \frac{1}{2}.$$

Theorem 1.3 (Necessary condition of convergence)**.** *If the sequence* $\sum_{n=1}^{\infty} a_n$*, converges, then* $\lim a_n = 0$.

Proof. Due to the assumption it holds for the sequence of partial sums

$$s_n := \sum_{k=1}^{n} a_k$$

that

$$s := \lim s_n \in \mathbb{R} \ (!),$$

and thus

$$\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = s - s = 0.$$

□

Examples 1.4.

a) $\sum\limits_{n=1}^{\infty}(-1)^n n^2$ diverges since $\lim(-1)^n n^2$ does not exist.

b) $\sum\limits_{n=1}^{\infty} n^2$ diverges since $\lim n^2 = +\infty$.

c) $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim \frac{1}{n} = 0$.

(The converse of the implication in Theorem 1.3 `does not hold`!)

Theorem 1.5 (Bolzano–Cauchy condition). *The sequence* $\sum\limits_{n=1}^{\infty} a_n$ *converges if and only if*

$$\left(\forall \varepsilon \in \mathbb{R}^+\right)(\exists n_0 \in \mathbb{N})\,(\forall m, n \in \mathbb{N};\ n_0 \le m < n) : \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon.$$

Proof. The theorem is a direct corollary of the assertion that a sequence of real numbers is convergent if and only if it is a Cauchy sequence and its equivalence to the fact that the sequence $s_n := \sum\limits_{k=1}^{n} a_k$ of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$ is a Cauchy sequence, i.e.

$$\left(\forall \varepsilon \in \mathbb{R}^+\right)(\exists n_0 \in \mathbb{N})\,(\forall n, m \in \mathbb{N};\ n, m \ge n_0) :\ |s_n - s_m| < \varepsilon.$$

□

Theorem 1.6. *If the series* $\sum\limits_{n=1}^{\infty} |a_n|$ *is convergent, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *converges as well.*

Proof. First we define (for every $n \in \mathbb{N}$):

$$\begin{aligned} a_n^+ &:= \max\{a_n, 0\} = \frac{1}{2}(|a_n| + a_n) \ge 0,\\ a_n^- &:= \max\{-a_n, 0\} = \frac{1}{2}(|a_n| - a_n) \ge 0;\\ s_n^+ &:= a_1^+ + a_2^+ + \cdots + a_n^+,\\ s_n^- &:= a_1^- + a_2^- + \cdots + a_n^-. \end{aligned}$$

We aim to prove that the sequence of partial sums

$$s_n := \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_k^+ - a_k^-) = \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^- = s_n^+ - s_n^-$$

is convergent, i.e. that its limit is finite. It is sufficient to prove convergence of the sequences (s_n^+) and (s_n^-). Both of these sequences are non-increasing and due to the assumption

$$\sum_{n=1}^{\infty} |a_n| =: s \in \mathbb{R}$$

and due to relations

$$s_n^+ = a_1^+ + a_2^+ + \cdots + a_n^+ \leq |a_1| + |a_2| + \cdots + |a_n| \leq \sum_{n=1}^{\infty} |a_n| = s,$$
$$s_n^- = a_1^- + a_2^- + \cdots + a_n^- \leq |a_1| + |a_2| + \cdots + |a_n| \leq \sum_{n=1}^{\infty} |a_n| = s,$$

holding for every $n \in \mathbb{N}$ the sequences are also bounded from above. Their convergence is thus a direct consequence of the known proposition on the limit of a monotone sequence. [4] □

Definition 1.7. If the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then the (convergent!) series $\sum\limits_{n=1}^{\infty} a_n$ is said to converge absolutely. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges and simultaneously the series $\sum\limits_{n=1}^{\infty} |a_n|$ diverges, the series $\sum\limits_{n=1}^{\infty} a_n$ is said to converge non-absolutely. [5]

Examples 1.8.

a) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n}$ ... `non-absolutely convergent` series.

(The assertion will be proven later by the `Leibniz criterion.`)

b) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ ... `absolutely convergent` series.

(The assertion will be proven later by the `integral criterion`.)

1.2 Absolute convergence tests

Convention. We say that

$$V(n) \textit{ holds for all sufficiently large } n \in \mathbb{N},$$

if

$$(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n \geq n_0):\ V(n).$$

[4]**Theorem** (on the limit of a monotone sequence).

If the sequence (α_n) is non-decreasing, it holds that

$$\lim \alpha_n = \sup\{\alpha_n :\ n \in \mathbb{N}\}.$$

If the sequence (β_n) is non-increasing, it holds that

$$\lim \beta_n = \inf\{\beta_n :\ n \in \mathbb{N}\}.$$

[5]Notice that the sum $\sum\limits_{n=1}^{\infty} |a_n|$ always exists (the corresponding sequence of partial sums is non-decreasing), it can be, however, equal to $+\infty$.

Theorem 1.9 (Direct comparison test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *denote series such that*

i) $|a_n| \leq b_n$ *for all sufficiently large* $n \in \mathbb{N}$*,*

ii) $\sum_{n=1}^{\infty} b_n$ *converges.*

Then $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

Proof. From the assumptions it follows that the sequence of partial sums of the sequence $\sum_{n=1}^{\infty} |a_n|$ is bounded from above, and since it is – as we found out earlier – non-decreasing, it has a `finite` limit. The limit is $\sum_{n=1}^{\infty} |a_n|$. [6] □

Example 1.10.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{1977}\right)^n$$

`converges absolutely`, since for every $n \in \mathbb{N}$ it holds that

$$\left|\frac{(-1)^n}{n} \left(\frac{1}{1977}\right)^n\right| \leq \left(\frac{1}{1977}\right)^n$$

and $\sum_{n=1}^{\infty} \left(\frac{1}{1977}\right)^n$ is a convergent (geometric) series ($-1 < q := \frac{1}{1977} < 1$).

Observation (and a direct corollary of Theorem 1.9.)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ denote series such that $0 \leq a_n \leq b_n$ for all sufficiently large $n \in \mathbb{N}$ and assume that $\sum_{n=1}^{\infty} a_n = +\infty$. Then it holds that $\sum_{n=1}^{\infty} b_n = +\infty$.

Example 1.11.

$$\sum_{n=1}^{\infty} \frac{\ln(1966+n)}{n}$$

diverges, because we have

$$0 \leq \frac{1}{n} \leq \frac{\ln(1966+n)}{n} \text{ (for all } n \in \mathbb{N})$$

and moreover $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

Theorem 1.12 (Ratio test (D'Alembert's criterion))**.** *For an arbitrary series* $\sum_{n=1}^{\infty} a_n$ *the following assertions hold.*

[6] If the proof is not clear enough, the reader is advised to think through all the steps carefully!

i) If there exists $q \in (0,1)$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q \quad \text{for all sufficiently large } n \in \mathbb{N},$$

then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii) If

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \quad \text{for all sufficiently large } n \in \mathbb{N},$$

then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

a) First we prove assertion i).

$$\begin{aligned} &|a_1| + |a_2| + \dots + |a_{n_0}| + |a_{n_0+1}| + |a_{n_0+2}| + \dots \\ \leq\ &|a_1| + |a_2| + \dots + |a_{n_0-1}| + |a_{n_0}| + q|a_{n_0}| + q^2|a_{n_0}| + \dots \\ =\ &|a_1| + |a_2| + \dots + |a_{n_0-1}| + |a_{n_0}|(1 + q + q^2 + \dots) \\ =\ &|a_1| + |a_2| + \dots + |a_{n_0-1}| + |a_{n_0}| \sum_{n=1}^{\infty} q^{n-1} \\ =\ &|a_1| + |a_2| + \dots + |a_{n_0-1}| + |a_{n_0}| \frac{1}{1-q} < +\infty. \end{aligned}$$

b) Also the proof of ii) is straightforward. From the assumption

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \quad \text{for all sufficiently large } n \in \mathbb{N}$$

it follows that

$$(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n \geq n_0): \ \ |a_{n+1}| \geq |a_n| > 0,$$

and thus

$$(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n \geq n_0): \ \ |a_n| \geq |a_{n_0}| > 0.$$

One can easily conclude that the necessary condition for series convergence, $\lim a_n = 0$ (see Theorem 1.3), does not hold for $\sum_{n=1}^{\infty} a_n$. The series $\sum_{n=1}^{\infty} a_n$ is thus divergent.

□

The following theorem is a direct corrolary of Theorem 1.12.

Theorem 1.13 (Limit ratio test (Limit d'Alembert criterion))**.**

i) If

$$\lim \left|\frac{a_{n+1}}{a_n}\right| < 1,$$

then the series $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

ii) If

$$\lim \left|\frac{a_{n+1}}{a_n}\right| > 1,$$

then the series $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

Proof.

a) First we investigate why assertion i) holds. Let us (arbitrarily) choose

$$q \in \left(\lim \left|\frac{a_{n+1}}{a_n}\right|, 1\right) \subset (0,1).$$

Then it is obvious that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q \quad \text{for all sufficiently large } n \in \mathbb{N},$$

and the assertion follows directly from the already proven assertion i) of Theorem 1.12.

b) Proof of assertion ii). If

$$\lim \left|\frac{a_{n+1}}{a_n}\right| > 1,$$

then it follows that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \quad \text{for all sufficiently large } n \in \mathbb{N}.$$

Thus, divergence of the series $\sum\limits_{n=1}^{\infty} a_n$ follows directly from assertion ii) of Theorem 1.12.

□

Examples 1.14.

1.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n} \quad \text{converges absolutely,}$$

because

$$\left|\frac{(-1)^{n+1} \frac{(n+1)^2}{3^{n+1}}}{(-1)^n \frac{n^2}{3^n}}\right| = \frac{1}{3}\frac{(n+1)^2}{n^2} \to \frac{1}{3} < 1.$$

2.
$$\sum_{n=1}^{\infty} \frac{n!}{10^n} \text{ diverges,}$$
since
$$\left|\frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}\right| = \frac{1}{10}\frac{(n+1)!}{n!} = \frac{1}{10}(n+1) \to +\infty > 1.$$

3. `Be careful!` The ratio test is not helpful for e.g. $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ since
$$1 > \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \frac{n}{n+1} \to 1.$$

Theorem 1.15 (Root test (Cauchy's criterion))**.** *For an arbitrary series* $\sum\limits_{n=1}^{\infty} a_n$ *the following assertions hold.*

i) If there exists $q \in (0,1)$ *such that*
$$\sqrt[n]{|a_n|} \le q \quad \textit{for all sufficiently large } n \in \mathbb{N},$$
then the series $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

ii) If it holds for infinitely many $n \in \mathbb{N}$
$$\sqrt[n]{|a_n|} \ge 1,$$
then the series $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

Proof.

a) First let us prove assertion i). From the assumptions it follows that
$$|a_n| \le q^n \quad \text{for all sufficiently large } n \in \mathbb{N}$$
and that the series $\sum_{n=1}^{\infty} q^n$ converges (since it is a geometric series with common ratio $q \in (0,1)$). Thus the assertion follows from the direct comparison test (see Theorem 1.9).

b) It remains to prove assertion ii). From the assumptions we have for infinitely many $n \in \mathbb{N}$ that $|a_n| \ge 1$. This, however, means that $\lim a_n = 0$ does not hold, i.e. the necessary condition for convergence of $\sum_{n=1}^{\infty} a_n$ is not satisfied (see Theorem 1.3). Thus the series $\sum_{n=1}^{\infty} a_n$ diverges.

□

The 'limit' version of the theorem follows.

Theorem 1.16 (Limit root test, (Limit Cauchy's criterion))**.**

i) If

$$\lim \sqrt[n]{|a_n|} < 1,$$

then the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

ii) If

$$\lim \sqrt[n]{|a_n|} > 1,$$

then the series $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof.

a) Proof of assertion i). Let us choose (arbitrarily)

$$q \in \left(\lim \sqrt[n]{|a_n|}, 1\right) \subset (0,1).$$

Then it obviously holds that

$$\sqrt[n]{|a_n|} \leq q \ \text{ for all sufficiently large } n \in \mathbb{N}.$$

The assertion then follows from the first part of Theorem 1.15.

b) Proof of assertion ii). If

$$\lim \sqrt[n]{|a_n|} > 1,$$

then

$$\sqrt[n]{|a_n|} \geq 1 \ \text{ for all sufficiently large } n \in \mathbb{N},$$

and thus

$$\sqrt[n]{|a_n|} \geq 1 \ \text{ for infinitely many } n \in \mathbb{N}.$$

The divergence of the series $\sum_{n=1}^{\infty} a_n$ then directly follows from assertion ii) of Theorem 1.15.

□

Examples 1.17.

1.

$$\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n-1}\right)^n \text{ converges absolutely,}$$

because

$$\sqrt[n]{\left(\frac{2n+1}{3n-1}\right)^n} = \frac{2n+1}{3n-1} \to \frac{2}{3} < 1.$$

2. $$\sum_{n=1}^{\infty} \frac{2^n}{n^{1993}} \text{ diverges,}$$
   since
   $$\sqrt[n]{\frac{2^n}{n^{1993}}} = \frac{2}{(\sqrt[n]{n})^{1993}} \to 2 > 1.$$

3. `Be careful!` Again, the root criterion is not helpful for testing $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ for convergence, because (for every $n \in \mathbb{N},\ n > 1$)
   $$1 > \sqrt[n]{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} \to 1.$$

Theorem 1.18 (Raabe's criterion)**.** *For an arbitrary series* $\sum\limits_{n=1}^{\infty} a_n$ *the following assertions hold.*

i) If there exists $q > 1$ such that
$$n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) \geq q \quad \textit{for all sufficiently large } n \in \mathbb{N},$$
then the series $\sum\limits_{n=1}^{\infty} a_n$ *converges absolutely.*

ii) If
$$n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) \leq 1 \quad \textit{for all sufficiently large } n \in \mathbb{N},$$
then the series $\sum_{n=1}^{\infty} a_n$ does not converge absolutely (i.e. it either converges non-absolutely or it diverges).

Proof.

a) First we prove assertion i).
From the condition $n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) \geq q$ it follows that $n(|a_n| - |a_{n+1}|) \geq q|a_n|$. Therefore, we can assume that there exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > n_0$, it holds that
$$\begin{aligned} n_0(|a_{n_0}| - |a_{n_0+1}|) &\geq q|a_{n_0}|, \\ (n_0+1)(|a_{n_0+1}| - |a_{n_0+2}|) &\geq q|a_{n_0+1}|, \\ &\cdots \\ n(|a_n| - |a_{n+1}|) &\geq q|a_n|. \end{aligned}$$
Summing up the inequalities leads to
$$n_0|a_{n_0}| + (|a_{n_0+1}| + \cdots + |a_n|) - n|a_{n+1}| \geq q|a_{n_0}| + q(|a_{n_0+1}| + \cdots + |a_n|),$$
and we easily derive that
$$(q-1)(|a_{n_0+1}| + \cdots + |a_n|) \leq n_0|a_{n_0}| - n|a_{n+1}| - q|a_{n_0}| \leq n_0|a_{n_0}|.$$

Taking into account that $q-1>0$ we obtain

$$|a_{n_0+1}|+\cdots+|a_n|\le\frac{n_0|a_{n_0}|}{q-1}\quad\text{for every } n\in\mathbb{N}, n>n_0.$$

We conclude that the sequence of partial sums of the series $\sum_{n=1}^{\infty}|a_n|$ is bounded from above, and thus the series $\sum_{n=1}^{\infty}a_n$ converges absolutely.

b) Now we show that also assertion ii) holds, i.e. (under the above assumptions) that the series $\sum_{n=1}^{\infty}|a_n|$ diverges.
The condition $n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)\le 1$ can be rewritten as $\left|\frac{a_{n+1}}{a_n}\right|\ge 1-\frac{1}{n}=\frac{n-1}{n}$. Thus, there exists $n_0\in\mathbb{N}$, $n_0\ge 2$, such that for every $n\in\mathbb{N}$, $n\ge n_0$ it holds that

$$\left|\frac{a_{n_0+1}}{a_{n_0}}\right|\ge\frac{n_0-1}{n_0},$$
$$\left|\frac{a_{n_0+2}}{a_{n_0+1}}\right|\ge\frac{n_0}{n_0+1},$$
$$\cdots$$
$$\left|\frac{a_{n+1}}{a_n}\right|\ge\frac{n-1}{n}.$$

Multiplying the above inequalities (comparing positive numbers) leads to

$$\left|\frac{a_{n+1}}{a_{n_0}}\right|\ge\frac{n_0-1}{n},$$

and thus

$$|a_{n+1}|\ge|a_{n_0}|(n_0-1)\frac{1}{n}\quad\text{for every } n\in\mathbb{N}, n\ge n_0.$$

Taking into account divergence of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ we conclude that $\sum_{n=1}^{\infty}|a_{n+1}|$ (and therefore also $\sum_{n=1}^{\infty}|a_n|$) diverges (see corollary of Theorem 1.9).

□

The following theorem is a direct corollary of Theorem 1.18.

Theorem 1.19 (Limit Raabe's criterion)**.**

i) If

$$\lim n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)>1,$$

then the series $\sum_{n=1}^{\infty}a_n$ converges absolutely.

ii) If

$$\lim n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)<1,$$

then the series $\sum_{n=1}^{\infty}a_n$ does not converge absolutely (i.e. it either converges non-absolutely or it diverges).

Proof. The proof follows the steps of the proof of Theorem 1.13 and is thus left to the diligent reader. □

Examples 1.20.

1. The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, because

$$\lim n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) = \lim n\left(1 - \frac{n^3}{(n+1)^3}\right)$$
$$= \lim \frac{n((n+1)^3 - n^3)}{(n+1)^3} = \lim \frac{3n^3 + 3n^2 + n}{n^3 + 3n^2 + 3n + 1} = 3 > 1.$$

2. The series $\sum_{n=1}^{\infty} \frac{(2n)!}{4^n (n!)^2}$ diverges, since

$$\lim n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) = \lim n\left(1 - \frac{(2n+2)(2n+1)}{4(n+1)^2}\right)$$
$$= \lim n\left(1 - \frac{2n+1}{2(n+1)}\right) = \lim \frac{n}{2n+2} = \frac{1}{2} < 1.$$

   Note that the ratio test is not applicable here since

$$1 > \left|\frac{a_{n+1}}{a_n}\right| \to 1.$$

Theorem 1.21 (Integral test)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ denote a function* non-increasing *in $[1, +\infty)$ and assume that for every $n \in \mathbb{N}$ it holds that $|a_n| = f(n)$.*

Then the series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely if and only if the improper integral $\int_1^\infty f(x)\,\mathrm{d}x$ converges (i.e. the limit $\lim_{c\to\infty} \int_1^c f(x)\ \mathrm{d}x$ exists and is finite).

Proof. First we define

$$s_n := \sum_{k=1}^{n} |a_k|$$

for every $n \in \mathbb{N}$. Notice that the limits

$$\lim s_n = \sum_{n=1}^{\infty} |a_n| \in \mathbb{R}^*,$$

$$\lim_{c\to\infty} \int_1^c f(x) = \lim \int_1^n f(x)\ \mathrm{d}x = \int_1^\infty f(x)\ \mathrm{d}x \in \mathbb{R}^*$$

exist[7].

We have to prove the equivalence

$$\sum_{n=1}^{\infty} |a_n| < +\infty \ \Leftrightarrow\ \int_1^\infty f(x)\ \mathrm{d}x < +\infty. \tag{1.2}$$

[7] A question to the reader: '*Why do they exist?*'

It follows from the assumptions that[8]

$$s_n = \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} f(k) \geq \int_1^{n+1} f(x)\ \mathrm{d}x \geq \sum_{k=2}^{n+1} f(k) = \sum_{k=2}^{n+1} |a_k| = s_{n+1} - |a_1|.$$

Passing to the limit ($n \to \infty$) leads to inequalities

$$\sum_{n=1}^{\infty} |a_n| \geq \int_1^{\infty} f(x)\ \mathrm{d}x \geq \sum_{n=1}^{\infty} |a_n| - |a_1|,$$

from which the equivalence (1.2) follows easily. □

Examples 1.22.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges absolutely,}$$
since
$$\int_1^{\infty} \frac{1}{x^2}\ \mathrm{d}x = \left[-\frac{1}{x}\right]_1^{\infty} = 0 - (-1) = 1 < +\infty.$$

2.
$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges,}$$
because
$$\int_1^{\infty} \frac{1}{x}\ \mathrm{d}x = [\ln x]_1^{\infty} = +\infty - 0 = +\infty.$$

The reader should think through for which $\alpha \in \mathbb{R}$ the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges.

1.3 Non-absolute convergence tests

Let us first note that the term 'non-absolute convergence tests' may sound misleading. The following theorems do not assert that the corresponding sequences (satisfying certain qualities) `converge non-absolutely`. Instead, the following tests ensure that the series `converge` (possibly absolutely).

First we provide a test of convergence for <u>alternating</u> series (i.e. series whose terms alternate between positive and negative).

---
[8] It is helpful to draw a figure!

Theorem 1.23 (Leibniz criterion)**.** *Let (a_n) denote a monotonic sequence defined in $\mathbb{N}$ such that* $\lim a_n = 0$.[9] *Then the sequence*

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\, a_n$$

converges.

Proof. Assume, for example, that

$$\forall n \in \mathbb{N}:\ 0 \le a_{n+1} \le a_n.$$

From the sequence

$$s_n := \sum_{k=1}^{n} (-1)^{k+1} a_k$$

of partial sums of the series in question we choose subsequences of odd elements (except for the first one) and of even elements, i.e.

$$s_n^* := s_{2n+1}, \quad s_n^{**} := s_{2n}.$$

Since we know (by assumption ii)) that for every $n \in \mathbb{N}$ it holds that

$$\begin{aligned} s_{n+1}^* &= s_{2n+3} = s_{2n+1} - a_{2n+2} + a_{2n+3} \le s_{2n+1} = s_n^*, \\ s_{n+1}^{**} &= s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \ge s_{2n} = s_n^{**}, \end{aligned}$$

the limits

$$\begin{aligned} &\lim s_n^* \in \mathbb{R} \cup \{-\infty\}, \\ &\lim s_n^{**} \in \mathbb{R} \cup \{+\infty\} \end{aligned}$$

exist[10]. Moreover, due to assumption iii) we have

$$\lim(s_n^* - s_n^{**}) = \lim(s_{2n+1} - s_{2n}) = \lim a_{2n+1} = 0,$$

and thus

$$\lim s_n^* = \lim s_n^{**} =: s \in \mathbb{R}\,!$$

Now it easily follows (the readers will think this through!), that

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, a_n = \lim s_n = s \in \mathbb{R},$$

which was to be proven.

□

[9] Notice that for a monotonic sequence (a_n) with a vanishing limit exactly one of the following possibilities holds.

i) $\forall n \in \mathbb{N}:\ 0 \le a_{n+1} \le a_n$,

ii) $\forall n \in \mathbb{N}:\ 0 \ge a_{n+1} \ge a_n$.

[10] See Theorem on monotonic sequences.

Example 1.24. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

is `non-absolutely convergent`, because it holds that

- $\forall n \in \mathbb{N}: \ \frac{1}{n+1} \leq \frac{1}{n}$,
- $\lim \frac{1}{n} = 0$;
- $\sum\limits_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n} = +\infty$.

Theorem 1.25 (Dirichlet's test)**.** *Let (a_n) denote a monotonic sequence defined in $\mathbb{N}$ such that $\lim a_n = 0$ and assume that the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$ is bounded. Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.*

Proof. Without loss of generality let us assume that (a_n) is non-increasing (for a non-decreasing sequence it would suffice to switch to $(-a_n)$). This means that $a_n \geq 0$ for all $n \in \mathbb{N}$ (taking into account $\lim a_n = 0$). By the assumptions it further follows that the sequence

$$s_n := \sum_{k=1}^{n} b_k$$

of partial sums of the series $\sum_{n=1}^{\infty} b_n$ satisfies

$$\left(\exists k \in \mathbb{R}^+\right) (\forall n \in \mathbb{N}): \ |s_n| \leq k.$$

Now it is sufficient to show (due to Theorem 1.5) that for the series $\sum_{n=1}^{\infty} a_n b_n$ the Bolzano–Cauchy condition

$$\left(\forall \varepsilon \in \mathbb{R}^+\right) (\exists n_0 \in \mathbb{N}) (\forall m, n \in \mathbb{N}; \ n_0 \leq m < n): \ \left|\sum_{k=m+1}^{n} a_k b_k\right| < \varepsilon$$

holds.

Let $\underline{\varepsilon > 0 \text{ be given}}$. From the assumption $\lim a_n = 0$ it follows that

$$(\underline{\exists n_0 \in \mathbb{N}}) (\forall n \in \mathbb{N}; \ n \geq n_0): \ a_n = |a_n| < \frac{\varepsilon}{2k}.$$

It remains to prove that $\underline{\text{for every } m, n \in \mathbb{N},\ n_0 \leq m < n, \text{ it holds that}}$ $\left|\sum\limits_{k=m+1}^{n} a_k b_k\right| < \varepsilon$.

Direct computation leads to

$$\begin{aligned}
\underline{|a_{m+1}b_{m+1} + \dots + a_n b_n|} &= |a_{m+1}(s_{m+1} - s_m) + \dots + a_n(s_n - s_{n-1})| = \\
&= |-a_{m+1}s_m + (a_{m+1} - a_{m+2})s_{m+1} + \dots + (a_{n-1} - a_n)s_{n-1} + a_n s_n| \\
&\leq a_{m+1}|s_m| + (a_{m+1} - a_{m+2})|s_{m+1}| + \dots + (a_{n-1} - a_n)|s_{n-1}| + a_n|s_n| \\
&\leq k a_{m+1} + k(a_{m+1} - a_{m+2}) + \dots + k(a_{n-1} - a_n) + k a_n = 2k a_{m+1} < \varepsilon.
\end{aligned}$$

□

Remark 1.26. Theorem 1.23 is now a direct corollary of Theorem 1.25. It is sufficient to define $b_n := (-1)^{n+1}$. Clearly, then the sequence of partial sums of the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n+1}$$

is bounded.

Example 1.27. The series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^{\alpha}}$$

is convergent for arbitrary $\alpha > 0$, because the sequence $\left(\frac{1}{n^{\alpha}}\right)$ is monotonic and converges to zero and the sequence of partial sums of the series $\sum_{n=1}^{\infty} \sin n$ is bounded[11] (see Theorem 1.25).

Theorem 1.28 (Abel's test)**.** *Let (a_n) denote a monotonic bounded sequence defined in $\mathbb{N}$ and assume that the series $\sum_{n=1}^{\infty} b_n$ converges. Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges as well.*

Proof. By the assumptions there exists `a finite` $\lim a_n =: a$. For every $n \in \mathbb{N}$ we define

$$a_n^{\star} := a_n - a.$$

The sequence $(a_n^{\star})$ is clearly monotonic and its limit vanishes; moreover, since the series $\sum_{n=1}^{\infty} b_n$ converges, its sequence of partial sums is bounded. From Dirichlet's test (see Theorem 1.25) it follows that the series $\sum_{n=1}^{\infty} a_n^{\star} b_n$ is convergent.

The rest is simple, since for the sequence (s_n) of partial sums of the series $\sum_{n=1}^{\infty} a_n b_n$ it holds that

$$s_n := \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (a_k^{\star} + a) b_k = \sum_{k=1}^{n} a_k^{\star} b_k + a \sum_{k=1}^{n} b_k \to \sum_{k=1}^{\infty} a_k^{\star} b_k + a \sum_{k=1}^{\infty} b_k \in \mathbb{R}.$$

□

Examples 1.29.

a) The series

$$\sum_{n=1}^{\infty} \left(\arctan n \, \frac{\sin n}{n^{\alpha}} \right)$$

[11] This assertion is not trivial. The interested reader can (e.g. by mathematical induction or using complex numbers) prove that for every $n \in \mathbb{N}$ it holds that

$$s_n := \sum_{k=1}^{n} \sin k = \frac{\sin \frac{n+1}{2} \sin \frac{n}{2}}{\sin \frac{1}{2}}, \text{ and thus } |s_n| \leq \frac{1}{\sin \frac{1}{2}}.$$

converges for an arbitrary $\alpha > 0$, because in Example 1.27 we showed that $\sum_{n=1}^{\infty} \frac{\sin n}{n^\alpha}$ is convergent. Furthermore, it is obvious that the sequence $(\arctan n)$ is monotonic and bounded. The assertion then follows directly by Theorem 1.28.

b) If $\sum_{n=1}^{\infty} b_n$ denotes an arbitrary convergent series, then also (see Theorem 1.28) the series $\sum_{n=1}^{\infty} \frac{n+1}{n} b_n$ converges, as the sequence $\left(\frac{n+1}{n}\right)$ is monotonic and bounded.

1.4 Some final remarks

Remark 1.30 (remainder of a series). For a series $\sum_{n=1}^{\infty} a_n$ and $n \in \mathbb{N}$ we define the remainder after the n^{th} element as[12]

$$a_{n+1} + a_{n+2} + a_{n+3} + \ldots = \sum_{k=n+1}^{\infty} a_k.$$

It is often useful (for a convergent series) to estimate the sum of the remainder.[13] However, this might not be easy. For illustration let us note that under the assumptions of the Leibniz criterion it holds for every $n \in \mathbb{N}$ that[14]

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}a_n - \sum_{k=1}^{n}(-1)^{k+1}a_k\right| = \left|\sum_{k=n+1}^{\infty}(-1)^{k+1}a_k\right| \leq |a_{n+1}|.$$

The reader can also attempt to estimate the remainder under the assumptions of other convergence tests.

Remark 1.31 (Rearranging series). If the mapping

$$\varphi : \mathbb{N} \to \mathbb{N}$$

is

- `defined in all` $\mathbb{N}$,
- `injective`,
- `surjective` (i.e. $\varphi(\mathbb{N}) = \mathbb{N}$),

[12]The symbol '$\sum_{n=\alpha}^{\infty} a_n$', where $1 < \alpha \in \mathbb{N}$, is used to denote 'whole' series, not only their remainders (after all, `a remainder of a sequence` is a 'whole' `sequence`). It will be clear for the reader which series are dealt with, if we write – for example – $\sum_{n=3}^{\infty} \frac{1}{n-2}$, $\sum_{n=18}^{\infty} \frac{\ln(n-17)}{n^5}$, ...

[13]Note the obvious assertion that

A series is convergent if and only if its remainder after the n^{th} element is convergent.

[14]It is a matter of honor for every reader to proof the estimate.

then the series

$$\sum_{n=1}^{\infty} a_{\varphi(n)}$$

is a rearrangement of the series $\sum_{n=1}^{\infty} a_n$.

It can be shown that the following holds.

i) If the series $\sum_{n=1}^{\infty} a_n$ is `absolutely convergent`, then also $\sum_{n=1}^{\infty} a_{\varphi(n)}$ converges absolutely and their sums are equal.

ii) If the sequence $\sum_{n=1}^{\infty} a_n$ is `non-absolutely convergent`, there exist rearrangements such that the new series sums to an a-priori given number in $\mathbb{R}^*$, or such that the sum does not exist at all.

Remark 1.32 (series of complex numbers). The expression

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n,$$

where $a_n \in \mathbb{C}$ for every $n \in \mathbb{N}$ is called a series of complex numbers.

Let us denote for every $n \in \mathbb{N}$:

$$\alpha_n = \text{Re } a_n,$$
$$\beta_n = \text{Im } a_n,$$

i.e.

$$a_n = \alpha_n + \beta_n i;$$
$$\alpha_n, \beta_n \in \mathbb{R}.$$

The series $\sum_{n=1}^{\infty} a_n$ converges if there exist `finite`(!) sums of the series

$$\sum_{n=1}^{\infty} \alpha_n =: \alpha \in \mathbb{R},$$

$$\sum_{n=1}^{\infty} \beta_n =: \beta \in \mathbb{R}.$$

and the sum of the series $\sum_{n=1}^{\infty} a_n$ is defined by the (complex) number

$$s := \alpha + \beta i.$$

The reader interested in series of complex numbers is referred to [2].

2 Sequences and series of functions

2.1 Pointwise and uniform convergence

Definition 2.1. A sequence of real functions (f_n) converges pointwise to a function f on a set $M \subset \mathbb{R}$ if it holds that

$$\forall x \in M: \lim f_n(x) = f(x),$$

i.e. if the following holds

$$(\forall x \in M)\left(\forall \varepsilon \in \mathbb{R}^+\right)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n \geq n_0): |f_n(x) - f(x)| < \varepsilon.$$

We write $f_n \to f$ on M.

Remark 2.2. In general, the natural number n_0 in the condition above depends on the choice of $x \in M$ and $\varepsilon \in \mathbb{R}^+$. If the number n_0 can be chosen independently of $x \in M$, the convergence is said to be uniform on M. More precisely:

Definition 2.3. A sequence of real functions (f_n) converges uniformly on a set $M \subset \mathbb{R}$ to a function f if it holds

$$\lim \left[\sup_{x \in M} |f_n(x) - f(x)|\right] = 0,$$

i.e. if the following holds

$$\left(\forall \varepsilon \in \mathbb{R}^+\right)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N},\ n \geq n_0)(\forall x \in M): |f_n(x) - f(x)| < \varepsilon.$$

We write $f_n \rightrightarrows f$ on M.

Remark 2.4. Notice that the implication below follows easily

$$\boxed{f_n \rightrightarrows f \text{ on } M \ \Rightarrow\ f_n \to f \text{ on } M.}$$

Example 2.5. Let f_n, where $n \in \mathbb{N}$, denote a function defined by the formula

$$f_n(x) := x^n - x^{2n}.$$

Determine if the sequence of functions(f_n) converges pointwise or uniformly in the interval $[0, 1]$.

Solution. It is not difficult to find the pointwise limit. It suffices to notice that for an arbitrary (but fixed) $x \in [0, 1]$ we have

$$\lim x^{2n} = \lim x^n = \begin{cases} 0, & \text{for } x \in [0, 1), \\ 1, & \text{for } x = 1, \end{cases}$$

and thus

$$\lim f_n(x) = \lim(x^n - x^{2n}) = 0.$$

Therefore, the sequence (f_n) converges in $[0,1]$ pointwise to the function

$$f(x) := 0.$$

It remains to determine (and due to Remark 2.4 it is sufficient) if $f_n \rightrightarrows 0$ in $[0,1]$, i.e. if it holds that

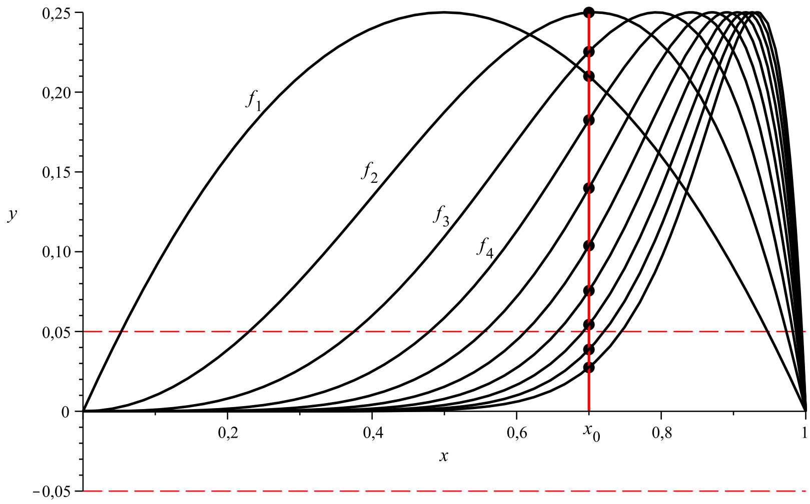
$$\lim\left(\sup_{x\in[0,1]}|f_n(x)-f(x)|\right) = \lim\left(\sup_{x\in[0,1]}|f_n(x)|\right) = 0.$$

It is not difficult to compute that for an arbitrary $n \in \mathbb{N}$ we have

$$\sup_{x\in[0,1]}|f_n(x)| = \sup_{x\in[0,1]}(x^n - x^{2n}) = \max_{x\in[0,1]}(x^n - x^{2n}) = \frac{1}{4},$$

and therefore the sequence (f_n) is not uniformly convergent in $[0,1]$.

Illustartion: The sequence (f_n) is depicted in the following figure,



from which it is clear that for an arbitrary (but fixed) $x_0 \in [0,1]$ the sequence $(f_n(x_0))$ goes to 0, i.e. that the pointwise limit (f_n) is the zero function (in $[0,1]$).

If we construct a band of the width $0 < \varepsilon < \frac{1}{4}$ around the limit (zero) function (in the figure we chose $\varepsilon = 0.05$), we find out that none of the graphs of functions f_n lies `fully` inside the band. This, however, means that the convergence of (f_n) to the function $f(x) := 0$ is not uniform in $[0,1]$.

Definition 2.6. Let f_n and f, $n \in \mathbb{N}$, denote functions defined on a set $M \subset \mathbb{R}$. The function series

$$f_1(x) + f_2(x) + \cdots + f_n(x) + \ldots := \sum_{n=1}^{\infty} f_n(x) \tag{2.1}$$

converges pointwise (or uniformly, respectively) on a set M to its sum f, if the sequence (s_n) of partial sums of the series (2.1) [15] converges pointwise (uniformly, respectively) to the function f on M.

2.2 Uniform convergence tests

The proofs of theorems presented in this section are technical and will be omitted. Interested readers can consult e.g. [1, 4].

Theorem 2.7 (Bolzano–Cauchy's criterion)**.** *A sequence of functions (f_n) converges uniformly on a set $M \subset \mathbb{R}$ if and only if*

$$\left(\forall \varepsilon \in \mathbb{R}^+\right) (\exists n_0 \in \mathbb{N}) (\forall m, n \in \mathbb{N};\ m, n \geq n_0) (\forall x \in M):\ |f_m(x) - f_n(x)| < \varepsilon.$$

Theorem 2.8 (Bolzano–Cauchy's criterion for series of functions)**.** *A series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on a set $M \subset \mathbb{R}$ if and only if*

$$\left(\forall \varepsilon \in \mathbb{R}^+\right) (\exists n_0 \in \mathbb{N}) (\forall m, n \in \mathbb{N};\ n_0 \leq m < n) (\forall x \in M):\ \left|\sum_{k=m+1}^{n} f_k(x)\right| < \varepsilon.$$

(Compare to Theorem 1.5.)

Theorem 2.9 (Weierstrass's criterion)**.** *Let $M \subset \mathbb{R}$ and let $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} f_n(x)$ denote such a series that*

i) $|f_n(x)| \leq b_n$ *for every $n \in \mathbb{N}$ and every $x \in M$,*

ii) $\sum_{n=1}^{\infty} b_n$ *converges.*

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on M.

(Compare to Theorem 1.9.)

Example 2.10. The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2 + x^2}$$

converges uniformly in $\mathbb{R}$, because

$$(\forall n \in \mathbb{N}) (\forall x \in \mathbb{R}):\ \left|\frac{\sin nx}{n^2 + x^2}\right| \leq \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}$$

and the real series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (e.g. according to the integral test – see Theorem 1.21).

Definition 2.11. A sequence of functions (f_n) is monotonic on a set $M \subset \mathbb{R}$ if one of the conditions below holds:

[15] $s_n(x) := \sum_{k=1}^{n} f_k(x)$.

i) $(\forall n \in \mathbb{N})\,(\forall x \in M): \ f_n(x) \le f_{n+1}(x)$,

ii) $(\forall n \in \mathbb{N})\,(\forall x \in M): \ f_n(x) \ge f_{n+1}(x)$.

Definition 2.12. A sequence of functions (f_n) is uniformly bounded on a set $M \subset \mathbb{R}$ if

$$\left(\exists c \in \mathbb{R}^+\right)(\forall n \in \mathbb{N})\,(\forall x \in M): \ |f_n(x)| \le c.$$

Theorem 2.13 (Dirichlet's criterion for series of functions)**.** *Let (f_n) denote sequence of functions on a set M, which satisfies $f_n \rightrightarrows 0$ on M, and assume that the sequence of partial sums of the series $\sum_{n=1}^{\infty} g_n(x)$ is uniformly bounded in M.*[16] *Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly in M.*

(Compare to Theorem 1.25.)

Example 2.14. Thanks to the Dirichlet test from Theorem 2.13 we know that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges uniformly in the interval

$$I_\alpha = [\alpha, 2\pi - \alpha],$$

where $\alpha \in (0, \pi)$. (The sequence of constant functions $\left(\frac{1}{n}\right)$ is monotonic, $\frac{1}{n} \rightrightarrows 0$ in I_α and the sequence of partial sums of the series

$$\sum_{n=1}^{\infty} \sin nx$$

is uniformly bounded in I_α.[17])

Remark 2.15. In the last example we showed that for an arbitrarily small $\alpha \in (0, \pi)$ the series of functions

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges uniformly in $[\alpha, 2\pi - \alpha]$. It can be shown that the series converges in $[0, 2\pi]$, however, the convergence is not uniform.

[16] I.e. $\left(\exists c \in \mathbb{R}^+\right)(\forall n \in \mathbb{N})\,(\forall x \in M): \ \left|\sum_{k=1}^{n} g_k(x)\right| \le c$.

[17] By the assertion

$$(\forall x \in I_\alpha)\,(\forall n \in \mathbb{N}): \ \sum_{k=1}^{n} \sin kx = \frac{\sin\left(\frac{n+1}{2}x\right)\,\sin\left(\frac{n}{2}x\right)}{\sin\frac{x}{2}},$$

which can be proven e.g. by mathematical induction, we easily obtain

$$(\forall n \in \mathbb{N})\,(\forall x \in I_\alpha): \ \left|\sum_{k=1}^{n} \sin kx\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{\sin\frac{\alpha}{2}} =: c \in \mathbb{R}^+.$$

This is exactly the above mentioned uniform boundedness of the sequence of partial sums of the series $\sum_{n=1}^{\infty} \sin nx$.

Theorem 2.16 (Abel's criterion for series of functions)**.** *Let* (f_n) *denote a monotonic and uniformly bounded sequence of functions in* M *and assume that the series* $\sum_{n=1}^{\infty} g_n(x)$ *is uniformly convergent in* M*. Then also the series* $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ *is uniformly convergent in* M*.*

(Compare to Theorem 1.28.)

2.3 Properties of uniformly convergent sequences and series of functions

Theorem 2.17. *Let the sequence of functions* (f_n) *converge uniformly to* f *in an interval* $I \subset \mathbb{R}$*. If the functions* f_n *are continuous in* I *for all sufficiently large* $n \in \mathbb{N}$*, then also the function* f *is continuous in* I*.*

Remark 2.18. The assumption of uniform convergence cannot be replaced by poitwise convergence. For example, consider the sequence of functions (f_n) defined in the interval $I = [0, 1]$ by

$$f_n(x) := x^n.$$

Obviously, for every $x \in I$ it holds that

$$\lim f_n(x) = f(x),$$

where

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, 1), \\ 1 & \text{for } x = 1. \end{cases}$$

All functions f_n are continuous in I, $f_n \to f$ in I, but the limit function f is not continuous in I.

Corollary 2.19. *Let* $I \subset \mathbb{R}$ *denote an interval and assume that the series of functions* $\sum_{n=1}^{\infty} f_n(x)$ *converges to its sum*

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

uniformly in I*. If the the functions* f_n *are continuous in* I *for every* $n \in \mathbb{N}$ *then the function* f *is continuous in* I *as well.*

Theorem 2.17 says that the uniform limit of continuous functions is continuous itself. In a sense we show below that the assertion can be (under additional requirements) conversed.

Theorem 2.20 (Dini)**.** *Assume that* $a, b \in \mathbb{R}$*,* $a < b$*, and*

i) (f_n) *is a monotonic sequence of functions continuous in* $[a, b]$*,*

ii) $f_n \to f$ *in* $[a, b]$*,*

iii) the function f *is continuous in* $[a, b]$*.*

Then $f_n \rightrightarrows f$ *in* $[a, b]$*.*

Corollary 2.21. *Let* (f_n) *denote a sequence of non-negative (or non-positive, respectively) functions continuous in the interval* $I = [a, b]$*, where* $a, b \in \mathbb{R}$*,* $a < b$*, and assume that the function* $f(x) := \sum_{n=1}^{\infty} f_n(x)$ *is continuous in* I*. Then the sequence of functions* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly to* f *in* I*.*

Theorem 2.22. *Let the sequence of functions* (f_n) *converge uniformly to* f *in the interval* $[a, b]$*, where* $a, b \in \mathbb{R}$*,* $a < b$*. If all the functions* f_n *are (Riemann) integrable in* $[a, b]$*, then also* f *is integrable in* $[a, b]$ *and it holds*

$$\int_a^b f(x)\ \mathrm{d}x = \lim \int_a^b f_n(x)\ \mathrm{d}x.$$

Remark 2.23. The previous theorem says that under the given assumptions we can interchange a limit and an integral, i.e.

$$\int_a^b \lim f_n(x)\ \mathrm{d}x = \lim \int_a^b f_n(x)\ \mathrm{d}x.$$

In the case of a pointwise convergence we generally cannot interchange the limit a integral operators. This is demonstrated in the following example.

Example 2.24. Consider a sequence of functions (f_n) defined in the interval $I = [0, 1]$ by

$$f_n(x) := \begin{cases} n^2 x & \text{for } x \in [0, \frac{1}{2n}], \\ n - n^2 x & \text{for } x \in (\frac{1}{2n}, \frac{1}{n}), \\ 0 & \text{for } x \in [\frac{1}{n}, 1]. \end{cases}$$

All functions f_n are continuous (and thus integrable) in I and it is not difficult to see that for every $x \in I$ it holds that

$$\lim f_n(x) = 0.$$

Direct computation, however, leads to

$$\int_0^1 \lim f_n(x)\ \mathrm{d}x = \int_0^1 0\ \mathrm{d}x = 0 \neq \frac{1}{4} = \lim \int_0^1 f_n(x)\ \mathrm{d}x.$$

Corollary 2.25. *Let the series of functions* $\sum_{n=1}^{\infty} f_n(x)$ *converge uniformly in the interval* $[a, b]$*, where* $a, b \in \mathbb{R}$*,* $a < b$*, to its sum*

$$f(x) := \sum_{n=1}^{\infty} f_n(x).$$

If all functions f_n *are (Riemann) integrable in* $[a, b]$*, then also the function* f *is integrable in* $[a, b]$ *and it holds that*

$$\int_a^b f(x)\ \mathrm{d}x = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\ \mathrm{d}x \right).$$

Remark 2.26. The corrollary above says that (under the given assumptions) we can interchange an integral and a sum (of a series), i.e.

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right)\ \mathrm{d}x = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\ \mathrm{d}x \right).$$

Theorem 2.27. *Let (f_n) denote a sequence of functions, each differentiable in an open interval $I \subset \mathbb{R}$, and assume that the sequence (f_n) converges (pointwise) to a function f in I and that the sequence of derivatives (f_n') converges uniformly in I. Then the function f is differentiable in I and it holds that*

$$f'(x) = \lim f_n'(x) \quad \textit{for every } x \in I,$$

i.e.

$$(\lim f_n)' = \lim f_n' \quad \textit{in } I.$$

Corollary 2.28. *Let (f_n) denote a sequence of functions differentiable in an open interval $I \subset \mathbb{R}$. Assume that $\sum_{n=1}^{\infty} f_n(x)$ converges (pointwise) to a function f in I and that the series of derivatives $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly in I. Then the function f is differentiable in I and it holds that*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \textit{for every } x \in I,$$

i.e.

$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x) \quad \textit{in } I.$$

2.4 Power and Taylor series

Definition 2.29. A function

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{2.2}$$

with $a_n \in \mathbb{R}$ for every $n \in \mathbb{N} \cup \{0\}$ is called a power series centered in $x_0 \in \mathbb{R}$.

Let us study the convergence of series (2.2), i.e. let us find out for which $x \in \mathbb{R}$ the corresponding series of `numbers` converges. Clearly, series (2.2) converges for $x = x_0$, i.e. in its center, where it sums to a_0. Now assume that series (2.2) converges in a point $x_1 \neq x_0$ and that a point $x \in \mathbb{R}$ satisfies $|x - x_0| < |x_1 - x_0|$. Then for every $n \in \mathbb{N}$ it holds that

$$|a_n(x - x_0)^n| = |a_n(x_1 - x_0)^n| \left|\frac{x - x_0}{x_1 - x_0}\right|^n. \tag{2.3}$$

From the assumption that the series $\sum\limits_{n=0}^{\infty} a_n(x_1 - x_0)^n$ converges it follows that (see the necessary condition of convergence in Theorem 1.3)

$$\lim \left(a_n(x_1 - x_0)^n\right) = 0,$$

and thus there exists $k \in \mathbb{R}^+$ such that for every $n \in \mathbb{N}$ it holds that $|a_n(x_1 - x_0)^n| \leq k$. Moreover, from the assumption $\left|\frac{x-x_0}{x_1-x_0}\right| < 1$ it follows that the (geometric) series

$$\sum_{n=0}^{\infty} k \left|\frac{x - x_0}{x_1 - x_0}\right|^n$$

converges and by (2.3) (and the comparison test from Theorem 1.9) we obtain that $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely. This observation is further generalized in the following theorem.

Theorem 2.30 (Abel). *Let series* (2.2) *converge in a point $x_1 \neq x_0$ and let us denote*

$$\varepsilon = |x_1 - x_0| > 0.$$

Then

(i) for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ series (2.2) *converges absolutely,*

(ii) power series (2.2) *converges locally uniformly in the interval* [18]

$$(x_0 - \varepsilon, x_0 + \varepsilon).$$

Corollary. *If the power series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ diverges in a point $x_2 \in \mathbb{R}$, it also diverges in every point of the set*

$$\{x \in \mathbb{R} : \ |x - x_0| > |x_2 - x_0|\}.$$

The assertion of Abel's theorem directly leads to the following definition.

Definition 2.31. The number

$$R := \sup\left\{ |x - x_0| : \ \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges} \right\}$$

is called the radius of convergence of the power series (2.2).

Remark 2.32. Note that these direct corollaries of Abel's theorem 2.30 and Definition 2.31 of radius of convergence $R \in [0, +\infty) \cup \{+\infty\}$ hold:

(i) if

$$R = 0,$$

then the series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ converges if and only if $x = x_0$;

(ii) if

$$R > 0,$$

then series (2.2) converges absolutely and locally uniformly in the interval[19]

$$(x_0 - R, x_0 + R);$$

(iii) series (2.2) diverges if $|x - x_0| > R$.

[18] Locally uniform convergence in an interval $I \subset \mathbb{R}$ is a convergence uniform in every `closed bounded` interval $[a, b] \subset I$.

[19] The so-called interval of convergence of the power series (2.2).

Remark 2.33. Assume that for the radius of convergence R of the power series (2.2) it holds that

$$0 < R < +\infty.$$

Note that `generally` we `cannot` judge on the convergence of this series in the boundary points of the interval of convergence, i.e. in points $x_0 - R$ and $x_0 + R$.

We illustrate this fact by the following three power series:[20]

$$\sum_{n=1}^{\infty} x^n, \quad \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Since for every $0 \neq x \in \mathbb{R}$ it holds that

$$\left|\frac{x^{n+1}}{x^n}\right| \to |x|, \quad \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| \to |x|, \quad \left|\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}\right| \to |x|,$$

each of the series is convergent for $|x| < 1$ and divergent for $|x| > 1$ (see d'Alembert's criterion in Theorem 1.13). Thus, (recall Remark 2.32) the radius of convergence of all series is 1 and the interval of convergence is $(-1, 1)$. We can say the following on the convergence of the series in points -1 and 1.

- series $\sum\limits_{n=1}^{\infty} x^n$ diverges for $x = -1$ and for $x = 1$ (in neither case the necessary condition of convergence is satisfied – see Theorem 1.3);
- the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ converges (non-absolutely) for $x = -1$ and diverges for $x = 1$ (see the Leibniz criterion and integral test in Theorems 1.23 and 1.21, respectively);
- series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$ converges (absolutely) for both $x = -1$, $x = 1$ (these assertions easily follow from the integral test in Theorem 1.21).

Theorem 2.34. *Assume that the limits*

$$\lim \left|\frac{a_{n+1}}{a_n}\right| := L \quad \textit{and} \quad \left(\lim \sqrt[n]{|a_n|} := K\right)$$

exist. Then it holds for the radius of convergence R of the power series $\sum\limits_{n=0}^{\infty} a_n(x - x_0)^n$ that

$$R = \begin{cases} \frac{1}{L}, & \textit{if } 0 < L < +\infty, \\ 0, & \textit{if } L = +\infty, \\ +\infty, & \textit{if } L = 0, \end{cases} \qquad \left(R = \begin{cases} \frac{1}{K}, & \textit{if } 0 < K < +\infty, \\ 0, & \textit{if } K = +\infty, \\ +\infty, & \textit{if } K = 0. \end{cases}\right)$$

[20] In all cases the power series are of the form $\sum\limits_{n=0}^{\infty} a_n(x - x_0)^n$, where $x_0 = 0$ and $a_0 = 0$.

Proof. It is sufficient to realize that for $x \neq x_0$ we have

$$\lim \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = L|x-x_0| \quad \text{and} \quad \lim \sqrt[n]{|a_n(x-x_0)^n|} = K|x-x_0|,$$

and we can use ratio and root tests, from Theorems 1.13, 1.16, respectively. □

Example 2.35. Find the interval of convergence of the power series [21] (centered in 1)

$$\sum_{n=0}^{\infty} \frac{n}{2^n} (x-1)^n.$$

Solution.

$$\lim \sqrt[n]{\frac{n}{2^n}} = \lim \frac{\sqrt[n]{n}}{2} = \frac{1}{2},$$

and thus $R = 2$; the given series converges (absolutely) for every $x \in (-1, 3)$ and diverges for every $x \in \mathbb{R}$ such that $|x-1| > 2$.

For $x = -1$ and $x = 3$ the series $\sum\limits_{n=0}^{\infty} \frac{n}{2^n} (x-1)^n$ does not converge, since for neither of the points the necessary condition of convergence is satisfied[22] (see Theorem 1.3).

The interval of convergence of the given series is $(-1, 3)$.

Example 2.36. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n.$$

Solution.

$$\frac{\frac{(2(n+1))!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \to 4,$$

and thus $R = \frac{1}{4}$.

The following very important theorem follows from Corollaries 2.28, 2.25, and Abel's theorem 2.30.

Theorem 2.37 (on the differentiation and integration of power series element by element)**.** *Let $R > 0$ denote the radius of convergence of the power series*

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots = \sum_{n=0}^{\infty} a_n(x-x_0)^n. \tag{2.4}$$

[21] I.e. the set of all $x \in \mathbb{R}$ for which the series converges.

[22] I.e. the equality

$$\lim \frac{n}{2^n}(x-1)^n = 0$$

does not hold.

Then the radii of convergence of the power series

$$a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1},$$

$$a_0(x - x_0) + \frac{a_1}{2}(x - x_0)^2 + \frac{a_2}{3}(x - x_0)^3 + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$$

(obtained by differentiating and integrating series (2.4) *'element by element') are equal to* R *and for the function* S *defined by*

$$S(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

and every $x \in (x_0 - R, x_0 + R)$ *it holds:*

$$S'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1},$$

$$\int_{x_0}^{x} S(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}.$$

Remark 2.38. Looking back to the previous theorem we should notice that (under the given assumptions) the following assertions hold for the sum S of the power series:

i) S is infinitely differentiable and for every $p \in \mathbb{N}$ and every $x \in (x_0 - R, x_0 + R)$ it holds

$$S^{(p)}(x) = \sum_{n=p}^{\infty} n\,(n-1)\,\ldots\,(n-p+1)\,a_n\,(x - x_0)^{n-p},$$

ii) the function

$$x \mapsto \int_{x_0}^{x} S(t)\,\mathrm{d}t$$

is the primitive function to S in the interval $(x_0 - R, x_0 + R)$.

Theorem 2.39 (Abel)**.** *Let* $0 < R < +\infty$ *and assume that the series* $\sum\limits_{n=0}^{\infty} a_n(x - x_0)^n$ *converges in point* $x = x_0 + R$ *(or in point* $x = x_0 - R$*, respectively). Then the function* S *defined by*

$$S(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is continuous from the left in $x = x_0 + R$ *(or from the right in* $x = x_0 - R$*, respectively), i.e.*

$$S(x_0 + R) = \lim_{x \to x_0 + R-} S(x), \quad \Big(S(x_0 - R) = \lim_{x \to x_0 - R+} S(x)\Big).$$

Example 2.40. Let us compute the sum of the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{n}.$$

Solution. First, notice that the Leibnitz criterion from Theorem 1.23 guarantees that the given series converges. Now consider a function S defined by

$$S(x):=\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{x^n}{n}.$$

Because (obviously) the radius of convergence of the given power series is 1) by Theorem 2.37 it follows that

$$\forall x\in(-1,1):\;\;S'(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\,x^{n-1}=\sum_{n=1}^{\infty}(-x)^{n-1}=\frac{1}{1+x}.$$

Using this result (and the obvious fact that $S(0)=0$) leads to

$$\forall x\in(-1,1):\;\;S(x)=\ln(1+x).$$

The rest easily follows from Abel's theorem 2.39:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{n}=S(1)=\lim_{x\to1-}S(x)=\lim_{x\to1-}\ln(1+x)=\ln 2.$$

Example 2.41. Express the function

$$S(x):=\arctan x$$

as a sum of a power series in the neighbourhood of 0 .

Solution. It is sufficient to observe that

$$\forall x\in(-1,1):\;\;S'(x)=\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots=\sum_{n=0}^{\infty}(-1)^n\,x^{2n},$$

and thus (see Theorem 2.37 and use the fact that $S(0)=\arctan 0=0$)

$$\forall x\in(-1,1):\;\;S(x)=\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{2n+1}.$$

Notice that the constructed power series converges in point $x=1$ (see the Leibniz criterion in Theorem 1.23), and thus we obtain an interesting bonus from Abel's theorem 2.39:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\sum_{n=0}^{\infty}(-1)^n\,\frac{1}{2n+1}.$$

We conclude the section on series of functions by a short note on a special kind of power series, namely the Taylor series.

Definition 2.42. Assume that the function $f:\ \mathbb{R}\to\mathbb{R}$ is infinitely differentiable in point $x_0\in\mathbb{R}$. The power series

$$f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{2.5}$$

is the Taylor series of f centered in x_0

(Notice the clear connection to Taylor polynomials of f in x_0.)

It is an interesting task to find out how the sum of Taylor series (2.5), i.e. a function S defined by the formula

$$S(x):=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

relates to the function f itself.

Example 2.43. Consider thee sum of Taylor series of function $f(x):=e^x$ centered in point $x_0=0$, i.e. the function

$$S(x):=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{2.6}$$

Since

$$\lim\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}=\lim\frac{1}{n+1}=0,$$

the radius of convergence of the given Taylor series is $R=+\infty$ (see Theorem 2.34), and thus we can by Theorem 2.37 say that for every $x\in\mathbb{R}$ it holds that

$$\underline{S'(x)}=\Big(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\Big)'=0+1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots=\underline{S(x)}.$$

This explains why (since we know that the unique solution to the Cauchy problem

$$\begin{cases} f'(x)=f(x),\\ f(0)=1\,(=S(0)),\end{cases}$$

in $\mathbb{R}$ is the exponential function $f(x):=e^x$) we can be sure that $S(x)=e^x$ for every $x\in\mathbb{R}$.[23]

Remark 2.44. Similarly as in the previous example it can be shown for many other functions that they are equal to the sum of their Taylor series. For example

[23] The proof given above that the function e^x is equal to the sum of its Taylor series and the assembly of the Taylor series itself is a bit problematic – it is not clear how we define function e^x. Often, the exponential function is `defined` by the sum of the power series from (2.6).

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for every $x \in \mathbb{R}$,
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for every $x \in \mathbb{R}$,
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for every $x \in (-1, 1]$,

...

`Be careful!` This does not hold in general. Consider the function

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} \text{ for } x \neq 0, \\ 0 \text{ for } x = 0. \end{cases}$$

It can be shown that all derivatives of f are continuous in $\mathbb{R}$ and that the Taylor series of f centered in 0 is given by

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = 0 + 0 + 0 + \cdots = \sum_{n=0}^{\infty} 0$$

with its sum vanishing in $\mathbb{R}$. The function f, however, vanishes only in 0.

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