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Line Integrals and Surface Integrals

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1 Vector-valued Functions

1.1 Vector-valued functions and operations on vector-valued functions

Recall that by $\mathbb{R}^n$ we denote a normed vector space whose elements are ordered n-tuples of real numbers (usually denoted by $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $\dots$) and that the (Euclidean) norm is defined by

$$\|x\| := \sqrt{x_1^2 + ... + x_n^2}\,.$$

For $\varepsilon \in \mathbb{R}^+$ we use the following notation:

$U(x,\varepsilon) := \{y \in \mathbb{R}^n :\ \|x - y\| < \varepsilon\}$ is an ε–neighbourhood of a point x,
$P(x,\varepsilon) := U(x,\varepsilon) \setminus \{x\}$ is a punctured ε–neighbourhood of a point x
(if the radius ε does not play a significant role, we will use the notation $U(x)$ and $P(x)$).

Let us further recall the definition of convergence of sequences in $\mathbb{R}^n$ and its properties:

$$a_k = (a_{k1}, ..., a_{kn}) \to a = (a_1, ..., a_n) \overset{\text{def.}}{\Leftrightarrow} \|a_k - a\| \to 0$$
$$\Leftrightarrow \big[\forall i \in \{1, ..., n\} :\ a_{ki} \to a_i \text{ for } k \to \infty\big].$$

Example 1.1. Determine if the sequence (a_k) in $\mathbb{R}^n$ converges and if so, find its limit:

a) $n = 2$, $a_k := \left(\dfrac{k^3 - k}{2k^3 + 1}, \dfrac{3^k + 2^k}{3^{k+1} + 2^{k+1}}\right)$;

b) $n = 4$, $a_k := \left(\dfrac{2k}{k^2 + 1}, \dfrac{(-1)^k}{k^2}, 0, \dfrac{2^k}{k}\right)$.

Solution. The task can be reduced to computing the limit of individual vector elements.

a) $$\lim \left(\frac{k^3 - k}{2k^3 + 1}, \frac{3^k + 2^k}{3^{k+1} + 2^{k+1}}\right) = \left(\lim \frac{k^3 - k}{2k^3 + 1}, \lim \frac{3^k + 2^k}{3^{k+1} + 2^{k+1}}\right)$$
$$= \left(\lim \frac{1 - \frac{1}{k^2}}{2 + \frac{1}{k^3}}, \lim \frac{1 + \left(\frac{2}{3}\right)^k}{3 + 2\left(\frac{2}{3}\right)^k}\right) = \left(\frac{1}{2}, \frac{1}{3}\right), \text{ thus } a_k \to \left(\frac{1}{2}, \frac{1}{3}\right).$$

b) Since $\lim \dfrac{2^k}{k} = \infty \notin \mathbb{R}$ (think this through!), the sequence (a_k) diverges.

Exercise 1.2. Determine if the sequence (a_k) in $\mathbb{R}^5$,

$$a_k := \left(\left(\frac{k+2}{k}\right)^k, \sqrt[k]{k}, \frac{\sin k}{k}, \frac{k+1}{\sqrt{4k^2+1}}, (-1)^k(\sqrt{k} - \sqrt{k+1})\right),$$

converges and if so, find its limit.

Definition 1.3. A mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a vector-valued function (in more detail: a real m–dimensional vector-valued function of n real variables).

If f is a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (denoted by $f:\ \mathbb{R}^n \to \mathbb{R}^m$), then every

$$x = (x_1, ..., x_n) \in \mathrm{D}f \subset \mathbb{R}^n$$

is mapped to a unique value

$$f(x) = (f_1(x), ..., f_m(x)) \in \mathrm{H}f \subset \mathbb{R}^m$$

($\mathrm{D}f$ is the domain of f, $\mathrm{Im}f$ is the image of f).

The functions $f_1, \dots, f_m:\ \mathbb{R}^n \to \mathbb{R}$ are the components of a vector-valued function f and we write $f = (f_1, ..., f_m)$.

Remark 1.4. If $m = n$ (and mostly $m = 2$ or $m = 3$), the function f is called the vector field; if $m = 1$, we use the term scalar field.

Convention 1.5. If the vector-valued function $f:\ \mathbb{R}^n \to \mathbb{R}^m$ is given only by a formula, i.e. without explicitly specifying its domain (e.g. $f(u, v, w) := (\sin u, v\sqrt{w})$), we assume that its domain is given by all $x \in \mathbb{R}^n$, for which the formula is valid (in our case: $\mathrm{D}f = \{(u, v, w) \in \mathbb{R}^3 : w \geq 0\}$).

Remark 1.6. Note that for $f = (f_1, \dots, f_m):\ \mathbb{R}^n \to \mathbb{R}^m$ given by its formula we have

$$\mathrm{D}f = \mathrm{D}f_1 \cap \mathrm{D}f_2 \cap \dots \cap \mathrm{D}f_m\,.$$

Definition 1.7. Let $c \in \mathbb{R}$ and $f, g:\ \mathbb{R}^n \to \mathbb{R}^m$.
We define the functions $f + g,\ f - g,\ c \cdot f:\ \mathbb{R}^n \to \mathbb{R}^m$ by:

$$(f \pm g)(x) := f(x) \pm g(x); \quad (c \cdot f)(x) := c \cdot f(x).$$

1.2 Limit of a vector-valued function

Definition 1.8. Let $f:\ \mathbb{R}^n \to \mathbb{R}^m,\ x_0 \in \mathbb{R}^n,\ a \in \mathbb{R}^m$. The limit of f in x_0 is a (and we write $\lim\limits_{x \to x_0} f(x) = a$), if for every sequence (x_k) in $\mathbb{R}^n$ it holds that

$$x_0 \neq x_k \to x_0 \Longrightarrow f(x_k) \to a.$$

Theorem 1.9. *Let $f:\ \mathbb{R}^n \to \mathbb{R}^m,\ x_0 \in \mathbb{R}^n,\ a \in \mathbb{R}^m$. Then it holds that*

$$\lim_{x \to x_0} f(x) = a \Longleftrightarrow (\forall U(a))\,(\exists P(x_0))\,(\forall x \in P(x_0)):\ f(x) \in U(a).$$

(Note that $U(a)$ and $P(x_0)$ are subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.)

Observation 1.10 (and a proof to the following theorem)**.** Let $f = (f_1, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$ and $a = (a_1, ..., a_m) \in \mathbb{R}^m$. Then

$$\begin{aligned}
&\lim_{x\to x_0} f(x) = a \Leftrightarrow \\
&\qquad \Leftrightarrow \big[x_0 \neq x_k \to x_0 \Rightarrow f(x_k) = (f_1(x_k), ..., f_m(x_k)) \to a = (a_1, ..., a_m)\big] \Leftrightarrow \\
&\qquad \Leftrightarrow \big[x_0 \neq x_k \to x_0 \Rightarrow \forall i \in \{1, ..., m\} : \ f_i(x_k) \to a_i \text{ for } k \to \infty\big].
\end{aligned}$$

Theorem 1.11. *Let $f = (f_1, \ldots, f_m) : \ \mathbb{R}^n \to \mathbb{R}^m, \ x_0 \in \mathbb{R}^n, \ a = (a_1, \ldots, a_m) \in \mathbb{R}^m$. Then it holds that*

$$\lim_{x\to x_0} f(x) = a \Longleftrightarrow \big[\forall i \in \{1, \ldots, m\} : \ \lim_{x\to x_0} f_i(x) = a_i\big].$$

In other words,

$$\lim_{x\to x_0} (f_1(x), ..., f_m(x)) = \big(\lim_{x\to x_0} f_1(x), \ldots, \lim_{x\to x_0} f_m(x)\big),$$

if at least one side of the equality is valid.

Exercise 1.12. Determine if the given limit exists and if it does, evaluate it:

a) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2+y^2}, \frac{(x+y)^2}{\sin(x^4+y^6)}\right)$;

b) $\displaystyle\lim_{x\to 0} \left(\frac{x}{\tan x}, \frac{\sqrt{5}-\sqrt{5+x^6}}{x^3}\right)$;

c) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x}{\tan x}, \frac{\sqrt{5}-\sqrt{5+y^6}}{y^3}\right)$.

1.3 Continuity of a vector-valued function

Definition 1.13. Let $f : \ \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$.
The function f is <u>continuous at point</u> x_0 if it holds $\lim_{x\to x_0} f(x) = f(x_0)$.

Theorem 1.14. *Let $f : \ \mathbb{R}^n \to \mathbb{R}^m, \ x_0 \in \mathbb{R}^n \cap Df$. Then it holds that*

$$\begin{aligned}
&f \text{ is continuous at point } x_0 \Longleftrightarrow [x_k \to x_0 \Rightarrow f(x_k) \to f(x_0)] \\
&\Longleftrightarrow (\forall U(f(x_0)))\,(\exists U(x_0))\,(\forall x \in U(x_0)) : \ f(x) \in U(f(x_0)).
\end{aligned}$$

Definition 1.15. Let $f : \ \mathbb{R}^n \to \mathbb{R}^m$, $M \subset \mathbb{R}^n$. We say that

- f is <u>continuous at a point</u> $x_0 \in M$ <u>on a set</u> M, if for every sequence (x_k) in $\mathbb{R}^n$ it holds that
$$M \ni x_k \to x_0 \Rightarrow f(x_k) \to f(x_0)\,;$$
- f is <u>continuous on a set</u> M, if it is continuous on M at every point $x_0 \in M$;
- f is <u>continuous</u>, if it is continuous on Df.

Theorem 1.16. *Let $f = (f_1, \dots, f_m) :\ \mathbb{R}^n \to \mathbb{R}^m$, $x_0 \in M \subset \mathbb{R}^n$. Then f is continuous at x_0, continuous at x_0 on M, or continuous on M if for every $i \in \{1, \dots, m\}$ the function f_i is continuous at x_0, continuous at x_0 on M, or continuous on M, respectively.*

Example 1.17. Linear mapping, i.e. vector-valued function $\mathcal{A} :\ \mathbb{R}^n \to \mathbb{R}^m$ given by the formula

$$\mathcal{A}(x) := \left(\mathbb{A}x^T\right)^T,$$

where

$$\mathbb{A} = \begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ & & \dots\dots & \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{pmatrix}$$

denotes a real matrix of the dimension $(m \times n)$ is an important example of a continuous vector-valued function.

Let us prove that the mapping $\mathcal{A}$ defined above is indeed continuous. Firstly notice that $\mathrm{D}\mathcal{A} = \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ denote an arbitrary but fixed point and (x_k) a sequence in $\mathbb{R}^n$ such that

$$x_k = (x_{k1}, ..., x_{kn}) \to x_0 = (x_1, ..., x_n).$$

Then for all $i \in \{1, ..., n\}$ we have $x_{ki} \to x_i$ (for $k \to \infty$), and thus

$$\mathcal{A}(x_k) = \left(\mathbb{A}x_k^T\right)^T = \left(\begin{pmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ & & \dots\dots & \\ a_{m1}, & a_{m2}, & \dots, & a_{mn} \end{pmatrix} \begin{pmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kn} \end{pmatrix}\right)^T$$

$$= (a_{11}x_{k1} + \dots + a_{1n}x_{kn},\ \dots\ , a_{m1}x_{k1} + \dots + a_{mn}x_{kn}) \to$$

$$\to (a_{11}x_1 + \dots + a_{1n}x_n,\ \dots\ , a_{m1}x_1 + \dots + a_{mn}x_n) = \left(\mathbb{A}x_0^T\right)^T = \mathcal{A}(x_0).$$

This concludes the proof (see Definition 1.15).

Exercise 1.18. If possible, define the vector-valued function f at point c so that it is continuous at c:

a) $f(x) := \left(\dfrac{\tan(x-\pi)}{x-\pi}, x^2 + 6, \dfrac{\sin x}{x}\right)$, $c = \pi$;

b) $f(x) := \left(\dfrac{1-\cos(2x)}{x^2}, \dfrac{1}{x^2}\left(\sin\left(\dfrac{1}{x}\right) - 6\right), \dfrac{\sin x}{x}\right)$, $c = 0$.

1.4 Differential of a vector-valued function

Definition 1.19. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The vector-valued function f is differentiable at point c if there exists a linear mapping $\mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m$ such that for the vector-valued function $\omega: \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$\omega(h) := f(c+h) - f(c) - \mathcal{A}(h)$$

it holds that

$$\lim_{h \to (0,...,0)} \frac{\omega(h)}{\|h\|} = (0, ..., 0).$$

The mapping $\mathcal{A}$ is denoted by $\mathrm{d}f_c$ and we call it the differential of the vector-valued function f at point c.

Remark 1.20 (illustrating the following theorem). Let $f = (f_1, f_2, f_3): \mathbb{R}^2 \to \mathbb{R}^3$ with f_1, f_2, f_3 differentiable at point $c = (c_1, c_2) \in \mathbb{R}^2$. Then for 'small' $h = (h_1, h_2) \in \mathbb{R}^2$ we have

$$\begin{aligned}
(f(c+h) - f(c))^T &= \begin{pmatrix} f_1(c+h) - f_1(c) \\ f_2(c+h) - f_2(c) \\ f_3(c+h) - f_3(c) \end{pmatrix} \doteq \begin{pmatrix} \mathrm{d}(f_1)_c(h) \\ \mathrm{d}(f_2)_c(h) \\ \mathrm{d}(f_3)_c(h) \end{pmatrix} = \\
&= \begin{pmatrix} \frac{\partial f_1}{\partial x}(c)h_1 + \frac{\partial f_1}{\partial y}(c)h_2 \\ \frac{\partial f_2}{\partial x}(c)h_1 + \frac{\partial f_2}{\partial y}(c)h_2 \\ \frac{\partial f_3}{\partial x}(c)h_1 + \frac{\partial f_3}{\partial y}(c)h_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(c), & \frac{\partial f_1}{\partial y}(c) \\ \frac{\partial f_2}{\partial x}(c), & \frac{\partial f_2}{\partial y}(c) \\ \frac{\partial f_3}{\partial x}(c), & \frac{\partial f_3}{\partial y}(c) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \\
&:= f'(c) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = f'(c)h^T.
\end{aligned}$$

Theorem 1.21. *If the vector-valued function $f = (f_1, ..., f_m): \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $c \in \mathbb{R}^n$, then there exist the first partial derivatives of all functions $f_1, ..., f_m$ at point c with respect to all variables and it holds that*

$$(\mathrm{d}f_c(h))^T = f'(c)h^T,$$

where

$$f'(c) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(c), & \frac{\partial f_1}{\partial x_2}(c), & \dots, & \frac{\partial f_1}{\partial x_n}(c) \\ & & \dots\dots & \\ \frac{\partial f_m}{\partial x_1}(c), & \frac{\partial f_m}{\partial x_2}(c), & \dots, & \frac{\partial f_m}{\partial x_n}(c) \end{pmatrix}, \quad f'(c) \text{ is the Jacobi matrix},$$

i.e.

$$\mathrm{d}f_c(h) = (\mathrm{d}(f_1)_c(h), ..., \mathrm{d}(f_m)_c(h)).$$

Theorem 1.22. *A vector-valued function $f = (f_1, ..., f_m): \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at point $c \in \mathbb{R}^n$ if and only if for every $i \in \{1, ..., m\}$ the function $f_i: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}^n$.*

Example 1.23. Let us determine $\mathrm{d}f_c$, where

$$f:\ \mathbb{R}^2 \to \mathbb{R}^3,\ f(u,v) := (\cos u, \sin u, v),\ \mathrm{D}f = \left[0, \frac{\pi}{2}\right] \times [0,2];\ c = \left(\frac{\pi}{4}, 1\right).$$

Solution. Note that

$$f'((u,v)) := f'(u,v) = \begin{pmatrix} -\sin u, & 0 \\ \cos u, & 0 \\ 0, & 1 \end{pmatrix},\quad f'(c) = \begin{pmatrix} -\frac{\sqrt{2}}{2}, & 0 \\ \frac{\sqrt{2}}{2}, & 0 \\ 0, & 1 \end{pmatrix},$$

and thus

$$\begin{aligned} \mathrm{d}f_c:\ h = (h_1,h_2) \mapsto (f'(c)h^T)^T &= \left(\begin{pmatrix} -\frac{\sqrt{2}}{2}, & 0 \\ \frac{\sqrt{2}}{2}, & 0 \\ 0, & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right)^T \\ &= \left(-\frac{\sqrt{2}}{2}h_1, \frac{\sqrt{2}}{2}h_1, h_2\right) = h_1\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) + h_2(0,0,1)\,. \end{aligned}$$

For points (u,v) 'close' to the point $c := (c_1, c_2)$ we have

$$\begin{aligned} f(u,v) &\doteq f(c_1,c_2) + \mathrm{d}f_c(u-c_1, v-c_2) = \\ &= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right) + \left(u - \frac{\pi}{4}\right)\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) + (v-1)(0,0,1)\,. \end{aligned}$$

The plane

$$\begin{aligned} \tau := \Big\{(x,y,z) \in \mathbb{R}^3:\ (x,y,z) =& \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right) + \left(u - \frac{\pi}{4}\right)\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \\ &+ (v-1)(0,0,1),\ (u,v) \in \mathbb{R}^2\Big\} \end{aligned}$$

is the <u>tangent plane to 'the surface'</u> $f(\mathrm{D}f)$ <u>at the point</u> $f(c) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$.

Exercise 1.24. Determine if the vector-valued function f is differentiable at point c and if so, evaluate $f'(c)$ and $\mathrm{d}f_c(h)$:

a) $f(x,y,z) := \left(x^3y^2z, \dfrac{x-y}{z}\right)$, $c = (1,2,3)$, $h = (h_1,h_2,h_3)$;

b) $f(x) := (\cos x, \sin x)$, $c = \dfrac{\pi}{4}$, $h = -\sqrt{2}$;

c) $f(x,y,z) := (xy, \sin(xy), \arcsin x)$, $c = (1,1,6)$, $h = (h_1,h_2,h_3)$.

Exercise 1.25. Let

$$f = (f_1, ..., f_m):\ \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad g = (g_1, ..., g_k):\ \mathbb{R}^m \to \mathbb{R}^k$$

denote vector-valued functions such that for every $i \in \{1, ..., m\}$ and every $j \in \{1, ..., k\}$ it holds that

$$f_i \in C^1(\mathbb{R}^n) \quad \text{and} \quad g_j \in C^1(\mathbb{R}^m).$$

Prove that for every $c \in \mathbb{R}^n$ it holds that

$$(g \circ f)'(c) = g'(f(c))\, f'(c).$$

Example 1.26. Evaluate $f'(c)$, $g'(f(c))$ and $(g \circ f)'(c)$, where $c = (1, 1)$,

$$f(x,y) := \left(x^2 + y^2, \ln x + \ln y, \frac{x}{y}\right), \ g(u,v,w) := \left(uv + 1, u^2 - v^2 + w, w - u\right).$$

Solution.

$$f'(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x}, & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x, & 2y \\ \frac{1}{x}, & \frac{1}{y} \\ \frac{1}{y}, & -\frac{x}{y^2} \end{pmatrix} \Rightarrow f'(c) = \begin{pmatrix} 2, & 2 \\ 1, & 1 \\ 1, & -1 \end{pmatrix},$$

$$f(c) = (2,0,1), \ g'(u,v,w) = \begin{pmatrix} v, & u, & 0 \\ 2u, & -2v, & 1 \\ -1, & 0, & 1 \end{pmatrix} \Rightarrow g'(f(c)) = \begin{pmatrix} 0, & 2, & 0 \\ 4, & 0, & 1 \\ -1, & 0, & 1 \end{pmatrix},$$

and thus

$$(g \circ f)'(c) = g'(f(c))\, f'(c) = \begin{pmatrix} 2, & 2 \\ 9, & 7 \\ -1, & -3 \end{pmatrix}.$$

2 Curves in $\mathbb{R}^m$

Definition 2.1. A continuous vector-valued function in $\mathbb{R}^m$

$$\varphi : \ I \to \mathbb{R}^m, \text{ where } I = \mathrm{D}\varphi \subset \mathbb{R} \text{ is an interval}$$

is called a <u>curve</u>. The set

$$\langle\varphi\rangle := \varphi(I) = \{\varphi(t) : \ t \in I\} \subset \mathbb{R}^m$$

is <u>the image of the curve</u> φ. If $M = \langle\varphi\rangle$, φ defines <u>a parametrization of the set</u> M.

A curve φ is called:

- <u>a simple curve</u>, if φ is injective;
- <u>a closed curve</u>, if $I = [a, b]$ $(a, b \in \mathbb{R};\ a < b)$ and $\varphi(a) = \varphi(b)$;
- <u>a simple closed curve</u>, if φ is closed and

$$\forall t_1, t_2 \in [a, b] : \ [0 < |t_1 - t_2| < b - a \Rightarrow \varphi(t_1) \neq \varphi(t_2)].$$

If $I = [a, b]$, the point $\varphi(a)$ $(\varphi(b))$ is called <u>the initial</u> (<u>the terminal</u>) <u>point of the curve</u> φ. For every curve $\varphi : \ I \to \mathbb{R}^m$ we define <u>the curve of opposite orientation</u> as

$$-\varphi : \ J \to \mathbb{R}^m, \text{ where } J = \{t \in \mathbb{R} : \ -t \in I\} \text{ and } (-\varphi)(t) := \varphi(-t).$$

Example 2.2. For

$$\varphi : \ [-1, 3] \to \mathbb{R}^2, \ \varphi(t) := (t, 1 + t),$$

the curve with the opposite orientation is given by

$$-\varphi : \ [-3, 1] \to \mathbb{R}^2, \ (-\varphi)(t) := (-t, 1 - t).$$

Definition 2.3. A curve $\varphi = (\varphi_1, ..., \varphi_m) : \ [a, b] \to \mathbb{R}^m$ is <u>regular</u> in $\mathbb{R}^m$, if the following conditions are met:

i) φ is injective (i.e. φ is simple);

ii) $\varphi \in C^1$ in $[a, b]$ (i.e. for every $i \in \{1, ..., m\}$ the function φ_i is continuously differentiable in $[a, b]$);

iii) $\varphi'(t) = (\varphi_1'(t), ..., \varphi_m'(t)) \neq (0, ..., 0)$ for every $t \in (a, b)$, [1]
$\varphi'(a) := ((\varphi_1)_+'(a), ..., (\varphi_m)_+'(a)) \neq (0, ..., 0)$,
$\varphi'(b) := ((\varphi_1)_-'(b), ..., (\varphi_m)_-'(b)) \neq (0, ..., 0)$.

[1] We abuse the notation by writing

$$\varphi'(t) = (\varphi_1'(t), ..., \varphi_m'(t))$$

instead of

$$(\varphi'(t))^T = (\varphi_1'(t), ..., \varphi_m'(t)).$$

Remark 2.4 (to the geometrical interpretation of $\varphi'(t)$).
Let $\varphi : [a,b] \to \mathbb{R}^m$ denote a regular curve. Then

$$\begin{aligned}\varphi'(t) &= \left(\lim_{h\to 0}\frac{\varphi_1(t+h)-\varphi_1(t)}{h}, \ \ldots \ , \lim_{h\to 0}\frac{\varphi_m(t+h)-\varphi_m(t)}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{\varphi_1(t+h)-\varphi_1(t)}{h}, \ \ldots \ , \frac{\varphi_m(t+h)-\varphi_m(t)}{h}\right) = \lim_{h\to 0}\frac{\varphi(t+h)-\varphi(t)}{h}.\end{aligned}$$

The line

$$\{\varphi(t) + h\varphi'(t) : \ h \in \mathbb{R}\}$$

is called the <u>tangent line to the curve</u> φ in point t; the vector $\varphi'(t)$ is called the <u>tangent vector to the curve</u> φ in point t.

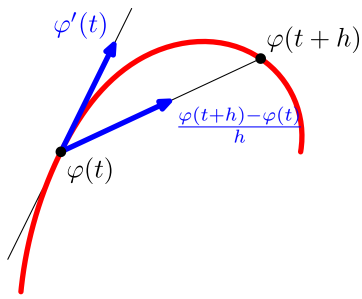

Figure 2.1: to illustrate the geometrical interpretation of $\varphi'(t)$

Definition 2.5. A curve $\varphi : \ [a,b] \to \mathbb{R}^m$ is <u>piecewise smooth</u>, if there exists a partitioning

$$D : \ a = t_0 < t_1 < \ldots < t_n = b$$

of the interval $[a,b]$ such that for every $i \in \{1,\ldots,n\}$ the curve

$$\psi_i := \varphi|_{[t_{i-1},t_i]}$$

$$(\text{i.e. } \mathrm{D}\psi_i = [t_{i-1},t_i], \ \psi_i(t) := \varphi(t))$$

is regular.

Example 2.6. Draw the image of the given curves and determine which of them are

- simple;
- closed;
- simple closed;
- regular;
- piecewise smooth:

a) $\varphi_a(t) := (3 + 2\cos t, 2 + 2\sin t)$, $D_{\varphi_a} = [0, 2\pi]$;

b) $\varphi_b(t) := (3 + 2\cos(2t), 2 + 2\sin(2t)),\ D_{\varphi_b} = [0, 2\pi];$

c) $\varphi_c(t) := \left(\frac{2000}{\sqrt{1+t^2}}, \frac{2000t}{\sqrt{1+t^2}}\right),\ D_{\varphi_c} = \mathbb{R};$

d) $\varphi_d(t) := (t, |t|),\ D_{\varphi_d} = [-2, 2].$

Solution.

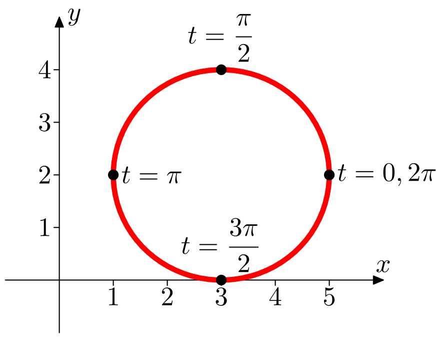

Figure 2.2: $\langle\varphi_a\rangle$ from Ex. 2.6a)

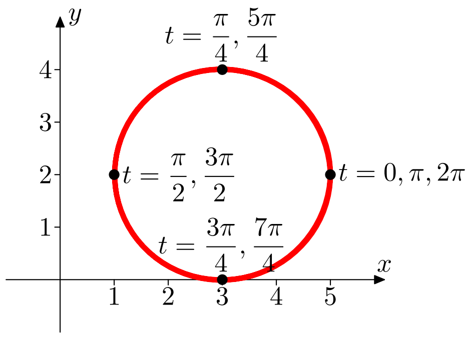

Figure 2.3: $\langle\varphi_b\rangle$ from Ex. 2.6b)

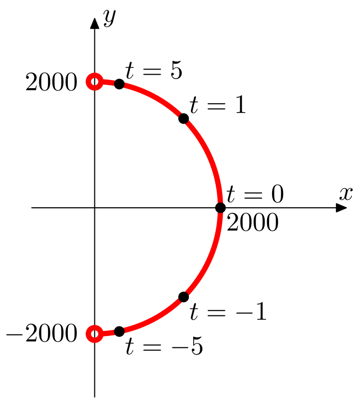

Figure 2.4: $\langle\varphi_c\rangle$ from Ex. 2.6c)

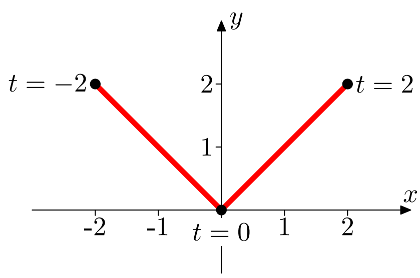

Figure 2.5: $\langle\varphi_d\rangle$ from Ex. 2.6d)

One can easily check (the images of the given curves in Figures 2.2-2.5 provide helpful hints) that

- φ_c and φ_d are simple;
- φ_a and φ_b are closed;
- φ_a is the only closed simple curve;
- none of the curves is regular;
- φ_a, φ_b, and φ_d are piecewise smooth.

Remarks 2.7 (to Ex. 2.6).

- for the curve $\varphi(t) := (3+2\cos t, 2+2\sin t)$, $D_\varphi = [0, 3\pi]$, it holds that $\langle\varphi\rangle = \langle\varphi_a\rangle = \langle\varphi_b\rangle$, but φ is not closed;
- there does not exist a regular curve that parametrizes $\langle\varphi_d\rangle$;
- for the curve $\varphi(t) = (t^3, |t^3|)$, $t \in [-\sqrt[3]{2}, \sqrt[3]{2}]$, it holds that $\langle\varphi\rangle = \langle\varphi_d\rangle$, but φ is not piecewise smooth.

Exercise 2.8. Draw the image of the curve φ defined in the interval I and determine whether it defines a simple, closed, regular, or piecewise smooth curve:

a) $\varphi(t) := (\cos t, 2+\arcsin(\cos t))$, $I = [-\pi, \pi]$;

b) $\varphi(t) := (2\sin^2 t, 4\cos^2 t)$, $I = [0, \frac{\pi}{2}]$;

c) $\varphi(t) := (t^2-2t+3, t^2-2t+1)$, $I = (1, +\infty)$.

Example 2.9. Parametrize Ω if

a) $\Omega = \{(x,y) \in \mathbb{R}^2 :\ 3x+2y = 1 \ \wedge\ x \in [1,3]\}$;

b) $\Omega = \{(x,y) \in \mathbb{R}^2 :\ \frac{x^2}{4} + \frac{y^2}{9} = 1\}$;

c) $\Omega = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 9 \ \wedge\ 2x+y-3z = 0\}$;

d) $\Omega = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 4 \ \wedge\ x^2+y^2 = 2x \ \wedge\ z \geq 0\}$.

Solution.

a) $\Omega = \{(x,y) \in \mathbb{R}^2 :\ y = \frac{1-3x}{2} \ \wedge\ x \in [1,3]\} = \langle\varphi\rangle$, where

$$\varphi(t) := \left(t, \frac{1-3t}{2}\right), \quad t \in [1,3].$$

b) The given set is an ellipse with axes of lengths 2 and 3. To parametrize it we make use of the generalized polar coordinates:

$$\begin{aligned}\Omega &= \Big\{(2r\cos t, 3r\sin t) :\ \frac{(2r\cos t)^2}{4} + \frac{(3r\sin t)^2}{9} = 1 \wedge r \geq 0 \wedge t \in [0,2\pi]\Big\}\\ &= \Big\{(2\cos t, 3\sin t) :\ t \in [0,2\pi]\Big\},\end{aligned}$$

and thus

$$\Omega = \langle\varphi\rangle, \text{ where } \varphi(t) := (2\cos t, 3\sin t),\ t \in [0,2\pi].$$

c) The set Ω defines a circle in space (centered in $s = (0,0,0)$ with the radius $r = 3$ and lying in the plane $2x+y-3z = 0$). In Example 2.6a) we showed that the set

$$\begin{aligned}\{(x,y) \in \mathbb{R}^2 : (x,y) &= (3+2\cos t, 2+2\sin t)\\ &= (3,2) + 2\cos t\,(1,0) + 2\sin t\,(0,1),\ t \in [0,2\pi]\}\end{aligned}$$

is a circle (in $\mathbb{R}^2$) centered in $(3,2)$ and with the radius of 2. Similarly it can be shown (think this through) that the set

$$\{(x,y,z) \in \mathbb{R}^3 :\ (x,y,z) = (s_1,s_2,s_3) + r\cos t\,(u_1,u_2,u_3) + r\sin t\,(v_1,v_2,v_3),\ t \in [0,2\pi]\}$$

is a circle (in $\mathbb{R}^3$) centered in $s = (s_1, s_2, s_3)$ and of the radius r, which 'lies' in the plane defined by the mutually orthogonal direction vectors

$$u = (u_1, u_2, u_3) \text{ and } v = (v_1, v_2, v_3).$$

Let us now return to our task. We already know the origin s and the radius r. It remains to find two (arbitrary) vectors u and v described above. It suffices to choose two (arbitrary) linearly independent vectors in the plane $2x + y - 3z = 0$, e.g. $\tilde{u} = (1, -2, 0)$ a $\tilde{v} = (3, 0, 2)$ and orthonormalize them:

$$u = \frac{\tilde{u}}{\|\tilde{u}\|} = \frac{1}{\sqrt{5}}(1, -2, 0), \quad v = \frac{\tilde{v} - (\tilde{v} \cdot u)u}{\|\tilde{v} - (\tilde{v} \cdot u)u\|} = \frac{1}{\sqrt{70}}(6, 3, 5),$$

where $\tilde{v} \cdot u = (3, 0, 2) \cdot (\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0) = \frac{3}{\sqrt{5}}$ is the inner product of the vectors $\tilde{v}$ and u.
Another option is to choose an arbitrary unit vector in the plane $2x + y - 3z = 0$, e.g. $u = \frac{\tilde{u}}{\|\tilde{u}\|} = \frac{1}{\sqrt{5}}(1, -2, 0)$, and determine v as the cross product of the vector u and a unit normal vector of the plane $2x + y - 3z = 0$, i.e. the vector $n = \frac{1}{\sqrt{14}}(2, 1, -3)$.
To conclude – one (of the infinitely many) parametrizations of the set Ω is given by the curve

$$\begin{aligned}\varphi(t) &:= (0, 0, 0) + 3\cos t\,\frac{1}{\sqrt{5}}(1, -2, 0) + 3\sin t\,\frac{1}{\sqrt{70}}(6, 3, 5) = \\ &= \left(\frac{3}{\sqrt{5}}\cos t + \frac{18}{\sqrt{70}}\sin t, -\frac{6}{\sqrt{5}}\cos t + \frac{9}{\sqrt{70}}\sin t, \frac{15}{\sqrt{70}}\sin t\right), \ t \in [0, 2\pi].\end{aligned}$$

d) Let us present two ways of tackling the task. The first one, making use of cylindrical coordinates, leads to

$$\begin{aligned}\Omega &= \{(r\cos t, r\sin t, z) \in \mathbb{R}^3 : \ r^2 + z^2 = 4 \wedge r^2 = 2r\cos t \wedge z \geq 0 \wedge \\ &\qquad \wedge r \geq 0 \wedge t \in [-\pi, \pi]\} \\ &= \{(r\cos t, r\sin t, z) \in \mathbb{R}^3 : \ z = \sqrt{4 - r^2} \wedge r = 2\cos t \wedge r \geq 0 \wedge t \in [-\pi, \pi]\} \\ &= \left\{(2\cos^2 t, \sin(2t), 2|\sin t|) \in \mathbb{R}^3 : \ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}\end{aligned}$$

and the parametrization

$$\varphi_1(t) := \left(2\cos^2 t, \sin(2t), 2|\sin t|\right), \quad \mathrm{D}\varphi_1 = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The second approach is based on the observation

$$\begin{aligned}\Omega &= \{(x, y, z) \in \mathbb{R}^3 : \ (x - 1)^2 + y^2 = 1 \wedge z = \sqrt{4 - 2x}\} \\ &= \{(\cos t + 1, \sin t, \sqrt{4 - 2(\cos t + 1)}) \in \mathbb{R}^3 : \ t \in [0, 2\pi]\},\end{aligned}$$

and thus $\Omega = \langle\varphi_2\rangle$, where

$$\varphi_2(t) := (\cos t + 1, \sin t, \sqrt{2 - 2\cos t}) = \left(\cos t + 1, \sin t, 2\sin\frac{t}{2}\right), \ \mathrm{D}\varphi_2 = [0, 2\pi].$$

Exercise 2.10. Parametrize the set Ω, if

a) $\Omega = \{(x, y) \in \mathbb{R}^2 : \ x^2 - y^2 = 1 \ \wedge \ x \geq 0\}$;

b) $\Omega = \{(x, y) \in \mathbb{R}^2 : \ y^2 = x \ \wedge \ x \leq 2\}$;

c) $\Omega = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 9 \ \wedge \ x^2 + y^2 - z^2 = 0 \ \wedge \ z \geq 0\}$;

d) $\Omega = \{(x, y, z) \in \mathbb{R}^3 : \ z = x^2 - y^2 \ \wedge \ x^2 + y^2 = 6\}$;

e) $\Omega = \{(x, y, z) \in \mathbb{R}^3 : \ y^2 = x \ \wedge \ z^2 = y\}$.

3 Line Integral

3.1 Line integral of the first kind

Motivation 3.1. Assume that $\varphi: [a,b] \to \mathbb{R}^2$ denotes a regular curve and $f: \mathbb{R}^2 \to \mathbb{R}$ denotes a positive continuous function in $\langle\varphi\rangle$. Let us compute the 'surface area' τ, where

$$\tau = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \langle\varphi\rangle \ \wedge \ 0 \le z \le f(x,y)\}.$$

For a partitioning D,

$$D: \ a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $[a,b]$ it is natural to approximate the area by

$$\begin{aligned}\sum_{k=0}^{n-1} f(\varphi(t_k)) \cdot \|\varphi(t_{k+1}) - \varphi(t_k)\| &\doteq \sum_{k=0}^{n-1} f(\varphi(t_k)) \cdot \|\varphi'(t_k) \cdot (t_{k+1} - t_k)\| \\ &= \sum_{k=0}^{n-1} f(\varphi(t_k)) \cdot \|\varphi'(t_k)\| \cdot (t_{k+1} - t_k) \approx \int_a^b f(\varphi(t)) \cdot \|\varphi'(t)\| \, \mathrm{d}t.\end{aligned}$$

Remark 3.2. Now we could continue in the same spirit as in the definition of Riemann integral, i.e. consider a function $f: \mathbb{R}^2 \to \mathbb{R}$ (only) bounded in $\langle\varphi\rangle$, define for each partitioning of $[a,b]$ the corresponding lower and upper sums, ...

However, to make things easier in the following we only define the line integral of the first kind for continuous functions.

Definition 3.3. Let $\varphi: [a,b] \to \mathbb{R}^m$ be a regular curve and let $f: \mathbb{R}^m \to \mathbb{R}$ be continuous in $\langle\varphi\rangle = \varphi([a,b])$. We define <u>the line integral of the first kind of the function</u> f <u>along the curve</u> φ by the equality

$$\int_\varphi f(x) \, \mathrm{d}s := \int_a^b f(\varphi(t)) \cdot \|\varphi'(t)\| \, \mathrm{d}t \,. \tag{3.1}$$

If $\varphi: [a,b] \to \mathbb{R}^m$ is a piecewise smooth curve (i.e. there exists a partitioning D,

$$D: \ a = t_0 < t_1 < ... < t_n = b$$

of the interval $[a,b]$ such that for every $i \in \{1,...,n\}$ the curve ψ_i,

$$\psi_i := \varphi|_{[t_{i-1},t_i]},$$

is a regular curve) and the function $f: \mathbb{R}^m \to \mathbb{R}$ is continuous in $\langle\varphi\rangle$, we define

$$\int_\varphi f(x) \, \mathrm{d}s := \sum_{i=1}^n \int_{\psi_i} f(x) \, \mathrm{d}s \,. \tag{3.2}$$

Remark 3.4. If $\varphi = (\varphi_1, ..., \varphi_m) : [a,b] \to \mathbb{R}^m$ defines a regular curve and $f : \mathbb{R}^m \to \mathbb{R}$ is continuous in $\langle\varphi\rangle$, then the function t,

$$t \mapsto f(\varphi(t)) \cdot \|\varphi'(t)\| = f(\varphi(t)) \cdot \sqrt{(\varphi_1'(t))^2 + \; ... \; + (\varphi_m'(t))^2} \in \mathbb{R}$$

from (3.1) is continuous, and thus integrable in $[a, b]$.

Remark 3.5. It can be shown that the definition $\int_\varphi f(x)\,\mathrm{d}s$ (see (3.2)) is independent of 'the partitioning' of the piecewise smooth curve φ into regular curves ψ_i.

Features of the line integral (linearity, aditivity, ...) follow from the features of the definite (Riemann) integral.

Example 3.6. Evaluate

a) $\int_\varphi (x^2 + y^2)\,\mathrm{d}s$, where $\varphi : [0, 2\pi] \to \mathbb{R}^2$, $\varphi(t) := (\cos(2t), \sin(2t))$;

b) $\int_\varphi (x^2 + y^2 + z^2)\,\mathrm{d}s$, where $\varphi : [0, 2\pi] \to \mathbb{R}^3$, $\varphi(t) := (2\cos t, 2\sin t, t)$;

c) $\int_\varphi (2z - \sqrt{x^2 + y^2})\,\mathrm{d}s$, where $\varphi : [0, 2\pi] \to \mathbb{R}^3$, $\varphi(t) := (t\cos t, t\sin t, t)$.

Solution.

a) $$\int_\varphi (x^2 + y^2)\,\mathrm{d}s = \int_0^{2\pi} (\cos^2(2t) + \sin^2(2t)) \cdot \|(-2\sin(2t), 2\cos(2t))\|\,\mathrm{d}t = \int_0^{2\pi} 2\,\mathrm{d}t = \underline{\underline{4\pi}}\,.$$

b) $$\begin{aligned}\int_\varphi (x^2 + y^2 + z^2)\,\mathrm{d}s &= \int_0^{2\pi} (4\cos^2 t + 4\sin^2 t + t^2) \cdot \|(-2\sin t, 2\cos t, 1)\|\,\mathrm{d}t \\ &= \int_0^{2\pi} (4 + t^2)\sqrt{5}\,\mathrm{d}t = \underline{\underline{\sqrt{5}\Big(8\pi + \frac{8\pi^3}{3}\Big)}}\,.\end{aligned}$$

c) $$\begin{aligned}\int_\varphi f(x,y,z)\,\mathrm{d}s &= \int_0^{2\pi} (2t - \sqrt{t^2}) \cdot \|(\cos t - t\sin t, \sin t + t\cos t, 1)\|\,\mathrm{d}t \\ &= \int_0^{2\pi} t\sqrt{t^2 + 2}\,\mathrm{d}t \overset{(s1)}{=} \frac{1}{2}\int_2^{2+4\pi^2} \sqrt{u}\,\mathrm{d}u = \frac{1}{2}\Big[\frac{2}{3}u^{\frac{3}{2}}\Big]_2^{2+4\pi^2} = \underline{\underline{\frac{2\sqrt{2}}{3}((1 + 2\pi^2)^{\frac{3}{2}} - 1)}}\,.\end{aligned}$$

$((s1)$: we substituted $t^2 + 2 = u)$.

Exercise 3.7. Evaluate

a) $\displaystyle\int_\varphi \sqrt{1 + 4x^2}\,\mathrm{d}s$, where $\varphi : [-1, 2] \to \mathbb{R}^2$, $\varphi(t) := (t, t^2)$;

b) $\displaystyle\int_\varphi \frac{z^2}{x^2 + y^2}\,\mathrm{d}s$, where $\varphi : [0, 2\pi] \to \mathbb{R}^3$, $\varphi(t) := (\cos t, \sin t, t)$.

Example 3.8. Evaluate

$$\int_\varphi x^3 y\,\mathrm{d}s, \quad \int_{-\varphi} x^3 y\,\mathrm{d}s$$

with

$$\varphi : [-1, 3] \to \mathbb{R}^2, \quad \varphi(t) := (t, 1 + t).$$

Solution.

$$\int_{\varphi} x^3 y \,\mathrm{d}s = \int_{-1}^{3} t^3(1+t)\cdot\|(1,1)\|\,\mathrm{d}t = \int_{-1}^{3}(t^3+t^4)\sqrt{2}\,\mathrm{d}t = \underline{\underline{\frac{344}{5}\sqrt{2}}}\,;$$

$$\int_{-\varphi} x^3 y \,\mathrm{d}s = \int_{-3}^{1} (-t)^3(1-t)\cdot\|(-1,-1)\|\,\mathrm{d}t = \int_{-3}^{1}(t^4-t^3)\sqrt{2}\,\mathrm{d}t = \underline{\underline{\frac{344}{5}\sqrt{2}}}\,.$$

Theorem 3.9 (on the independence of parametrization)**.** *Let φ and ψ denote* `simple` *or* `simple closed` *piecewise smooth curves in $\mathbb{R}^m$, let $\langle\varphi\rangle = \langle\psi\rangle$ and assume that $f:\ \mathbb{R}^m \to \mathbb{R}$ is continuous in $\langle\varphi\rangle$. Then*

$$\int_{\varphi} f(x)\,\mathrm{d}s = \int_{\psi} f(x)\,\mathrm{d}s\,.$$

Convention 3.10. Note that under the assumptions from Theorem 3.9 the value $\int_{\varphi} f(x)\,\mathrm{d}s$ is uniquely determined by the function f, the set $\langle\varphi\rangle$ and the fact that φ is a simple (or simple closed) piecewise smooth curve.

In literature it is customary to write

$$\int_{k} f(x)\,\mathrm{d}s,\ \text{ where } k \subset \mathbb{R}^m,$$

or speak of a line integral of the first kind of f along 'the curve' k. This means that $k = \langle\varphi\rangle$ for a simple (or simple closed) piecewise smooth curve φ and that

$$\int_{k} f(x)\,\mathrm{d}s := \int_{\varphi} f(x)\,\mathrm{d}s\,.$$

If there does not exist any curve φ of the required qualities, the symbol $\int_k f(x)\,\mathrm{d}s$ is meaningless!

Example 3.11. Evaluate

a) $\int_k \sqrt{x^2+y^2}\,\mathrm{d}s$, where $k = \{(x,y)\in\mathbb{R}^2:\ x^2+y^2 = 6x\}$;

b) $\int_k (x+y)\,\mathrm{d}s$, where $k \subset \mathbb{R}^2$ is the boundary of a triangle given by the vertices $(0,0)$, $(1,0)$, $(0,1)$;

c) $\int_k x^2 y\,\mathrm{d}s$, where k is the boundary of the circular sector given by the circle $x^2+y^2 = R^2$ $(R>0)$, the positive semi-axis x and the half-line $y = x$, $x \geq 0$;

d) $\int_{\langle\varphi\rangle} y^2\,\mathrm{d}s$, $\varphi(t) := \big(2(t-\sin t), 2(1-\cos t)\big)$, $\mathrm{D}\varphi = [0,2\pi]$.

Solution.

a) First we have to find a curve φ of requested qualities. Since

$$k = \{(x,y)\in\mathbb{R}^2:\ (x-3)^2+y^2 = 3^2\},$$

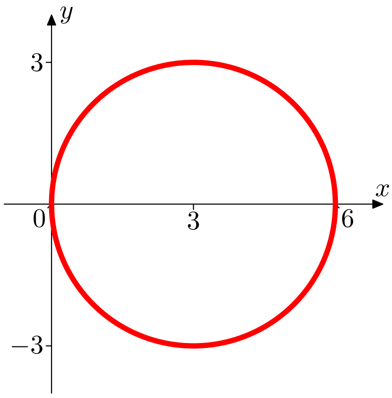

Figure 3.1: k from Example 3.11a)

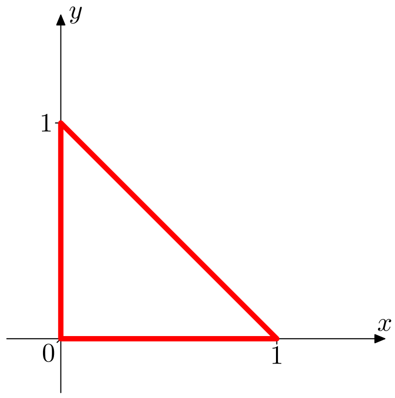

Figure 3.2: k from Example 3.11b)

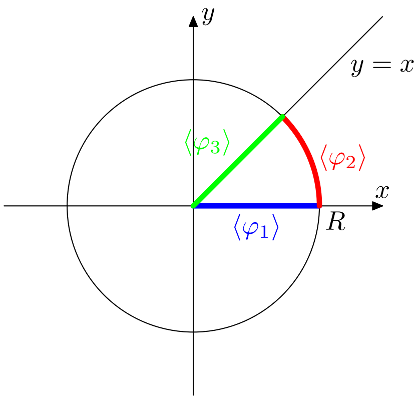

Figure 3.3: $k = \langle\varphi_1\rangle \cup \langle\varphi_2\rangle \cup \langle\varphi_3\rangle$ from Example 3.11c)

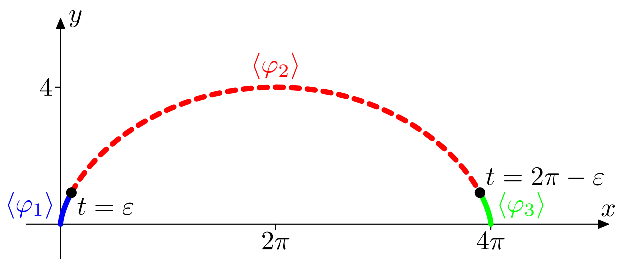

Figure 3.4: $\langle\varphi\rangle = \langle\varphi_1\rangle \cup \langle\varphi_2\rangle \cup \langle\varphi_3\rangle$ from Example 3.11d)

the set k is a circle illustrated in Figure 3.1 and to parametrize it one may use (for example) the 'shifted' polar coordinates, i.e. $k = \langle\varphi\rangle$, where

$$\varphi : [0, 2\pi] \to \mathbb{R}^2, \quad \varphi(t) := (3\cos t + 3, 3\sin t).$$

Then

$$\begin{aligned}
\int_k \sqrt{x^2+y^2}\,\mathrm{d}s &= \int_0^{2\pi} \sqrt{\left(3\cos t + 3\right)^2 + \left(3\sin t\right)^2}\,\|(-3\sin t, 3\cos t)\|\,\mathrm{d}t \\
&= 9\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos t}\,\mathrm{d}t = 9\sqrt{2}\int_0^{2\pi}\sqrt{2\cos^2\left(\frac{t}{2}\right)}\,\mathrm{d}t = 18\int_0^{2\pi}\left|\cos\frac{t}{2}\right|\mathrm{d}t = 36\int_0^{\pi}\cos\frac{t}{2}\,\mathrm{d}t \\
&= \underline{\underline{72}}\,.
\end{aligned}$$

b) The given triangle is depicted in Figure 3.2. We define

$$\varphi(t) := \begin{cases} (t,0), & t \in [0,1], \\ (2-t, t-1), & t \in [1,2], \\ (0, 3-t), & t \in [2,3]. \end{cases}$$

Then (see the convention above):

$$\int_k (x+y)\,\mathrm{d}s = \int_\varphi (x+y)\,\mathrm{d}s$$
$$= \int_0^1 t \cdot \|(1,0)\|\,\mathrm{d}t + \int_1^2 1 \cdot \|(-1,1)\|\,\mathrm{d}t + \int_2^3 (3-t)\cdot\|(0,-1)\|\,\mathrm{d}t = \underline{\underline{1+\sqrt{2}}}\,.$$

Note that it is possible to compute the integral more comfortably; we may split k into individual segments and parametrize these as:

$$\int_k (x+y)\,\mathrm{d}s = \int_{k_1} (x+y)\,\mathrm{d}s + \int_{k_2} (x+y)\,\mathrm{d}s + \int_{k_3} (x+y)\,\mathrm{d}s\,,$$

where

$$\begin{aligned} k_1 &= \langle\varphi_1\rangle; \quad \varphi_1(t) := (t,0),\ t \in [0,1]; \\ k_2 &= \langle\varphi_2\rangle; \quad \varphi_2(t) := (t,1-t),\ t \in [0,1]; \\ k_3 &= \langle\varphi_3\rangle; \quad \varphi_3(t) := (0,t),\ t \in [0,1]. \end{aligned}$$

This leads to

$$\int_k (x+y)\,\mathrm{d}s = \int_0^1 t \cdot \|(1,0)\|\,\mathrm{d}t + \int_0^1 1 \cdot \|(1,-1)\|\,\mathrm{d}t + \int_0^1 t \cdot \|(0,1)\|\,\mathrm{d}t = \underline{\underline{1+\sqrt{2}}}\,.$$

Readers are advised to think about the correctness of the computation above.

c) Clearly

$$\int_k x^2 y\,\mathrm{d}s = \int_{\varphi_1} x^2 y\,\mathrm{d}s + \int_{\varphi_2} x^2 y\,\mathrm{d}s + \int_{\varphi_3} x^2 y\,\mathrm{d}s, \tag{3.3}$$

where

$$\begin{aligned} \varphi_1 &:= (t,0),\ t \in [0,R], \\ \varphi_2 &:= (R\cos t, R\sin t),\ t \in \left[0, \frac{\pi}{4}\right], \\ \varphi_3 &:= (t,t),\ t \in \left[0, \frac{R}{\sqrt{2}}\right], \end{aligned}$$

(see Figure 3.3). It now remains to evaluate the integrals

$$\int_{\varphi_1} x^2 y \, \mathrm{d}s = \int_0^R t^2 \cdot 0 \cdot 1 \, \mathrm{d}t = 0,$$

$$\int_{\varphi_2} x^2 y \, \mathrm{d}s = \int_0^{\frac{\pi}{4}} R^3 \cos^2 t \, \sin t \sqrt{R^2(\sin^2 t + \cos^2 t)} \, \mathrm{d}t$$

$$\overset{\text{(s1)}}{=} -R^4 \int_1^{\frac{\sqrt{2}}{2}} u^2 \, \mathrm{d}u = -\frac{R^4}{3} \left[u^3\right]_1^{\frac{\sqrt{2}}{2}} = \frac{R^4}{3}\left(1 - \frac{\sqrt{2}}{4}\right)$$

(we used the substitution (s1): $\cos t = u$),

$$\int_{\varphi_3} x^2 y \, \mathrm{d}s = \int_0^{\frac{R}{\sqrt{2}}} t^3 \sqrt{2} \, \mathrm{d}t = \sqrt{2} \left[\frac{t^4}{4}\right]_0^{\frac{R}{\sqrt{2}}} = \frac{\sqrt{2}}{4} R^4 \frac{1}{4} = \frac{\sqrt{2}}{16} R^4,$$

and altogether (3.3) gives:

$$\int_k x^2 y \, \mathrm{d}s = \int_{\varphi_1} x^2 y \, \mathrm{d}s + \int_{\varphi_2} x^2 y \, \mathrm{d}s + \int_{\varphi_3} x^2 y \, \mathrm{d}s = \underline{\underline{\frac{16 - \sqrt{2}}{48} R^4}}.$$

d) An illustration of the cycloid $\langle \varphi \rangle$ is depicted in Figure 3.4. Proceeding mechanically we obtain

$$\int_{\langle \varphi \rangle} y^2 \, \mathrm{d}s = \int_0^{2\pi} 4(1 - \cos t)^2 \sqrt{\big(2(1 - \cos t)\big)^2 + (2 \sin t)^2} \; \mathrm{d}t$$

$$= 4 \int_0^{2\pi} (1 - \cos t)^2 \sqrt{8 - 8 \cos t} \; \mathrm{d}t = 8\sqrt{2} \int_0^{2\pi} (1 - \cos t)^{\frac{5}{2}} \; \mathrm{d}t$$

$$= 8\sqrt{2} \int_0^{2\pi} \left(2 \sin^2 \frac{t}{2}\right)^{\frac{5}{2}} \; \mathrm{d}t = 64 \int_0^{2\pi} \sin^5 \frac{t}{2} \; \mathrm{d}t = 128 \int_0^{\pi} \left(1 - \cos^2 u\right)^2 \sin u \; \mathrm{d}u$$

$$= 128 \int_1^{-1} \left(1 - z^2\right)^2 (-1) \; \mathrm{d}z = \underline{\underline{\frac{2048}{15}}} \; .$$

Notice that the calculation above is not correct (although it leads to the correct result). The problem is that in the endpoints we have

$$\varphi'(0) = \varphi'(2\pi) = (0, 0),$$

and thus φ is not a piecewise smooth curve. The question is whether $\langle \varphi \rangle$ can be parametrized by a piecewise smooth curve. If not, the assignment itself would be incorrect.
We 'split' (for some $\varepsilon \in (0, \pi)$[2]) the curve φ into three pieces:

$$\varphi_1 := \varphi|_{[0,\varepsilon]}, \; \varphi_2 := \varphi|_{[\varepsilon, 2\pi - \varepsilon]}, \text{ and } \varphi_3 := \varphi|_{[2\pi - \varepsilon, 2\pi]}.$$

[2]Figure 3.4 depicts such a case for $\varepsilon = 1$.

Clearly φ_2 is a regular curve and in Figure 3.4 one can see that the sets $\langle\varphi_1\rangle$ and $\langle\varphi_3\rangle$ are graphs of smooth functions g and h of variable y, [3] i.e.

$$\langle\varphi_1\rangle = \{(g(y),y) \in \mathbb{R}^2 :\ y \in [0, 2(1-\cos\varepsilon)]\},$$
$$\langle\varphi_3\rangle = \{(h(y),y) \in \mathbb{R}^2 :\ y \in [\,0, 2(1-\underbrace{\cos(2\pi-\varepsilon)}_{=\cos\varepsilon})\,]\}.$$

This leads to smooth parametrizations $\langle\varphi_1\rangle = \langle\psi_1\rangle$ and $\langle\varphi_3\rangle = \langle\psi_3\rangle$, where

$$\psi_1(t) := (g(t),t),\ t \in [0, 2(1-\cos\varepsilon)],$$
$$\psi_3(t) := (h(t),t),\ t \in [\,0, 2(1-\cos\varepsilon)\,].$$

We conclude that $\langle\varphi\rangle$ can be parametrized by a piecewise smooth curve.
If the readers lost confidence in the accuracy of the result above, we can find it again by writing

$$\int_{\langle\varphi\rangle} y^2\,\mathrm{d}s = \int_{\psi_1} y^2\,\mathrm{d}s + \int_{\varphi_2} y^2\,\mathrm{d}s + \int_{\psi_3} y^2\,\mathrm{d}s$$

and verifying that for $\varepsilon \to 0+$ it holds

$$\int_{\psi_1} y^2\,\mathrm{d}s \to 0,\ \int_{\psi_3} y^2\,\mathrm{d}s \to 0,$$

$$\int_{\varphi_2} y^2\,\mathrm{d}s = \int_{\varepsilon}^{2\pi-\varepsilon} \dots\ \mathrm{d}t \to \int_0^{2\pi} \dots\ \mathrm{d}t = \underline{\underline{\frac{2048}{15}}}\,.$$

3.2 Applications of line integral of the first kind

a) **Arc length.**

If φ denotes a piecewise smooth curve, we define its length by the number

$$l(\varphi) := \int_\varphi 1\,\mathrm{d}s.$$

b) **Cylindrical surface area.**

For a ‘surface’[4]

$$\tau := \{(x,y,z) \in \mathbb{R}^3 :\ (x,y) \in \langle\varphi\rangle\ \wedge\ 0 \le z \le f(x,y)\},$$

with a piecewise smooth simple (or simple closed curve) $\varphi :\ [a,b] \to \mathbb{R}^2$, and for a function $f :\ \mathbb{R}^2 \to \mathbb{R}$ continuous and non-negative in $\langle\varphi\rangle$ we define the surface ‘area’ of τ by

$$\sigma(\tau) := \int_\varphi f(x,y)\,\mathrm{d}s\,.$$

[3] Readers can make use of their knowledge of cyclometric functions and find explicit formulae for g and h.

[4] The inverted commas highlight the fact that the term `surface` has not been defined yet. The interpretation thus relies solely on intuition and geometrical imagination.

c) Let $k = \langle\varphi\rangle$ where φ denotes a simple (or simple closed) piecewise smooth curve in $\mathbb{R}^2$, and assume that the (linear) density of the 'curve' k is given by a function $h: \mathbb{R}^2 \to \mathbb{R}$ continuous and non-negative on k. Then we define the relations:

$$m(k) = \int_\varphi h(x,y)\,\mathrm{d}s \text{ ... } \underline{\textbf{mass of the 'curve'}}\ k,$$

$$S_x(k) = \int_\varphi y\,h(x,y)\,\mathrm{d}s \text{ ... } \underline{\textbf{moment of rotation of}}\ k\ \underline{\textbf{with respect to the}}\ x\text{-axis},$$

$$S_y(k) = \int_\varphi x\,h(x,y)\,\mathrm{d}s \text{ ... } \underline{\textbf{moment of rotation of}}\ k\ \underline{\textbf{with respect to the}}\ y\text{-axis},$$

$$T(k) = \left(\frac{S_y(k)}{m(k)}, \frac{S_x(k)}{m(k)}\right) \text{ ... } \underline{\textbf{center of mass of}}\ k,$$

$$I_x(k) = \int_\varphi y^2 h(x,y)\,\mathrm{d}s \text{ ... } \underline{\textbf{moment of intertia of}}\ k\ \underline{\textbf{with respect to the}}\ x\text{-axis},$$

$$I_y(k) = \int_\varphi x^2 h(x,y)\,\mathrm{d}s \text{ ... } \underline{\textbf{moment of intertia of}}\ k\ \underline{\textbf{with respect to the}}\ y\text{-axis}.$$

The formulae can be defined analogously for curves in $\mathbb{R}^3$.

Exercise 3.12. Compute

a) the arc length of (a single turn of a helix) $\varphi(t) := (\cos t, \sin t, 2t)$, $t \in [0, 2\pi]$;

b) the cylindrical surface area τ, where

$$\tau := \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 = 1 \wedge 0 \le z \le xy \wedge x \ge 0 \wedge y \ge 0\};$$

c) coordinates of the center of mass of a quarter of a circle $x^2 + y^2 = 4$ lying in the second quadrant, whose linear density in each point is given by a square of its distance to the point $(2, 0)$.

3.3 Line integral of the second kind

Motivation 3.13.

1. Let $(k) = [\alpha; \beta]$ denote <u>an oriented line segment</u> in $\mathbb{R}^2$ (i.e. with initial point $\alpha \in \mathbb{R}^2$ and terminal point $\beta \in \mathbb{R}^2$) and let $f:\ \mathbb{R}^2 \to \mathbb{R}^2$ define <u>a constant vector field</u> (i.e. $f(x,y) := f_0 \in \mathbb{R}^2$). In physics it is known that <u>work of a vector field</u> f <u>along the oriented curve</u> (k) is given by the inner product

$$f_0 \cdot (\beta - \alpha).$$

2. Let $\varphi:\ [a,b] \to \mathbb{R}^2$ denote a regular curve and let $f:\ \mathbb{R}^2 \to \mathbb{R}^2$ define a continuous vector field in $\langle\varphi\rangle$. Let us compute the work done by a vector field f along 'the oriented curve'$\langle\varphi\rangle$. We consider a partitioning $D:\ a = t_0 < t_1 < ... < t_n = b$ of the interval $[a,b]$. Approximating 'the oriented curves' $\varphi([t_k, t_{k+1}])$ by oriented line segments $\langle\varphi(t_k); \varphi(t_{k+1})\rangle$

and the vector field f by a piecewise constant function given by $f(\varphi(t_k))$ in $[\varphi(t_k);\varphi(t_{k+1})]$, we obtain this approximation of the work of field f:

$$\sum_{k=0}^{n-1} f(\varphi(t_k))\cdot(\varphi(t_{k+1})-\varphi(t_k)) \doteq \sum_{k=0}^{n-1} f(\varphi(t_k))\cdot(\varphi'(t_k)\,(t_{k+1}-t_k)) \approx \int_a^b f(\varphi(t))\cdot\varphi'(t)\,\mathrm{d}t.$$

Definition 3.14. Let $\varphi:\ [a,b]\to\mathbb{R}^m$ denote a regular curve and $f:\ \mathbb{R}^m\to\mathbb{R}^m$ a vector field continuous in $\langle\varphi\rangle=\varphi([a,b])$. The line integral of the second kind of the vector field f along the curve φ is defined as

$$\int_{(\varphi)} f(x)\,\mathrm{d}s := \int_a^b f(\varphi(t))\cdot\varphi'(t)\,\mathrm{d}t. \tag{3.4}$$

If $\varphi:\ [a,b]\to\mathbb{R}^m$ denotes a piecewise smooth curve and the vector field $f:\ \mathbb{R}^m\to\mathbb{R}^m$ is continuous in $\langle\varphi\rangle$, we define

$$\int_{(\varphi)} f(x)\,\mathrm{d}s := \sum_{i=1}^{n}\int_{(\psi_i)} f(x)\,\mathrm{d}s, \tag{3.5}$$

with regular curves ψ_i given by 'a partitioning' of the curve φ (see Definition 2.5 of a piecewise smooth curve).

Remark 3.15 (analogous to Remark 3.4). If the regular curve $\varphi=(\varphi_1,...,\varphi_m)$ and the vector field $f=(f_1,\dots,f_m)$ are of the requested qualities, it holds that

$$\begin{aligned}\int_{(\varphi)} f(x)\,\mathrm{d}s &:= \int_{(\varphi)} f_1(x)\,\mathrm{d}x_1+\dots\ +f_m(x)\,\mathrm{d}x_m\\ &= \int_a^b f_1(\varphi(t))\cdot\varphi_1'(t)+\dots\ +f_m(\varphi(t))\cdot\varphi_m'(t)\,\mathrm{d}t\ ,\end{aligned}$$

with

$$t\mapsto f_1(\varphi(t))\cdot\varphi_1'(t)+\dots\ +f_m(\varphi(t))\cdot\varphi_m'(t)\in\mathbb{R}$$

continuous (and thus integrable) in $[a,b]$.

Remark 3.16 (analogous to Remark 3.5). Again, it can be shown that the definition of $\int_{(\varphi)} f(x)\,\mathrm{d}s$ (see (3.5)) is independent of the partitioning of the piecewise smooth curve φ into regular curves ψ_i.

Example 3.17. Evaluate

a) $I=\int_{(\varphi)} f(x,y)\,\mathrm{d}s$, where $f(x,y):=(y-1,x)$, $\varphi(t):=(3\cos t,2\sin t)$, $t\in[0,\frac{\pi}{2}]$;

b) $I=\int_{(\varphi)}(x^2+y^2)\,\mathrm{d}x+(x^2-y^2)\,\mathrm{d}y$, where $\varphi(t):=\big(t,1-|1-t|\big)$, $t\in[0,2]$;

c) $I=\int_{(\varphi)} x\,\mathrm{d}x+y\,\mathrm{d}y+(xz-y)\,\mathrm{d}z$, where $\varphi(t):=(t^2,2t,4t^3)$, $t\in[0,1]$.

Solution.

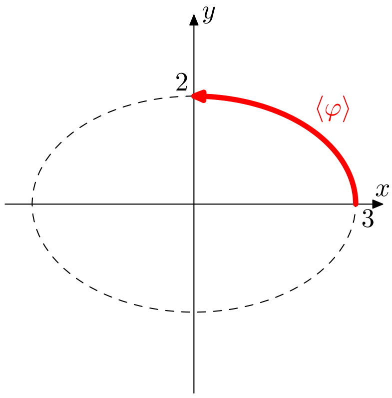

Figure 3.5: $\langle\varphi\rangle$ from Example 3.17 a)

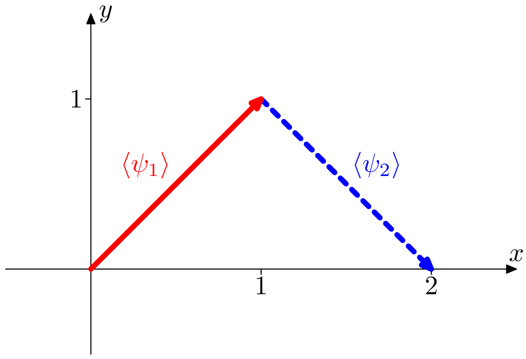

Figure 3.6: $\langle\varphi\rangle = \langle\psi_1\rangle \cup \langle\psi_2\rangle$ from Example 3.17 b)

a)

$$I = \int_0^{\frac{\pi}{2}} (2\sin t - 1)(-3\sin t) + (3\cos t)(2\cos t)\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} -6\sin^2 t + 3\sin t + 6\cos^2 t\,\mathrm{d}t$$
$$= \int_0^{\frac{\pi}{2}} 6\cos(2t) + 3\sin t\,\mathrm{d}t = 3\big[\sin(2t) - \cos t\big]_0^{\frac{\pi}{2}} = \underline{\underline{3}}\,.$$

The geometrical image of φ, i.e. a quarter of the ellipse $x^2/9 + y^2/4 = 1$ lying in the first quadrant, is depicted in Figure 3.5.

b) Let us choose (see Figure 3.6)

$$\psi_1 := \varphi|_{[0,1]}, \quad \psi_2 := \varphi|_{[1,2]}.$$

Then (see Definition 3.14)

$$I = \int_{\langle\psi_1\rangle} (x^2+y^2)\,\mathrm{d}x + (x^2-y^2)\,\mathrm{d}y + \int_{\langle\psi_2\rangle} (x^2+y^2)\,\mathrm{d}x + (x^2-y^2)\,\mathrm{d}y$$
$$= \int_0^1 (t^2+t^2)\cdot 1 + 0\cdot 1\,\mathrm{d}t + \int_1^2 (t^2+(2-t)^2)\cdot 1 + (t^2-(2-t)^2)(-1)\,\mathrm{d}t$$
$$= \Big[2\frac{t^3}{3}\Big]_0^1 + \Big[2\frac{(2-t)^3}{-3}\Big]_1^2 = \frac{2}{3} + 0 - \frac{2}{-3} = \underline{\underline{\frac{4}{3}}}\,.$$

c)

$$I = \int_0^1 t^2\cdot 2t + 2t\cdot 2 + (4t^5 - 2t)\cdot 12t^2\,\mathrm{d}t = \Big[2\frac{t^4}{4} + 4\frac{t^2}{2} + 48\frac{t^8}{8} - 24\frac{t^4}{4}\Big]_0^1 = \underline{\underline{\frac{5}{2}}}\,.$$

Example 3.18. Let

$$f(x,y) := (x^2+y, 2y);$$
$$\varphi(t):\ [-1,3]\to\mathbb{R}^2,\ \varphi(t) := (t, 1+t).$$

Then

$$\int_{(\varphi)} f(x,y)\,\mathrm{d}s = \int_{(\varphi)} (x^2+y)\,\mathrm{d}x + 2y\,\mathrm{d}y = \int_{-1}^{3} (t^2+(1+t), 2(1+t))\cdot(1,1)\,\mathrm{d}t$$
$$= \int_{-1}^{3} (t^2+3t+3)\,\mathrm{d}t = \underline{\underline{\frac{100}{3}}}\,,$$

$$\int_{(-\varphi)} f(x,y)\,\mathrm{d}s = \int_{-3}^{1} ((-t)^2+(1-t), 2(1-t))\cdot(-1,-1)\,\mathrm{d}t = \int_{-3}^{1} (-t^2+3t-3)\,\mathrm{d}t = \underline{\underline{-\frac{100}{3}}}\,.$$

Theorem 3.19 (on the independence of parametrization)**.** *Let* $\varphi : [a,b] \to \mathbb{R}^m$ *and* $\psi : [c,d] \to \mathbb{R}^m$ *denote* `simple` *or* `simple closed` *piecewise smooth curves in* $\mathbb{R}^m$, $\langle\varphi\rangle = \langle\psi\rangle$ *and let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be continuous in* $\langle\varphi\rangle$. *Then it holds that*

$$\int_{(\varphi)} f(x)\,\mathrm{d}s = \int_{(\psi)} f(x)\,\mathrm{d}s,$$

if the curves φ *and* ψ *are of* <u>*the same orientation*</u> *(i.e. there exist real numbers* $t \in (a,b)$, $t^* \in (c,d)$ *and* $\underline{e > 0}$ *such that* $\varphi(t) = \psi(t^*)$ *and* $\varphi'(t) = e\,\psi'(t^*)$*).*

If the curves φ *and* ψ *are of* <u>*opposite orientation*</u> *(i.e. there exist real numbers* $t \in (a,b)$, $t^* \in (c,d)$ *and* $\underline{e < 0}$ *such that* $\varphi(t) = \psi(t^*)$ *and* $\varphi'(t) = e\,\psi'(t^*)$*) it holds that*

$$\int_{(\varphi)} f(x)\,\mathrm{d}s = -\int_{(\psi)} f(x)\,\mathrm{d}s.$$

Convention 3.20 (analogous to Convention 3.10)**.** If $(k) \subset \mathbb{R}^m$ denotes a set for which there exists a simple (or simple closed) piecewise smooth curve φ such that $\langle\varphi\rangle = (k)$, we use the notation $\int_{(k)} f(x)\,\mathrm{d}s$ instead of the proper $\int_{(\varphi)} f(x)\,\mathrm{d}s$. For completeness, it is necessary to accompany the 'curve' (k) with its 'orientation' (i.e. its 'direction') and choose the curve φ so that it is of 'the same orientation'. The example below will make the idea clear.

Example 3.21. Evaluate

a) $I = \int_{(k)} (e^x + y)\,\mathrm{d}x + xy^2\,\mathrm{d}y$, where $(k) = \{(x,y) \in \mathbb{R}^2 :\ y^2 = x \wedge x \le 3\}$ and 'the orientation' (k) is determined by the points $(3,-\sqrt{3})$, $(3,\sqrt{3})$ in this order;

b) $\int_{(k)} f(x,y)\,\mathrm{d}s$, where $f(x,y) := (x+2y, y)$ and $(k) \subset \mathbb{R}^2$ is the oriented boundary of the triangle given by the vertices $(0,0)$, $(1,0)$, $(0,1)$, whose orientation is given by this order of vertices;

c) $\int_{(k)} f(x,y,z)\,\mathrm{d}s$, where

$$f(x,y,z) := (y^2-z^2, z^2-x^2, x^2-y^2),$$
$$(k) = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 1 \wedge x \ge 0 \wedge y \ge 0 \wedge z \ge 0 \wedge xyz = 0\}$$

and the orientation of (k) is given by the order of the points: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

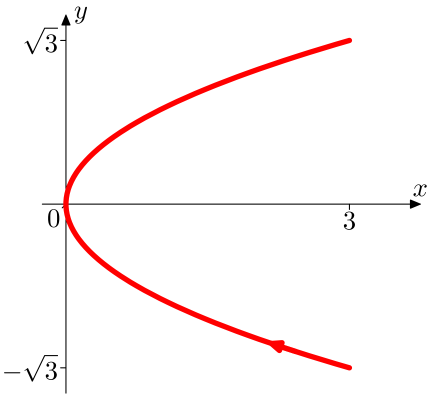

Figure 3.7: (k) from Example 3.21 a)

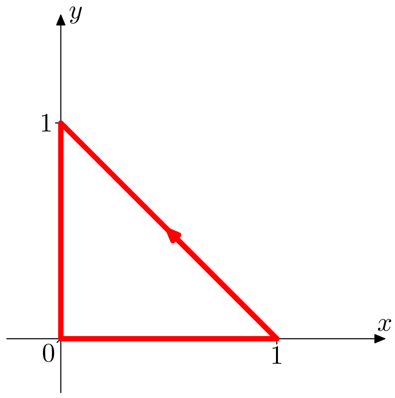

Figure 3.8: (k) from Example 3.21 b)

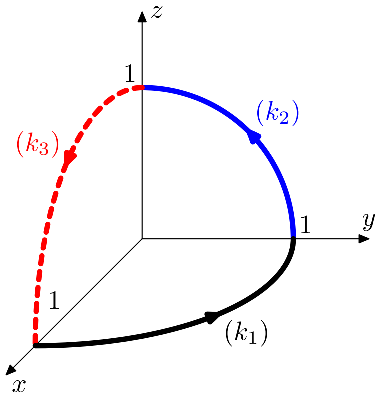

Figure 3.9: $(k) = (k_1) \cup (k_2) \cup (k_3)$ from Example 3.21 c)

Solution.

a) Clearly, we may write (see Figure 3.7)

$$(k) = \{(t^2, t) : \ t \in [-\sqrt{3}, \sqrt{3}]\},$$

and thus

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} (e^{t^2} + t)2t + t^4 \,\mathrm{d}t = \left[e^{t^2} + \frac{2t^3}{3} + \frac{t^5}{5}\right]_{-\sqrt{3}}^{\sqrt{3}} = \underline{\underline{\frac{38\sqrt{3}}{5}}}.$$

b) We choose

$$\varphi(t) := \begin{cases} (t, 0), & t \in [0,1], \\ (2-t, t-1), & t \in [1,2], \\ (0, 3-t), & t \in [2,3], \end{cases}$$

and notice that for this parametrization the 'orientation is consistent'. Thus,

$$\int_{(k)} f(x,y)\,\mathrm{d}s = \int_0^1 (t+0,0)\cdot(1,0)\,\mathrm{d}t + \int_1^2 (2-t+2(t-1),t-1)\cdot(-1,1)\,\mathrm{d}t$$
$$+\int_2^3 (0+2(3-t),3-t)\cdot(0,-1)\,\mathrm{d}t = \int_0^1 t\,\mathrm{d}t + \int_1^2 (-1)\,\mathrm{d}t + \int_2^3 (-3+t)\,\mathrm{d}t = \underline{\underline{-1}}.$$

We may proceed in a more comfortable way (detailed argumentation is left to the diligent reader):

$$\int_{(k)} f(x,y)\,\mathrm{d}s = \int_{(\varphi_1)} f(x,y)\,\mathrm{d}s - \int_{(\varphi_2)} f(x,y)\,\mathrm{d}s - \int_{(\varphi_3)} f(x,y)\,\mathrm{d}s,$$

where

$$\varphi_1(t) := (t,0),\ t\in[0,1];$$
$$\varphi_2(t) := (t,1-t),\ t\in[0,1];$$
$$\varphi_3(t) := (0,t),\ t\in[0,1];$$

and thus

$$\int_{(k)} f(x,y)\,\mathrm{d}s = \int_0^1 (t,0)\cdot(1,0)\,\mathrm{d}t - \int_0^1 (2-t,1-t)\cdot(1,-1)\,\mathrm{d}t$$
$$-\int_0^1 (2t,t)\cdot(0,1)\,\mathrm{d}t = \underline{\underline{-1}}.$$

c) 'The curve' (k) is clearly a union of quarter circles (k_1), (k_2) and (k_3) (see Figure 3.9). These quarter circles can be parametrized, e.g. by:

(k1):	$x=\cos t$, $y=\sin t$, $z=0$, $t\in[0,\frac{\pi}{2}]$; orientation is consistent,	(k2):	$x=0$, $y=\cos t$, $z=\sin t$, $t\in[0,\frac{\pi}{2}]$; orientation is consistent,	(k3):	$x=\sin t$, $y=0$, $z=\cos t$, $t\in[0,\frac{\pi}{2}]$; orientation is consistent,

and thus

$$\int_{(k)} f\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} (\sin^2 t)(-\sin t) + (-\cos^2 t)(\cos t)\,\mathrm{d}t$$
$$+\int_0^{\frac{\pi}{2}} (\sin^2 t)(-\sin t) + (-\cos^2 t)(\cos t)\,\mathrm{d}t$$
$$+\int_0^{\frac{\pi}{2}} (-\cos^2 t)(\cos t) + (\sin^2 t)(-\sin t)\,\mathrm{d}t$$
$$= \int_0^{\frac{\pi}{2}} -3\sin^3 t - 3\cos^3 t\,\mathrm{d}t = \underline{\underline{-4}}.$$

Exercise 3.22. Evaluate

a)
$$\int_{(k)} \frac{1}{|x|+|y|}\,\mathrm{d}x + \frac{1}{|x|+|y|}\,\mathrm{d}y\,,$$
where (k) is the oriented boundary of a square given by its vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ in this order;

b) $\int_{(k)} y^2\,\mathrm{d}x + z^2\,\mathrm{d}y + x^2\,\mathrm{d}z$ where the orientation of 'the curve'
$$(k) = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 9\ \wedge\ x^2+y^2 = 3x\ \wedge\ z \geq 0\}$$
is given by the order of the points: $(0,0,3)$, $(\frac{3}{2}, \frac{3}{2}, \frac{3}{\sqrt{2}})$, $(3,0,0)$.

3.4 The Green theorem

Definition 3.23. A set $\Omega \subset \mathbb{R}^m$ is called a domain, if it holds that:

- Ω is open (i.e. $(\forall x \in \Omega)(\exists U(x)) :\ x \in U(x) \subset \Omega$);
- Ω is connected (i.e. each pair of points in Ω can be connected by a curve lying in Ω; or more precisely: for each pair of points $\alpha,\ \beta \in \Omega$ there exists a curve $\varphi :\ [a,b] \to \Omega \subset \mathbb{R}^m$ such that $\varphi(a) = \alpha$ and $\varphi(b) = \beta$).

Theorem 3.24 (Jordan)**.** *Let φ denote a simple closed curve in $\mathbb{R}^2$. Then there exist* `domains` *$\Omega_1,\ \Omega_2 \subset \mathbb{R}^2$ such that*

- $\mathbb{R}^2 \setminus \langle\varphi\rangle = \Omega_1 \cup \Omega_2$,
- $\Omega_1 \cap \Omega_2 = \emptyset$,
- $\partial\Omega_1 = \partial\Omega_2 = \langle\varphi\rangle$,
- *Ω_1 is bounded and Ω_2 is unbounded in $\mathbb{R}^2$.*

(The domain $\Omega_1 := \operatorname{int}\varphi$ is called the interior of the curve φ, the domain $\Omega_2 := \operatorname{ext}\varphi$ is called the exterior of the curve φ.)

Definition 3.25. Let $\varphi :\ [a,b] \to \mathbb{R}^2$ denote a simple closed piecewise smooth curve and $t \in (a,b)$ a real number such that there exists a (non-zero) tangent vector $\varphi'(t) \in \mathbb{R}^2$.

`A non-zero` vector $n(t) \in \mathbb{R}^2$ is called an exterior normal vector to the curve φ in point t, if it holds:

- $n(t) \cdot \varphi'(t) = 0$,
- $(\exists\delta > 0)(\forall h \in (0,\delta)) :\ \varphi(t) + h\,n(t) \in \operatorname{ext}\varphi$.

A curve φ is positively oriented if the ordered pair $[n(t), \varphi'(t)]$ is oriented as the ordered pair $[e_1, e_2]$;[5] or more precisely if

$$\begin{vmatrix} n_1, & n_2 \\ \tau_1, & \tau_2 \end{vmatrix} > 0,$$

where

$$(n_1, n_2) := n(t), \quad (\tau_1, \tau_2) := \varphi'(t).$$

'When following a positively oriented curve φ its interior lies to the left'.

If

$$\begin{vmatrix} n_1, & n_2 \\ \tau_1, & \tau_2 \end{vmatrix} < 0,$$

the curve φ is negatively oriented. [6]

Remark 3.26. The above definition of positively (or negatively) oriented curve is correct; it is independent of the choice of $t \in (a, b)$ such that there exists $\varphi'(t)$.

Example 3.27 (oriented curves)**.**

a) The curve $\varphi_1(t) := (\cos t, \sin t),\ t \in [0, 2\pi]$ is positively oriented.
b) The curve $\varphi_2(t) := (\sin t, \cos t),\ t \in [0, 2\pi]$is negatively oriented.

Theorem 3.28 (Green)**.** *Let φ denote s simple closed positively oriented and piecewise smooth curve in $\mathbb{R}^2$ and assume that $f = (f_1, f_2) :\ \mathbb{R}^2 \to \mathbb{R}^2$ with $f_1, f_2, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x} :\ \mathbb{R}^2 \to \mathbb{R}$ continuous in $\Omega := \operatorname{int} \varphi \cup \langle \varphi \rangle$.*

Then it holds

$$\iint_\Omega \left(\frac{\partial f_2}{\partial x}(x, y) - \frac{\partial f_1}{\partial y}(x, y) \right) \mathrm{d}x\, \mathrm{d}y = \int_{(\varphi)} f_1(x, y)\mathrm{d}x + f_2(x, y)\mathrm{d}y = \int_{(\varphi)} f(x, y)\mathrm{d}s\,.$$

Proof. We prove the theorem only for a special case of Ω denoting a rectangular domain, i.e.

$$\Omega = [a, b] \times [c, d]$$
$$(a, b, c, d \in \mathbb{R};\quad a < b,\ c < d).$$

[5] $e_1 := (1, 0), \quad e_2 := (0, 1)$.
[6] Notice that the number

$$\begin{vmatrix} n_1, & n_2 \\ \tau_1, & \tau_2 \end{vmatrix}$$

defines the third coordinate of the cross product

$$(n_1, n_2, 0) \times (\tau_1, \tau_2, 0).$$

First, we make use of the Fubini theorem to modify the double integral on the left-hand side of the equality:

$$\iint_\Omega \Big(\frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega \frac{\partial f_2}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_\Omega \frac{\partial f_1}{\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_c^d \left(\int_a^b \frac{\partial f_2}{\partial x}(x,y)\,\mathrm{d}x\right)\mathrm{d}y - \int_a^b \left(\int_c^d \frac{\partial f_1}{\partial y}(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_c^d [f_2(x,y)]_{x=a}^b\,\mathrm{d}y - \int_a^b [f_1(x,y)]_{y=c}^d\,\mathrm{d}x$$

$$= \underline{\int_c^d (f_2(b,y) - f_2(a,y))\,\mathrm{d}y - \int_a^b (f_1(x,d) - f_1(x,c))\,\mathrm{d}x.}$$

The line integral on the right-hand side can be modified by parametrizing individual sides of the rectangle Ω as follows:

$$\int_{(\varphi)} f(x,y)\,\mathrm{d}s = \int_a^b (f_1(t,c)\cdot 1 + f_2(t,c)\cdot 0)\,\mathrm{d}t + \int_c^d (f_1(b,t)\cdot 0 + f_2(b,t)\cdot 1)\,\mathrm{d}t$$

$$- \int_a^b (f_1(t,d)\cdot 1 + f_2(t,d)\cdot 0)\,\mathrm{d}t - \int_c^d (f_1(a,t)\cdot 0 + f_2(a,t)\cdot 1)\,\mathrm{d}t$$

$$= \underline{\int_a^b (f_1(t,c) - f_1(t,d))\,\mathrm{d}t + \int_c^d (f_2(b,t) - f_2(a,t))\,\mathrm{d}t.}$$

Finally, notice that the underscored numbers are equal. □

Example 3.29. Making use of the Green theorem evaluate

a)
$$\int_{(k)} (x+y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y,$$
where (k) is 'a positively oriented' ellipse
$$\left\{(x,y)\in\mathbb{R}^2:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}\quad (a,b>0);$$

b)
$$\int_{(k)} \frac{1}{x}\,\mathrm{arctg}\,\frac{y}{x}\,\mathrm{d}x + \frac{2}{y}\,\mathrm{arctg}\,\frac{x}{y}\,\mathrm{d}y,$$
where (k) is 'a positively oriented' boundary of the domain
$$\Omega = \{(x,y)\in\mathbb{R}^2:\ (1 < x^2+y^2 < 4)\wedge(x < y < x\sqrt{3})\};$$

c)
$$\int_{(k)} yx^2\,\mathrm{d}x + xy\,\mathrm{d}y,$$
where (k) is a boundary of a square given by vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ in this order;

d) are of a disk of the radius $r > 0$

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq r^2\}.$$

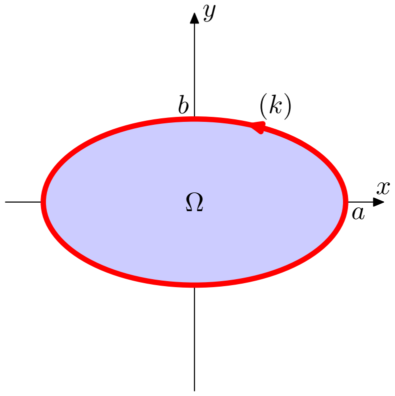

Figure 3.10: Illustration to Example 3.29a)

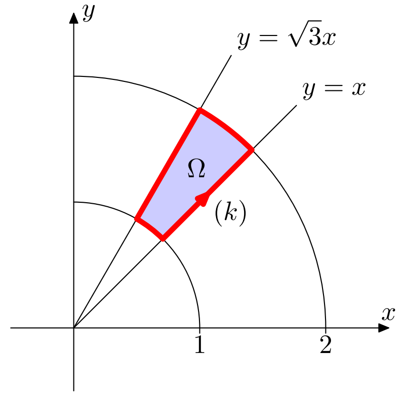

Figure 3.11: Illustration to Example 3.29b)

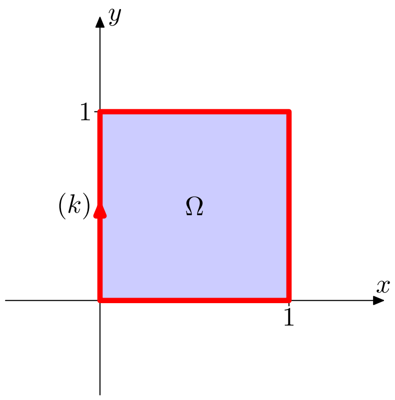

Figure 3.12: Illustration to Example 3.29c)

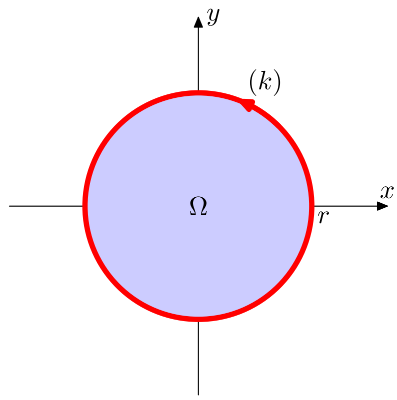

Figure 3.13: Illustration to Example 3.29d)

Solution.

a) We compute the integral using the Green theorem

$$\left(\text{where } \Omega = \left\{(x, y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}\right),$$

substitution to generalized polar coordinates

$$(x = ar\cos t\,, \quad y = br\sin t\,; \quad J(r, t) = abr)\,,$$

and the Fubini theorem:

$$\int_{(k)} (x + y)\,\mathrm{d}x + (y - x)\,\mathrm{d}y = \iint_{\Omega} (-1 - 1)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \left(\int_0^1 (-2abr)\,\mathrm{d}r \right)\,\mathrm{d}t = \underline{\underline{-2\pi ab}}\,.$$

b) The domain Ω and 'curve' (k) are illustrated in Figure 3.11. Since

$$\frac{\partial}{\partial x}\Big(\frac{2}{y}\ \arctan\frac{x}{y}\Big) = \frac{2}{y}\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y} = \frac{2}{x^2+y^2},$$
$$\frac{\partial}{\partial y}\Big(\frac{1}{x}\ \arctan\frac{y}{x}\Big) = \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{1}{x^2+y^2},$$

it holds that

$$\int_{(k)} \frac{1}{x}\ \mathrm{arctg}\ \frac{y}{x}\ \mathrm{d}x + \frac{2}{y}\ \mathrm{arctg}\ \frac{x}{y}\ \mathrm{d}y = \iint_{\Omega} \frac{1}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$$
$$\stackrel{\star}{=} \int_1^2 \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{r^2}\, r\ \mathrm{d}\varphi\right)\mathrm{d}r = \Big(\frac{\pi}{3}-\frac{\pi}{4}\Big)\,[\ln r]_1^2 = \underline{\underline{\frac{\pi}{12}\ln 2}}.$$

In $(\star)$ we used polar coordinates with the Jacobian r and the Fubini theorem.

c) Since (k) is 'a negatively oriented' boundary of the domain $\Omega = (0,1)\times(0,1)$ (see Figure 3.12), we have

$$\int_{(k)} yx^2\,\mathrm{d}x + xy\,\mathrm{d}y = -\iint_{\Omega}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(yx^2)\right)\mathrm{d}x\,\mathrm{d}y$$
$$= -\int_0^1\left(\int_0^1 y - x^2\,\mathrm{d}x\right)\mathrm{d}y = -\int_0^1 y-\frac{1}{3}\ \mathrm{d}y = -(\frac{1}{2}-\frac{1}{3}) = \underline{\underline{-\frac{1}{6}}}.$$

d) Let

$$\varphi:\ [0,2\pi]\to\mathbb{R}^2,\quad \varphi(t) := (r\cos t, r\sin t).$$

For the area of the disk Ω (more precisely: for `the measure of the set` Ω) it holds

$$\lambda(\Omega) = \iint_{\Omega} 1\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_{(\varphi)} (-y)\,\mathrm{d}x + x\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^{2\pi} \big((-r\sin t)(-r\sin t) + (r\cos t)(r\cos t)\big)\,\mathrm{d}t = \pi r^2.$$

This should not surprise the reader.

3.5 Path independence

Observation 3.30. Let $\Omega\subset\mathbb{R}^2$ denote a domain, $\varphi = (\varphi_1,\varphi_2):\ [a,b]\to\Omega$ a regular curve and $f=(f_1,f_2):\ \mathbb{R}^2\to\mathbb{R}^2$ a continuous vector field. Assume that there exists a function (later on `a potential`) $V:\ \mathbb{R}^2\to\mathbb{R}$ of class C^1 in Ω such that for every $(x,y)\in\Omega$ it holds that

$$\mathrm{grad}\ V(x,y) = \left(\frac{\partial V(x,y)}{\partial x}, \frac{\partial V(x,y)}{\partial y}\right) = (f_1(x,y), f_2(x,y)).$$

Consider a function $F:\ \mathbb{R}\to\mathbb{R}$ given by

$$F(t) := V(\varphi(t)) = V(\varphi_1(t),\varphi_2(t)).$$

Since

$$F'(t) = \frac{\partial V}{\partial x}(\varphi_1(t), \varphi_2(t))\, \varphi_1'(t) + \frac{\partial V}{\partial y}(\varphi_1(t), \varphi_2(t))\, \varphi_2'(t)$$
$$= f_1(\varphi_1(t), \varphi_2(t))\, \varphi_1'(t) + f_2(\varphi_1(t), \varphi_2(t))\, \varphi_2'(t),$$

we have

$$\int_{(\varphi)} f(x, y)\, \mathrm{d}s = \int_a^b f_1(\varphi_1(t), \varphi_2(t))\, \varphi_1'(t) + f_2(\varphi_1(t), \varphi_2(t))\, \varphi_2'(t)\, \mathrm{d}t$$
$$= [F(t)]_a^b = V(\varphi(b)) - V(\varphi(a)).$$

If φ is a piecewise smooth curve, it holds that[7]

$$\int_{(\varphi)} f(x, y)\, \mathrm{d}s = \sum_{k=1}^{n} \int_{(\psi_k)} f(x, y)\, \mathrm{d}s = \sum_{k=1}^{n} \Big(V(\varphi(t_k)) - V(\varphi(t_{k-1}))\Big) = V(\varphi(b)) - V(\varphi(a)).$$

Definition 3.31. Let $f: \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function in $\Omega \subset \mathbb{R}^m$. The vector field f is conservative in Ω, if there exists a function $V: \mathbb{R}^m \to \mathbb{R}$ (a so-called potential) such that

$$\operatorname{grad} V(x) = f(x) \quad \text{for every } x \in \Omega.$$

Theorem 3.32 (on path independence)**.** *Let the vector field $f: \mathbb{R}^m \to \mathbb{R}^m$ be continuous in $\Omega \subset \mathbb{R}^m$. Then the following statements are equivalent:*

(a) f is conservative in Ω,

(b) for every connected piecewise smooth curve $\varphi: [a, b] \to \Omega$ it holds that

$$\int_{(\varphi)} f(x)\, \mathrm{d}s = 0,$$

(c) the line integral of the second kind of f is path independent in Ω; i.e. if $\varphi_1: [a, b] \to \Omega$ and $\varphi_2: [c, d] \to \Omega$ denote piecewise smooth curves such that

$$\varphi_1(a) = \varphi_2(c), \quad \varphi_1(b) = \varphi_2(d),$$

then it holds

$$\int_{(\varphi_1)} f(x)\, \mathrm{d}s = \int_{(\varphi_2)} f(x)\, \mathrm{d}s.$$

Moreover, if V is a potential of f in Ω, then the following holds

$$\int_{(\varphi)} f(x)\, \mathrm{d}s = V(\varphi(b)) - V(\varphi(a)) := \int_{\varphi(a)}^{\varphi(b)} f(x)\, \mathrm{d}s$$

[7] Here we consider that same ‘partitioning’ of the curve φ into regular curves as in Definition 2.5.

for every piecewise smooth curve $\varphi : [a,b] \to \Omega$.[8]

One may thus ask:

How to determine whether a given field is conservative?

A partial answer is given in the two remarks presented below.

Remark 3.33. Assume for a while that $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^1 and is conservative (with a potential V) in a domain $\Omega \subset \mathbb{R}^2$. Then (see theorem on symmetry of second derivatives) for every $(x,y) \in \Omega$ it holds that

$$\underline{\frac{\partial f_1}{\partial y}(x,y)} = \frac{\partial}{\partial y}\frac{\partial V}{\partial x}(x,y) = \frac{\partial^2 V}{\partial y \partial x}(x,y) = \frac{\partial^2 V}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\frac{\partial V}{\partial y}(x,y) = \underline{\frac{\partial f_2}{\partial x}(x,y)}.$$

We have thus found `a necessary condition` of the existence of a potential. It can be shown that for <u>a simply connected domain</u> $\Omega \subset \mathbb{R}^2$ (i.e. a domain Ω such that for every simple closed curve $\varphi : [a,b] \to \Omega \subset \mathbb{R}^2$ it holds that $\operatorname{int} \varphi \subset \Omega$), (and a vector function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1 in Ω) the equality

$$\frac{\partial f_1}{\partial y}(x,y) = \frac{\partial f_2}{\partial x}(x,y)$$

also defines `a sufficient condition`.

Remark 3.34. For $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ of class C^1 in a domain $\Omega \subset \mathbb{R}^3$ one can similarly obtain `necessary conditions` of the existence of a potential:

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \quad \text{in } \Omega.$$

Also in this case it holds that for simply connected domains[9] $\Omega \subset \mathbb{R}^3$ the `condition` is also `sufficient`.

In the examples below we present a procedure of finding a potential and its utilization for the evaluation of the line integral of the second kind.

Example 3.35. Evaluate

$$\int_{(\varphi)} y\,\mathrm{d}x + x\,\mathrm{d}y, \quad \text{where } \varphi(t) := (\cos t, \sin t),\ t \in \left[0, \frac{\pi}{4}\right].$$

[8] If we use the symbol

$$\int_\alpha^\beta f(x)\,\mathrm{d}s,$$

with $\alpha, \beta \in \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$, then f must either be conservative in $\mathbb{R}^m$ or it must be clear form the context that we integrate along a curve (with an initial point α and a terminal point β) lying in the domain Ω, where the vector field f is conservative.

[9] A rigorous definition of a simply connected domain in $\mathbb{R}^3$ would be too tedious, we make do with and intuitive understanding that it is a domain ‘without holes’.

Solution. Since $\mathbb{R}^2$ is a simply connected domain, where it holds that

$$\frac{\partial y}{\partial y} = 1 = \frac{\partial x}{\partial x},$$

the vector field $f(x,y) := (y,x)$ is conservative in $\mathbb{R}^2$.

The aim is to find a potential f, i.e. a function $V : \mathbb{R}^2 \to \mathbb{R}$ such that in $\mathbb{R}^2$ it holds that

$$\frac{\partial V}{\partial x}(x,y) = y \quad \text{a} \quad \frac{\partial V}{\partial y}(x,y) = x\,.$$

Integrating the equality

$$\frac{\partial V}{\partial x}(x,y) = y$$

leads to

$$V(x,y) = xy + \psi(y)$$

for a yet unknown function $\psi : \mathbb{R} \to \mathbb{R}$ and every $(x,y) \in \mathbb{R}^2$. Substituting into the second condition $(\frac{\partial V}{\partial y}(x,y) = x)$ yields

$$x = \frac{\partial}{\partial y}(xy + \psi(y)) = x + \psi'(y),$$

and thus $\psi'(y) = 0$. We conclude that $\psi(y) = c$ for some $c \in \mathbb{R}$. Think over carefully (!) that the functions

$$V(x,y) := xy + c, \quad \text{with } c \in \mathbb{R},$$

define all potentials of f in $\mathbb{R}^2$.

To compute the integral one can use any of the potentials above (see Theorem 3.32). For simplicity we choose $c = 0$ to obtain

$$\int_{(\varphi)} y\,\mathrm{d}x + x\,\mathrm{d}y = V\Big(\varphi\Big(\frac{\pi}{4}\Big)\Big) - V(\varphi(0)) = V\bigg(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\bigg) - V(1,0) = \underline{\underline{\frac{1}{2}}}\,.$$

Example 3.36. Evaluate

a) $I = \int\limits_{(2,1)}^{(-1,-2)} (9x^2y + 24xy^2 + 6 + 5y)\,\mathrm{d}x + (3x^3 + 24x^2y + 8 + 5x)\,\mathrm{d}y;$

b) $I = \int\limits_{(0,\frac{\pi}{4})}^{(\frac{\pi}{4},2)} (2xy - y\sin(xy))\,\mathrm{d}x + (x^2 + 2 - x\sin(xy))\,\mathrm{d}y;$

c) $I = \int\limits_{(1,0,0)}^{(2,-1,3)} 2xy\,\mathrm{d}x + (x^2 - z)\,\mathrm{d}y + (1-y)\,\mathrm{d}z;$

d) $I = \int\limits_{(0,0,0)}^{(1,0,0)} (2x + 3y + \sin(z^2))\,\mathrm{d}x + (2x)\,\mathrm{d}y + (2xz\cos(z^2))\,\mathrm{d}z.$

Solution.

a) $V(x,y) = 3x^3y + 12x^2y^2 + 6x + 5xy + 8y \Rightarrow I = V(-1,-2) - V(2,1) = \underline{\underline{-60}}\,.$

b) $V(x,y) = x^2y + \cos(xy) + 2y \Rightarrow I = V(\frac{\pi}{4},2) - V(0,\frac{\pi}{4}) = \underline{\underline{\frac{\pi^2}{8} - \frac{\pi}{2} + 3}}\,.$

c) Notice that the assignment is correct since the vector field

$$f(x,y,z) := (2xy, x^2 - z, 1 - y)$$

is conservative in the simply connected domain $\mathbb{R}^3$

$$\left(\frac{\partial(2xy)}{\partial y} = 2x = \frac{\partial(x^2 - z)}{\partial x}, \quad \frac{\partial(2xy)}{\partial z} = 0 = \frac{\partial(1-y)}{\partial x}, \frac{\partial(x^2-z)}{\partial z} = -1 = \frac{\partial(1-y)}{\partial y}\right).$$

Seek a potential V:

$$\frac{\partial V}{\partial x}(x,y,z) = 2xy \Rightarrow \underline{V(x,y,z) = x^2y + \psi(y,z)},$$

for a – yet unknown – function $\psi: \ \mathbb{R}^2 \to \mathbb{R}$;

$$\frac{\partial V}{\partial y}(x,y,z) = x^2 - z = \frac{\partial}{\partial y}(x^2y + \psi(y,z)) = x^2 + \frac{\partial \psi}{\partial y}(y,z)$$

$$\Rightarrow \frac{\partial \psi}{\partial y}(y,z) = -z \Rightarrow \underline{\psi(y,z) = -zy + \xi(z)},$$

for a – yet unknown – function $\xi: \ \mathbb{R} \to \mathbb{R}$;

$$\frac{\partial V}{\partial z}(x,y,z) = 1 - y = \frac{\partial}{\partial z}(x^2y - zy + \xi(z)) = -y + \xi'(z) \Rightarrow \xi'(z) = 1 \Rightarrow \underline{\xi(z) = z + c},$$

for some $c \in \mathbb{R}$. Again, if we choose $c = 0$, we have

$$V(x,y,z) = x^2y - zy + z,$$

and thus

$$\int_{(1,0,0)}^{(2,-1,3)} 2xy\,\mathrm{d}x + (x^2 - z)\,\mathrm{d}y + (1-y)\,\mathrm{d}z = V(2,-1,3) - V(1,0,0) = \underline{\underline{2}}\,.$$

d) The assignment is not correct since

$$\frac{\partial}{\partial z}(2x + 3y + \sin(z^2)) = 3 \neq 2 = \frac{\partial}{\partial x}(2x),$$

and thus the integrated field is not conservative. (Read the footnote after Theorem 3.32.)

Exercise 3.37. Evaluate I, if

a) $I = \int\limits_{(\frac{\pi}{2},1)}^{(2,0)} (3x^2y + y\cos(xy))\,\mathrm{d}x + (x^3 + 1 + x\cos(xy))\,\mathrm{d}y;$

b) $I = \int\limits_{(2,0)}^{(1,1)} (2ye^{xy} + 2x + 2y^2)\,\mathrm{d}x + (2xe^{xy} + 4xy + 2y)\,\mathrm{d}y;$

c) $I = \int\limits_{(-1,3,0)}^{(0,1,2)} 3x^2y^2z\,\mathrm{d}x + (2x^3yz - z^2)\,\mathrm{d}y + (x^3y^2 - 2yz + 3z^2)\,\mathrm{d}z;$

d) $I = \int\limits_{(0,0,1)}^{(1,1,1)} (y^2z^2 + 2z)\,\mathrm{d}x + (2xyz^2 + 2y)\,\mathrm{d}y + (2xy^2z + 2x + 1)\,\mathrm{d}z.$

Exercise 3.38. Prove that the vector field

$$f(x,y) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

is not conservative in the domain $\mathbb{R}^2 \setminus \{(0,0)\}$ even though in $\mathbb{R}^2 \setminus \{(0,0)\}$ it holds that that

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right).$$

(Do not miss the connection to Remark 3.33.)

3.6 Applications of the line integral of the second kind

a) **Work of a vector field along an oriented curve.**
Let $f:\ \mathbb{R}^m \to \mathbb{R}^m$ be continuous on 'and oriented curve' $(k) \subset \mathbb{R}^m$, where $(k) = \langle\varphi\rangle$ for a simple (or simple closed) piecewise smooth curve φ.
Work of a vector field f along 'an oriented curve' (k) is defined by the number

$$\mathcal{A}(k) := \int_{(\varphi)} f(x)\,\mathrm{d}s.$$

Obviously, we assume that the orientation of (k) is 'consistent' with the curve φ.

b) **Area (more precisely measure) of plane shapes.**
Let $\Omega = \operatorname{int}\varphi \cup \langle\varphi\rangle$, where φ is a simple closed positively oriented piecewise smooth curve in $\mathbb{R}^2$. Then it holds (see Green theorem 3.28)

$$\lambda(\Omega) = \frac{1}{2}\int_{(\varphi)} (-y)\,\mathrm{d}x + x\,\mathrm{d}y = \int_{(\varphi)} x\,\mathrm{d}y = -\int_{(\varphi)} y\,\mathrm{d}x.$$

Exercise 3.39.

a) Evaluate the work of the vector field $f(x,y,z) = -(0,0,mg)$ along five turns of the helix

$$\varphi(t) := \left(r\cos t, r\sin t, -\frac{h}{10\pi}\,t\right), \quad t \in [0, 10\pi].$$

Amount of energy obtained by a person of the weight $m > 0$ sliding five turns on a slide of the height $h > 0$ and turn radius $r > 0$; the constant $g > 0$ denotes the gravity of Earth.

b) For $a, b > 0$ compute the area (measure) of the ellipse

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

4 Surfaces

Definition 4.1. A continuous vector-valued function

$$\psi:\ G \to \mathbb{R}^3,$$

for which there exists a non-empty domain $\Omega \subset \mathbb{R}^2$ such that $\Omega \subset G = \mathrm{D}\psi \subset \overline{\Omega}$ is called a surface (in $\mathbb{R}^3$).

The set

$$\langle \psi \rangle := \psi(G) = \{\psi(u,v):\ (u,v) \in G\} \subset \mathbb{R}^3$$

is called the geometrical image of the surface ψ. If $M = \langle \psi \rangle$, ψ defines a parametrization of the set M.

Exercise 4.2. Parametrize M if

a) $M = \{(x,y,z) \in \mathbb{R}^3:\ x^2+y^2+z^2 = 100 \wedge -8 \le z \le 6\}$;

b) $M = \{(x,y,z) \in \mathbb{R}^3:\ x^2+y^2+z^2 = 4 \wedge x^2+y^2 \le 2x \wedge z \ge 0\}$.

Similarly as in the case of curves this definition is too general for our purposes. Therefore, in the following we will consider additional 'differential' conditions.

Definition 4.3. A bounded domain $\Omega \subset \mathbb{R}^2$ is a regular domain if there exists a simple closed piecewise smooth curve (in $\mathbb{R}^2$) φ such that $\Omega = \operatorname{int} \varphi$.

Definition 4.4. A surface $\psi = (\psi_1, \psi_2, \psi_3):\ \overline{\Omega} \to \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^2$ denoting a regular domain is called a regular surface if all the following holds:

i) ψ is injective;

ii) there exists a vector-valued function $h = (h_1, h_2, h_3):\ \mathbb{R}^2 \to \mathbb{R}^3$ of class C^1 in an open set $M \supset \overline{\Omega}$ satisfying [10] $\psi = h|_{\overline{\Omega}}$;

iii) for every $(u,v) \in \overline{\Omega}$ the vectors

$$\frac{\partial \psi}{\partial u}(u,v) := \left(\frac{\partial h_1}{\partial u}(u,v), \frac{\partial h_2}{\partial u}(u,v), \frac{\partial h_3}{\partial u}(u,v)\right),$$

$$\frac{\partial \psi}{\partial v}(u,v) := \left(\frac{\partial h_1}{\partial v}(u,v), \frac{\partial h_2}{\partial v}(u,v), \frac{\partial h_3}{\partial v}(u,v)\right)$$

are linearly independent.[11]

[10] I.e. all partial derivatives of the first order $h_1,\ h_2$ and h_3 are continuous in M.

[11] Notice that for $(u,v) \in \Omega$ we have

$$\frac{\partial \psi}{\partial u}(u,v) := \left(\frac{\partial \psi_1}{\partial u}(u,v), \frac{\partial \psi_2}{\partial u}(u,v), \frac{\partial \psi_3}{\partial u}(u,v)\right), \quad \frac{\partial \psi}{\partial v}(u,v) := \left(\frac{\partial \psi_1}{\partial v}(u,v), \frac{\partial \psi_2}{\partial v}(u,v), \frac{\partial \psi_3}{\partial v}(u,v)\right).$$

The set

$$\mathcal{O}\psi := \psi(\partial\Omega) = \{\psi(u,v) : \ (u,v) \in \partial\Omega\}$$

is the boundary of the regular surface ψ.

Example 4.5 (of regular surfaces)**.**

- $\psi_1(u,v) := (\cos u, \sin u, v)$, $\mathrm{D}\psi_1 = [0, \frac{\pi}{2}] \times [0,2]$.
- $\psi_2(u,v) := (\cos u \cos v, \sin u \cos v, \sin v)$, $\mathrm{D}\psi_2 = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{4}]$.

Remark 4.6 (to the geometrical interpretation of the vectors $\frac{\partial\psi}{\partial u}(u,v)$, $\frac{\partial\psi}{\partial v}(u,v)$).
Notice that

$$\frac{\partial\psi}{\partial u}(u,v) \ \text{ and } \ \frac{\partial\psi}{\partial v}(u,v)$$

define `direction vectors of the tangent plane to 'surface'` $\langle\psi\rangle$ `in point` $\psi(u,v)$.
The attentive reader knows we have already used this in Example 1.23.

5 Surface Integral

5.1 Surface integral of the first kind over a regular surface

Motivation 5.1. Let $\psi = (\psi_1, \psi_2, \psi_3) : \ \overline{\Omega} \to \mathbb{R}^3$ denote a regular surface. Our aim is to compute 'the weight of the surface' $\langle\psi\rangle$ if the (surface) density in $\langle\psi\rangle$ is given by a continuous and non-negative function $f : \ \mathbb{R}^3 \to \mathbb{R}$.

Consider a two-dimensional interval $[a,b]\times[c,d]$ such that $\overline{\Omega} \subset [a,b]\times[c,d]$, and its partitioning $D = (D_u, D_v)$, i.e. a system of two-dimensional intervals

$$J_{kl} = [u_k, u_{k+1}] \times [v_l, v_{l+1}] \,.$$

Our goal is to approximate the weight $m(\langle\psi\rangle)$ by the sum $\sum_{k,l} m(\psi(J_{kl}))$ over k and l such that $J_{kl} \subset \overline{\Omega}$.

Since

$$\psi(u_{k+1}, v_l) - \psi(u_k, v_l) \doteq \mathrm{d}\psi_{(u_k,v_l)}(u_{k+1} - u_k, 0) = \left(\psi'(u_k, v_l) \cdot \begin{pmatrix} u_{k+1} - u_k \\ 0 \end{pmatrix}\right)^T$$

$$= \left(\begin{pmatrix} \frac{\partial\psi_1}{\partial u}(u_k,v_l), & \frac{\partial\psi_1}{\partial v}(u_k,v_l) \\ \frac{\partial\psi_2}{\partial u}(u_k,v_l), & \frac{\partial\psi_2}{\partial v}(u_k,v_l) \\ \frac{\partial\psi_3}{\partial u}(u_k,v_l), & \frac{\partial\psi_3}{\partial v}(u_k,v_l) \end{pmatrix} \cdot \begin{pmatrix} u_{k+1} - u_k \\ 0 \end{pmatrix}\right)^T = (u_{k+1} - u_k)\,\frac{\partial\psi}{\partial u}(u_k, v_l)$$

and (by an analogous computation)

$$\psi(u_k, v_{l+1}) - \psi(u_k, v_l) \doteq (v_{l+1} - v_l)\,\frac{\partial\psi}{\partial v}(u_k, v_l),$$

it is natural to approximate 'the area of the surface' $\psi(J_{kl})$ by [12]

$$\left\|(u_{k+1} - u_k)\frac{\partial\psi}{\partial u}(u_k, v_l) \times (v_{l+1} - v_l)\frac{\partial\psi}{\partial v}(u_k, v_l)\right\| =$$
$$= \left\|\frac{\partial\psi}{\partial u}(u_k, v_l) \times \frac{\partial\psi}{\partial v}(u_k, v_l)\right\| (u_{k+1} - u_k)(v_{l+1} - v_l).$$

Approximating f on every $\psi(J_{kl})$ by the constant $f(\psi(u_k, v_l))$ leads to the following approximation of the desired weight:

$$m(\langle\psi\rangle) \doteq \sum_{k,l} m(\psi(J_{kl})) \doteq \sum_{k,l} f(\psi(u_k, v_l)) \left\|\frac{\partial\psi}{\partial u}(u_k, v_l) \times \frac{\partial\psi}{\partial v}(u_k, v_l)\right\| (u_{k+1} - u_k)(v_{l+1} - v_l)$$
$$\approx \iint_{\overline{\Omega}} f(\psi(u,v)) \left\|\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v)\right\| \,\mathrm{d}u\,\mathrm{d}v.$$

One should notice that for $f \equiv 1$ we compute 'the surface area' of $\langle\psi\rangle$.

[12] We use '$u \times v$', with $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$, to denote the cross product of the vectors u and v,

$$u \times v := (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Definition 5.2. Let $\psi: \overline{\Omega} \to \mathbb{R}^3$ denote a regular surface and let $f: \mathbb{R}^3 \to \mathbb{R}$ denote a function continuous in $\langle\psi\rangle = \psi(\overline{\Omega})$. <u>The surface integral of the first kind of</u> f <u>over a regular surface</u> ψ is defined by the equality

$$\iint\limits_{\psi} f(x,y,z)\,\mathrm{d}\sigma := \iint_{\overline{\Omega}} f(\psi(u,v)) \left\| \frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) \right\| \,\mathrm{d}u\,\mathrm{d}v.$$

Remark 5.3 (to Definition 5.2). Under the assumptions above the function

$$(u,v) \mapsto f(\psi(u,v)) \left\| \frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) \right\|$$

is continuous in the closed set $\overline{\Omega}$, and thus integrable in $\overline{\Omega}$.

Example 5.4. Evaluate $\iint_\psi f(x,y,z)\,\mathrm{d}\sigma$ for

a) $f(x,y,z) := x+y+z, \quad \psi(u,v) := (1,u,v), \quad \overline{\Omega} = \mathrm{D}\psi = [0,1]\times[0,1]$;

b) $f(x,y,z) := z\sqrt{x^2+y^2}, \quad \psi(u,v) := (\cos u\cos v, \sin u\cos v, \sin v), \overline{\Omega} = \mathrm{D}\psi = [0,\frac{\pi}{2}]\times[0,\frac{\pi}{4}]$.

Solution.

a) We evaluate

$$\frac{\partial\psi}{\partial u}(u,v) = (0,1,0)\,, \quad \frac{\partial\psi}{\partial v}(u,v) = (0,0,1)\,,$$

$$\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) = (1,0,0)\,, \quad \left\| \frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) \right\| = 1,$$

and thus

$$\iint_\psi f(x,y,z)\,\mathrm{d}\sigma = \int_0^1 \Big(\int_0^1 1+u+v\,\mathrm{d}u \Big)\,\mathrm{d}v = \int_0^1 1 + \frac{1}{2} + v\,\mathrm{d}v = \underline{\underline{2}}\,.$$

b) Since for $(u,v) \in \overline{\Omega}$ it holds that

$$\frac{\partial\psi}{\partial u}(u,v) = (-\sin u\cos v, \cos u\cos v, 0),$$

$$\frac{\partial\psi}{\partial v}(u,v) = (-\cos u\sin v, -\sin u\sin v, \cos v),$$

$$\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) = (\cos^2 v\cos u, \cos^2 v\sin u, \sin v\cos v),$$

$$\left\| \frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) \right\| = \sqrt{\cos^2 v} = |\cos v| = \cos v,$$

we have

$$\iint_\psi z\sqrt{x^2+y^2}\,\mathrm{d}\sigma = \iint_{\overline{\Omega}} \sin v\cos v\cos v \;\,\mathrm{d}u\,\mathrm{d}v = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \cos^2 v\sin v\,\mathrm{d}v =$$

$$= \frac{\pi}{2}\int_1^{\frac{\sqrt{2}}{2}} -w^2\,\mathrm{d}w = \underline{\underline{\frac{\pi}{24}(4-\sqrt{2})}}\,.$$

Remark 5.5. For the evaluation of $\|a \times b\|$, where $a, b \in \mathbb{R}^3$, one can use the equality

$$\|a \times b\| = \sqrt{\|a\|^2 \|b\|^2 - (a \cdot b)^2}\,.$$

The proof is left to the reader.

5.2 Surface integral of the first kind over a piecewise smooth surface

In practice we have to deal with more complicated surfaces than just regular ones; e.g. parametrizations of the boundary of a cuboid, sphere, pyramid, ... These can be described by piecewise smooth surfaces. It is reasonable to expect that the definition of such surfaces follows the same lines as the definition of piecewise smooth curves. However, for simplicity we proceed in an alternative way and define a piecewise smooth surface as a `set of points` of certain qualities.

Definition 5.6. The set $S \subset \mathbb{R}^3$ is a piecewise smooth surface, if there exist regular surfaces $\psi_1, \psi_2, \ldots, \psi_n$ such that:

i)
$$S = \bigcup_{i=1}^{n} \langle \psi_i \rangle = \langle \psi_1 \rangle \cup \langle \psi_2 \rangle \cup \ \ldots \ \cup \langle \psi_n \rangle\,;$$

ii)
$$i \neq j \Rightarrow \langle \psi_i \rangle \cap \langle \psi_j \rangle \subset \mathcal{O}\psi_i \cap \mathcal{O}\psi_j$$

and $\langle \psi_i \rangle \cap \langle \psi_j \rangle$ can either be parametrized by a simple or simple closed piecewise smooth curve (then ψ_i and ψ_j are neighbouring surfaces), or $\langle \psi_i \rangle \cap \langle \psi_j \rangle$ is a singleton, or is empty;

iii) $i \neq j \neq k \neq i \Rightarrow \langle \psi_i \rangle \cap \langle \psi_j \rangle \cap \langle \psi_k \rangle$ is a singleton or is empty;

iv) $i \neq 1 \Rightarrow$ the surface ψ_i neighbours to at least one of the surfaces $\psi_1, \psi_2, \ldots, \psi_{i-1}$.

(Under this setting the regular surfaces $\psi_1, \ldots, \psi_n$ define a partitioning of a piecewise smooth surface S into regular surfaces.)

Definition 5.7. A simple curve $\varphi: \ [a, b] \to \mathbb{R}^3$ is a part of the boundary S if there exists a partitioning of S into regular surfaces $\psi_1, \psi_2, \ldots, \psi_n$ and `a unique` index $i \in \{1, \ldots, n\}$ such that

$$\emptyset \neq \varphi(a, b) \cap \langle \psi_i \rangle \subset \mathcal{O}\psi_i\,.$$

Definition 5.8. Boundary of a surface S is defined by the equality

$$\mathcal{O}S := \bigcup_{\substack{\varphi \text{ is part} \\ \text{of the boundary of } S}} \langle \varphi \rangle\,.$$

If $\mathcal{O}S = \emptyset$, i.e. there exists no curve φ being a part of the boundary of S, the surface S is closed.

Definition 5.9. A point $p \in S$ for which there exists a partitioning of S into regular surfaces $\psi_1, \psi_2, \ldots, \psi_n$ and an index $i \in \{1, ..., n\}$ such that $p \in \langle \psi_i \rangle \setminus \mathcal{O}\psi_i$ is a regular point of S.

Observation 5.10. Notice that in a regular point $p = \psi_i(u,v)$ there exists a plane tangent to the surface S with unit normal vector

$$n(p) = \frac{\frac{\partial \psi_i}{\partial u}(u,v) \times \frac{\partial \psi_i}{\partial v}(u,v)}{\left\| \frac{\partial \psi_i}{\partial u}(u,v) \times \frac{\partial \psi_i}{\partial v}(u,v) \right\|}.$$

Definition 5.11. Let ψ_1, ψ_2, …, ψ_n denote a partitioning of a piecewise smooth surface $S \subset \mathbb{R}^3$ into regular surfaces and let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function continuous in S.

The surface integral of the first kind of f over a piecewise smooth surface S is defined by the equality

$$\iint_S f(x,y,z)\,\mathrm{d}\sigma := \sum_{i=1}^{n} \iint_{\psi_i} f(x,y,z)\,\mathrm{d}\sigma.$$

Remark 5.12 (to Definition 5.11). It can be shown that the definition is independent of the partitioning of the piecewise smooth surface S into regular surfaces ψ_i.

In particular, if ψ_1 and ψ_2 are two regular surfaces such that $\langle\psi_1\rangle = \langle\psi_2\rangle$ and the function $f: \mathbb{R}^3 \to \mathbb{R}$ is continuous in $\langle\psi_1\rangle$, it holds that

$$\iint_{\psi_1} f(x,y,z)\,\mathrm{d}\sigma = \iint_{\psi_2} f(x,y,z)\,\mathrm{d}\sigma.$$

Compare this assertion to Theorem 3.9.

Example 5.13. Evaluate $\iint\limits_S f(x,y,z)\,\mathrm{d}\sigma$, where

a) $f(x,y,z) := \frac{1}{(1+x+y)^2}$ and S is the boundary of the tetrahedron defined by the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$;

b) $f(x,y,z) := x^2 + y^2$ and S is a boundary of the body

$$\left\{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \le z \le 2 \wedge z \ge 1\right\};$$

c) $f(x,y,z) := z^2$, $S = \{(x,y,z) \in \mathbb{R}^3 : z = xy \wedge x^2 + y^2 \le 1\}$;

d) $f(x,y,z) := xy$, $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 4z \wedge x \ge 0 \wedge y \ge 0 \wedge z \le 1\}$;

e) $f(x,y,z) := z$,
$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 9 \wedge x \ge 0 \wedge y \ge 0 \wedge z \ge 0 \wedge x + y \le 3\}$.

Solution.

a) We partition the tetrahedron S into regular surfaces $S = \langle\psi_1\rangle \cup \langle\psi_2\rangle \cup \langle\psi_3\rangle \cup \langle\psi_4\rangle$, where

$$\begin{aligned}
&\psi_1(x,y) := (x,y,0),\ (x,y) \in \Omega = \{(u,v) \in \mathbb{R}^2 : u \in [0,1] \wedge v \in [0,1-u]\},\\
&\psi_2(x,z) := (x,0,z),\ (x,z) \in \Omega,\\
&\psi_3(y,z) := (0,y,z),\ (y,z) \in \Omega,\\
&\psi_4(x,y) := (x,y,1-x-y),\ (x,y) \in \Omega.
\end{aligned}$$

Then[13]

$$\frac{\partial\psi_1}{\partial x}=(1,0,0),\ \frac{\partial\psi_1}{\partial y}=(0,1,0),\ \ldots,\ \frac{\partial\psi_4}{\partial x}=(1,0,-1),\ \frac{\partial\psi_4}{\partial y}=(0,1,-1),$$

and every reader surely obtains the same result that

$$\left\|\frac{\partial\psi_1}{\partial x}\times\frac{\partial\psi_1}{\partial y}\right\|=\left\|\frac{\partial\psi_2}{\partial x}\times\frac{\partial\psi_2}{\partial z}\right\|=\left\|\frac{\partial\psi_3}{\partial y}\times\frac{\partial\psi_3}{\partial z}\right\|=1,$$
$$\left\|\frac{\partial\psi_4}{\partial x}\times\frac{\partial\psi_4}{\partial y}\right\|=\|(1,1,1)\|=\sqrt{3}\,.$$

We thus obtain (see Definitions 5.11 and 5.2)

$$\begin{aligned}\iint_S f\,\mathrm{d}\sigma &= \iint_\Omega \frac{1}{(1+x+y)^2}\cdot 1\,\mathrm{d}x\,\mathrm{d}y+\iint_\Omega\frac{1}{(1+x)^2}\cdot 1\,\mathrm{d}x\,\mathrm{d}z\\
&\quad+\iint_\Omega\frac{1}{(1+y)^2}\cdot 1\,\mathrm{d}y\,\mathrm{d}z+\iint_\Omega\frac{1}{(1+x+y)^2}\cdot\sqrt{3}\,\mathrm{d}x\,\mathrm{d}y\\
&=\iint_\Omega(1+\sqrt{3})\frac{1}{(1+x+y)^2}+2\frac{1}{(1+x)^2}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_0^1\Big(\int_0^{1-x}\frac{1+\sqrt{3}}{(1+x+y)^2}+\frac{2}{(1+x)^2}\,\mathrm{d}y\Big)\,\mathrm{d}x\\
&=\int_0^1\frac{2}{(1+x)^2}(1-x)-(1+\sqrt{3})\left[\frac{1}{1+x+y}\right]_{y=0}^{1-x}\mathrm{d}x\\
&=\int_0^1\frac{2(1-x)}{(1+x)^2}-(1+\sqrt{3})\Big(\frac{1}{2}-\frac{1}{1+x}\Big)\,\mathrm{d}x=\underline{\underline{(\sqrt{3}-1)\ln 2-\frac{\sqrt{3}}{2}+\frac{3}{2}}}\,.\end{aligned}$$

b) Since

$$S=\langle\psi_1\rangle\cup\langle\psi_2\rangle\cup\langle\psi_3\rangle,$$

where

$$\begin{aligned}\psi_1(r,t)&:=(r\cos t,r\sin t,r),\ (r,t)\in[1,2]\times[0,2\pi],\\
\psi_2(r,t)&:=(r\cos t,r\sin t,1),\ (r,t)\in[0,1]\times[0,2\pi],\\
\psi_3(r,t)&:=(r\cos t,r\sin t,2),\ (r,t)\in[0,2]\times[0,2\pi],\end{aligned}$$

and moreover (as can be easily checked) it holds that

$$\left\|\frac{\partial\psi_1}{\partial r}(r,t)\times\frac{\partial\psi_1}{\partial t}(r,t)\right\|=\sqrt{2}r,$$
$$\left\|\frac{\partial\psi_2}{\partial r}(r,t)\times\frac{\partial\psi_2}{\partial t}(r,t)\right\|=\left\|\frac{\partial\psi_3}{\partial r}(r,t)\times\frac{\partial\psi_3}{\partial t}(r,t)\right\|=r,$$

[13]We use the shorthand notation – also in the following text – '$\frac{\partial\psi_1}{\partial x}=\ldots$' instead of the proper '$\frac{\partial\psi_1}{\partial x}(x,y)=\ldots$'.

we have

$$\begin{aligned}\iint_S (x^2+y^2)\,\mathrm{d}\sigma &= \iint_{\psi_1} (x^2+y^2)\,\mathrm{d}\sigma + \iint_{\psi_2} (x^2+y^2)\,\mathrm{d}\sigma + \iint_{\psi_3} (x^2+y^2)\,\mathrm{d}\sigma \\ &= \int_0^{2\pi} \left(\int_1^2 (r^2\sqrt{2}r)\,\mathrm{d}r \right)\,\mathrm{d}t + \int_0^{2\pi} \left(\int_0^1 (r^2 r)\,\mathrm{d}r \right)\,\mathrm{d}t + \int_0^{2\pi} \left(\int_0^2 (r^2 r)\,\mathrm{d}r \right)\,\mathrm{d}t \\ &= \underline{\underline{\frac{\pi}{2}(15\sqrt{2}+17)}}\,.\end{aligned}$$

The attentive reader must feel very uncomfortable as the above computation is not correct: the surfaces ψ_1, ψ_2, and ψ_3 are not regular. The situation is similar as in the case of double integrals: injectivity and linear independence of vectors $\frac{\partial \psi_i}{\partial r}(r,t)$, $\frac{\partial \psi_i}{\partial t}(r,t)$ is only `violated` on a set (in $\mathbb{R}^2$) of measure zero, which is 'insignificant' for the evaluation of double integrals.
We recommend the reader a calming exercise:

evaluate the given integral correctly;

this should lead to understanding that (and why) the above presented procedure leads to the correct result.

c) Obviously $S = \langle \psi \rangle$, where

$$\psi(r,t) := (r\cos t, r\sin t, r^2 \sin t \cos t),\ \ (r,t) \in [0,1] \times [0, 2\pi].$$

It holds that

$$\frac{\partial \psi}{\partial r}(r,t) = (\cos t, \sin t, 2r \sin t \cos t), \qquad \frac{\partial \psi}{\partial t}(r,t) = (-r\sin t, r\cos t, r^2 \cos 2t),$$

$$\begin{aligned}\left\| \frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial t} \right\| &= \left\| (r^2 \sin t \cos 2t - r^2 \cos t \sin 2t, -r^2 \cos t \cos 2t - r^2 \sin t \sin 2t, r) \right\| \\ &= \sqrt{r^4(\sin^2 t \cos^2 2t + \cos^2 t \sin^2 2t + \cos^2 t \cos^2 2t + \sin^2 t \sin^2 2t) + r^2} \\ &= \sqrt{r^4(\cos^2 2t + \sin^2 2t) + r^2} = r\sqrt{1+r^2}\,,\end{aligned}$$

and thus

$$\begin{aligned}\iint\limits_S f(x,y,z)\,\mathrm{d}\sigma &= \int_0^{2\pi} \Big(\int_0^1 \frac{r^4}{4} \sin^2(2t)\, r\, \sqrt{1+r^2}\,\mathrm{d}r \Big)\,\mathrm{d}t \\ &= \Big(\int_0^{2\pi} \frac{1-\cos 4t}{8}\,\mathrm{d}t \Big) \Big(\int_0^1 r^4 \sqrt{1+r^2}\, r\,\mathrm{d}r \Big) = \frac{\pi}{4} \int_1^2 (w-1)^2 \sqrt{w}\, \frac{1}{2}\,\mathrm{d}w \\ &= \frac{\pi}{8} \int_1^2 w^{\frac{5}{2}} - 2w^{\frac{3}{2}} + w^{\frac{1}{2}}\,\mathrm{d}w = \underline{\underline{\pi\Big(\frac{11\sqrt{2}}{210} - \frac{2}{105} \Big)}}\,.\end{aligned}$$

Below we provide an alternative way of computing the surface integral. We define

$$\psi(u,v) := (u, v, uv),\ \text{ where } (u,v) \in \Omega = \{(u,v) \in \mathbb{R}^2 :\ u^2 + v^2 \le 1\}.$$

Then $S = \langle\psi\rangle$ (S is a graph of a bivariate function $(u,v) \mapsto uv$),

$$\frac{\partial\psi}{\partial u} = (1,0,v), \quad \frac{\partial\psi}{\partial u} = (0,1,u), \quad \|\frac{\partial\psi}{\partial u} \times \frac{\partial\psi}{\partial v}\| = \|(-v,-u,1)\| = \sqrt{1+u^2+v^2}\,,$$

and thus

$$\begin{aligned}\iint_S f\,\mathrm{d}\sigma &= \iint_\Omega u^2v^2\,\sqrt{1+u^2+v^2}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^{2\pi}\Big(\int_0^1 r^4\sin^2 t\cos^2 t\,\sqrt{1+r^2}\,r\,\mathrm{d}r\Big)\,\mathrm{d}t\\ &= \int_0^{2\pi}\Big(\int_0^1 \frac{r^4}{4}\sin^2(2t)\,\sqrt{1+r^2}\,r\,\mathrm{d}r\Big)\,\mathrm{d}t = \underline{\underline{\pi\Big(\frac{11\sqrt{2}}{210} - \frac{2}{105}\Big)}}\,.\end{aligned}$$

In the evaluation of the double integral we used the polar coordinates

$$\Omega = \big\{(r\cos t, r\sin t) \in \mathbb{R}^2 :\ r \in [0,1]\ \wedge\ t \in [0,2\pi]\big\}$$

and the Fubini theorem. The resulting integral has already been evaluated before.

d) Clearly

$$\begin{aligned}S = \Big\{\Big(r\cos t, r\sin t, \frac{r^2}{4}\Big) \in \mathbb{R}^3 :\ r \in [0,\infty]\ \wedge\ t \in [0,2\pi] \wedge\\ \wedge\ \cos t \ge 0\ \wedge\ \sin t \ge 0\ \wedge\ \frac{r^2}{4} \le 1\Big\} = \langle\psi\rangle,\end{aligned}$$

where

$$\psi(r,t) := \Big(r\cos t, r\sin t, \frac{r^2}{4}\Big), \quad (r,t) \in [0,2] \times \Big[0, \frac{\pi}{2}\Big].$$

It follows that

$$\frac{\partial\psi}{\partial r} = \Big(\cos t, \sin t, \frac{r}{2}\Big), \quad \frac{\partial\psi}{\partial t} = \big(-r\sin t, r\cos t, 0\big),$$

$$\|\frac{\partial\psi}{\partial r} \times \frac{\partial\psi}{\partial t}\| = \Big\|\Big(-\frac{r^2}{2}\cos t, -\frac{r^2}{2}\sin t, r\Big)\Big\| = \sqrt{\frac{r^4}{4} + r^2} = r\sqrt{1+\frac{r^2}{4}}\,.$$

Combining the intermediate steps leads to the final result

$$\begin{aligned}\iint_S f\,\mathrm{d}\sigma &= \int_0^{\frac{\pi}{2}}\Big(\int_0^2 r^2\cos t\sin t\sqrt{1+\frac{r^2}{4}}\;r\,\mathrm{d}r\Big)\,\mathrm{d}t =\\ &= \Big(\int_0^1 u\,\mathrm{d}u\Big)\Big(\int_1^2 4(v-1)\sqrt{v}\;2\,\mathrm{d}v\Big) = 4\Big[\frac{v^{\frac{5}{2}}}{\frac{5}{2}} - \frac{v^{\frac{3}{2}}}{\frac{3}{2}}\Big]_1^2 = \underline{\underline{\frac{16}{15}(1+\sqrt{2})}}\,.\end{aligned}$$

We made use of substitutions $\sin t = u$ and $1 + \frac{r^2}{4} = v$.

e) If we parametrize

$$S = \left\{ \left(x, y, \sqrt{9 - x^2 - y^2}\right) \in \mathbb{R}^3 :\ x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ x + y \leq 3 \right\} = \langle \psi \rangle,$$

where

$$\psi(x, y) := (x, y, \sqrt{9 - x^2 - y^2}),$$
$$(x, y) \in \Omega = \{(x, y) \in \mathbb{R}^2 :\ x \in [0, 3] \ \wedge \ y \in [0, 3 - x]\},$$

we have

$$\frac{\partial \psi}{\partial x} = \left(1, 0, \frac{-x}{\sqrt{9 - x^2 - y^2}}\right), \quad \frac{\partial \psi}{\partial y} = \left(0, 1, \frac{-y}{\sqrt{9 - x^2 - y^2}}\right),$$
$$\left\| \frac{\partial \psi}{\partial x} \times \frac{\partial \psi}{\partial y} \right\| = \left\| \left(\frac{x}{\sqrt{9 - x^2 - y^2}}, \frac{y}{\sqrt{9 - x^2 - y^2}}, 1 \right) \right\| = \frac{3}{\sqrt{9 - x^2 - y^2}},$$

and thus

$$\iint_S f \, \mathrm{d}\sigma = \iint_\Omega \sqrt{9 - x^2 - y^2} \, \frac{3}{\sqrt{9 - x^2 - y^2}} \, \mathrm{d}x \, \mathrm{d}y = 3\lambda(\Omega) = \underline{\underline{\frac{27}{2}}}.$$

A question to the reader: *Why is the above computation incorrect and still leads to the correct result?*

Exercise 5.14. Evaluate $\iint\limits_S f(x, y, z) \, \mathrm{d}\sigma$, where

a) $f(x, y, z) := xy + yz + zx$, $S = \{(x, y, z) \in \mathbb{R}^3 :\ z = \sqrt{x^2 + y^2} \ \wedge \ x^2 + y^2 \leq 2x\}$;

b) $f(x, y, z) := xyz$, $S = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 = z^2 \ \wedge \ x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ 0 \leq z \leq 1\}$;

c) $f(x, y, z) := x^2 + y^2 + z$ and S is the boundary of the set

$$\{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 \leq 4 \ \wedge \ z \geq 0\}.$$

5.3 Applications of the surface integral of the first kind

a) **Surface area.**
For $P \subset \mathbb{R}^3$ denoting a piecewise smooth surface we define its area by

$$\sigma(P) := \iint_P 1 \, \mathrm{d}\sigma.$$

b) Let $P \subset \mathbb{R}^3$ denote a piecewise smooth surface, whose (surface) density is given by the function $h :\ \mathbb{R}^3 \mapsto \mathbb{R}$, which is continuous and non-negative on P. Then it is reasonable to define the following quantities:

$$m(P) := \iint_P h(x,y,z)\,\mathrm{d}\sigma \quad \dots\ \underline{\text{weight of the surface}}\ P,$$

$$S_{yz}(P) := \iint_P x\,h(x,y,z)\,\mathrm{d}\sigma$$
$$\dots\ \underline{\text{moment of rotation of}}\ P\ \underline{\text{with respect to the plane}}\ x=0,$$

$$S_{zx}(P) := \iint_P y\,h(x,y,z)\,\mathrm{d}\sigma$$
$$\dots\ \underline{\text{moment of rotation of}}\ P\ \underline{\text{with respect to the plane}}\ y=0,$$

$$S_{xy}(P) := \iint_P z\,h(x,y,z)\,\mathrm{d}\sigma$$
$$\dots\ \underline{\text{moment of rotation of}}\ P\ \underline{\text{with respect to the plane}}\ z=0,$$

$$T(P) := \left(\frac{S_{yz}(P)}{m(P)}, \frac{S_{zx}(P)}{m(P)}, \frac{S_{xy}(P)}{m(P)}\right) \quad \dots\ \underline{\text{center of mass}}\ P.$$

One can analogously define moments of inertia of P with respect to the coordinate axes.

Example 5.15. Compute the surface area S using surface integral if

a) $S = \{(x,y,z) \in \mathbb{R}^3 :\ (x-8)^2 + (y-7)^2 + (6-z)^2 = 25\}$;

b) $S = \{(x,y,z) \in \mathbb{R}^3 :\ z = \frac{1}{2}(x^2+y^2)\ \wedge\ x^2+y^2 \le 1\}$.

Solution.

a) We parametrize the sphere S by

$$\psi(u,v) := (8,7,6) + (5\cos u\cos v, 5\sin u\cos v, 5\sin v),\ (u,v) \in [0,2\pi] \times \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big].$$

For this parametrization it holds that

$$\frac{\partial\psi}{\partial u} = 5(-\sin u\cos v, \cos u\cos v, 0), \qquad \frac{\partial\psi}{\partial v} = 5(-\cos u\sin v, -\sin u\sin v, \cos v),$$
$$\left\|\frac{\partial\psi}{\partial u} \times \frac{\partial\psi}{\partial v}\right\| = \|25(\cos u\cos^2 v, \sin u\cos^2 v, \sin v\cos v)\| = 25|\cos v| = 25\cos v$$

since $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. For the surface area S we have

$$\sigma(S) = \iint_S 1\,\mathrm{d}\sigma = \int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 25\cos v\,\mathrm{d}v\right)\mathrm{d}u = 25\cdot 2\pi\cdot\big[\sin v\big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \underline{\underline{100\pi}}\,.$$

b) The surface S, which is a part of a rotational paraboloid, can be parametrized by the vector-valued function

$$\psi(r,t) := \left(r\cos t, r\sin t, \frac{r^2}{2}\right),\ (r,t) \in [0,1] \times [0,2\pi],$$

for which the following holds:

$$\frac{\partial\psi}{\partial r} = (\cos t, \sin t, r), \qquad \frac{\partial\psi}{\partial t} = (-r\sin t, r\cos t, 0),$$
$$\left\|\frac{\partial\psi}{\partial r} \times \frac{\partial\psi}{\partial t}\right\| = \|(-r^2\cos t, -r^2\sin t, r)\| = \sqrt{r^4+r^2} = r\sqrt{r^2+1}\,.$$

The resulting surface area reads

$$\sigma(S) = \int_0^1 \left(\int_0^{2\pi} r\,\sqrt{1+r^2}\;\mathrm{d}t \right) \mathrm{d}r = 2\pi \int_1^2 \sqrt{u}\,\frac{1}{2}\,\mathrm{d}u = \pi \left[\frac{2}{3} u^{\frac{3}{2}} \right]_1^2 = \underline{\underline{\frac{2\pi}{3}(\sqrt{8}-1)}}\,.$$

In the calculations we used the substitution $u = 1 + r^2$.

Exercise 5.16. Determine the coordinates of the center of mass of S if

a) $S = \left\{ (x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 36 \,\wedge\, z \geq 0 \right\}$, and the (surface) density is given by the function $h(x,y,z) := \sqrt{x^2+y^2}$;

b) $S = \left\{ (x,y,z) \in \mathbb{R}^3 :\ z^2 = x^2+y^2 \,\wedge\, 1 \leq z \leq 2 \right\}$, and the (surface) density in each point is given by its distance from the z-axis.

5.4 Surface integral of the second kind over a regular surface

Motivation 5.17.

1. Consider incompressible fluid flowing through a plane surface τ in the direction of its normal vector. Assume that the velocity of the flow is constant in space and time and is given by a vector $f_0 \in \mathbb{R}^3$. The amount of fluid which 'flows' through the surface τ with area (= measure) $\lambda(\tau)$, corresponds to the value

$$\left(f_0 \cdot \frac{n}{\|n\|} \right) \lambda(\tau).$$

2. Let $\psi :\ \overline{\Omega} \to \mathbb{R}^3$ denote a regular surface and $f :\ \mathbb{R}^3 \to \mathbb{R}^3$ a vector field continuous in $\langle\psi\rangle$. Again: f defines the flow velocity constant in time. We aim to compute how much fluid flows through the surface $\langle\psi\rangle$ in the direction given by the normal vectors $\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v)$ in a time unit.
   Proceeding analogously as in the beginning of Section 5.1, we obtain this approximation[14] of the desired volume

$$\begin{aligned}
\mathcal{K}(\langle\psi\rangle) &\doteq \sum_{k,l} \mathcal{K}(\psi(J_{kl})) \\
&\doteq \sum_{k,l} \left(f(\psi(u_k,v_l)) \cdot \left(\frac{\partial\psi}{\partial u}(u_k,v_l) \times \frac{\partial\psi}{\partial v}(u_k,v_l) \right) \right) (u_{k+1}-u_k)\,(v_{l+1}-v_l) \\
&\approx \iint_{\Omega} f(\psi(u,v)) \cdot \left(\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v) \right) \mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

[14] Here we used the equality:

$$\begin{aligned}
&\left(f\left(\psi(u_k,v_l)\right) \cdot \frac{\frac{\partial\psi}{\partial u}(u_k,v_l) \times \frac{\partial\psi}{\partial v}(u_k,v_l)}{\left\| \frac{\partial\psi}{\partial u}(u_k,v_l) \times \frac{\partial\psi}{\partial v}(u_k,v_l) \right\|} \right) \left\| \frac{\partial\psi}{\partial u}(u_k,v_l) \times \frac{\partial\psi}{\partial v}(u_k,v_l) \right\| \\
&= f\left(\psi(u_k,v_l)\right) \cdot \left(\frac{\partial\psi}{\partial u}(u_k,v_l) \times \frac{\partial\psi}{\partial v}(u_k,v_l) \right)
\end{aligned}$$

Definition 5.18. Let $\psi : \overline{\Omega} \to \mathbb{R}^3$ denote a regular surface and $f : \mathbb{R}^3 \to \mathbb{R}^3$ a function continuous on $\langle\psi\rangle = \psi(\overline{\Omega})$. The surface integral of the second kind of function f over a regular surface ψ is defined by the equality

$$\iint\limits_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma := \iint\limits_{\overline{\Omega}} f(\psi(u,v)) \cdot \Big(\frac{\partial\psi}{\partial u}(u,v) \times \frac{\partial\psi}{\partial v}(u,v)\Big)\,\mathrm{d}u\,\mathrm{d}v.$$

Remark 5.19. For $f = (f_1, f_2, f_3)$ one can use the notation

$$\iint_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma := \iint_{(\psi)} f_1(x,y,z)\,\mathrm{d}y \wedge \mathrm{d}z + f_2(x,y,z)\,\mathrm{d}z \wedge \mathrm{d}x + f_3(x,y,z)\,\mathrm{d}x \wedge \mathrm{d}y.$$

Example 5.20. Compute $\iint\limits_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma$ for

a) $f(x,y,z) := (0,0,x^2+y^2)$, $\psi(r,t) := (r\cos t, r\sin t, 0)$, $\mathrm{D}\psi = [1,2] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$;

b) $f(x,y,z) := (x-y, y-z, z-x+1)$, $\psi(u,v) := (u, v, 1-u-v)$,
$\mathrm{D}\psi = \overline{\Omega} = \{(u,v) \in \mathbb{R}^2 :\ u+v \le 1 \ \wedge\ u \ge 0 \ \wedge\ v \ge 0\}$.

Solution.

a) Since we have

$$\frac{\partial\psi}{\partial r} = (\cos t, \sin t, 0)\,, \quad \frac{\partial\psi}{\partial t} = (-r\sin t, r\cos t, 0)\,, \quad \frac{\partial\psi}{\partial r} \times \frac{\partial\psi}{\partial t} = (0,0,r)\,,$$

it holds that

$$\iint_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma = \int_1^2 \Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0,0,r^2)\cdot(0,0,r)\,\mathrm{d}t\Big)\,\mathrm{d}r = \pi\Big[\frac{r^4}{4}\Big]_1^2 = \underline{\underline{\frac{15}{4}\pi}}\,.$$

b) Proceeding analogously as in the previous assignment we get

$$\frac{\partial\psi}{\partial u} = (1,0,-1)\,, \quad \frac{\partial\psi}{\partial v} = (0,1,-1)\,, \quad \frac{\partial\psi}{\partial u} \times \frac{\partial\psi}{\partial v} = (1,1,1)\,,$$

and thus

$$\begin{aligned}\iint_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma &= \iint_{\overline{\Omega}} (u-v, u+2v-1, -2u-v+2)\cdot(1,1,1)\,\mathrm{d}u\,\mathrm{d}v = \\ &= \int_0^1 \Big(\int_0^{1-u} 1\,\mathrm{d}v\Big)\,\mathrm{d}u = \underline{\underline{\frac{1}{2}}}\,.\end{aligned}$$

Exercise 5.21. Evaluate $\iint\limits_{(\psi)} f(x,y,z)\,\mathrm{d}\sigma$ for

$$\begin{aligned} &f(x,y,z) := (-x^2 z, y, 2xy)\,, \\ &\quad \psi(u,v) := (\cos u\cos v, 2\sin u\cos v, \sin v)\,, \quad \mathrm{D}\psi = \Big[0, \frac{\pi}{2}\Big] \times \Big[0, \frac{\pi}{2}\Big]\,. \end{aligned}$$

5.5 Surface integral of the second kind over a piecewise smooth surface

We intentionally keep the following definition inaccurate (its proper formulation is rather complicated; it is similar to the definition of a positively oriented curve).

'Definition' 5.22. Let ψ denote a regular surface and let the curve φ be a part of the boundary $\langle\psi\rangle$. The surfaces ψ and φ are of the same orientation if it holds that '*when walking along $\langle\varphi\rangle$ in the direction of the orientation of φ and the head pointing in the direction of the vector $\frac{\partial\psi}{\partial u} \times \frac{\partial\psi}{\partial v}$, the surface $\langle\psi\rangle$ is on the left-hand side*'.

Definition 5.23. Let ψ_1 and ψ_2 denote neighbouring surfaces and let φ be a simple curve such that

$$\langle\varphi\rangle \subset \langle\psi_1\rangle \cap \langle\psi_2\rangle.$$

Then ψ_1 and ψ_2 are of the same orientation if the following holds:

$$\psi_1 \text{ and } \varphi \text{ are of the same orientation} \;\Leftrightarrow\; \psi_2 \text{ and } (-\varphi) \text{ are of the same orientation.}$$

A piecewise smooth surface S is orientable (or two-sided) if there exists its partitioning into regular surfaces $\psi_1, \psi_2, \dots, \psi_n$ such that each pair of neighbouring surfaces ψ_i and ψ_j of this decomposition are of the same orientation (the partitioning of S is then called orientable).

To orient an orientable surface S means to choose a unit vector $n(p) \in \mathbb{R}^3$ perpendicular to the surface S in each of its regular points p such that for every `oriented` partitioning $\psi_1, \psi_2, \dots, \psi_n$ of the surface S exactly one of the following implications holds true:

$$\psi_i(u,v) = p \in \langle\psi_i\rangle \setminus \mathcal{O}\psi_i \Rightarrow n(p) = \frac{\frac{\partial\psi_i}{\partial u}(u,v) \times \frac{\partial\psi_i}{\partial v}(u,v)}{\left\|\frac{\partial\psi_i}{\partial u}(u,v) \times \frac{\partial\psi_i}{\partial v}(u,v)\right\|},$$

$$\psi_i(u,v) = p \in \langle\psi_i\rangle \setminus \mathcal{O}\psi_i \Rightarrow n(p) = -\frac{\frac{\partial\psi_i}{\partial u}(u,v) \times \frac{\partial\psi_i}{\partial v}(u,v)}{\left\|\frac{\partial\psi_i}{\partial u}(u,v) \times \frac{\partial\psi_i}{\partial v}(u,v)\right\|}.$$

Remark 5.24 (to Definition 5.23).

i) For an orientable surface $S \subset \mathbb{R}^3$ there exist exactly two vector-valued functions n and n^* defining its orientation (since it clearly holds that $n = -n^*$, we speak of opposite orientations of S); to orient S it is thus sufficient to determine $n(p)$ in a single (arbitrarily chosen) regular point $p \in S$.

ii) There exist non-orientable (also called single-sided) piecewise smooth surfaces. The Möbius strip is an example of such a surface (see Figure 5.1):

$$\Big\{\Big(\cos v + u\cos\Big(\frac{v}{2}\Big)\cos v, \sin v + u\cos\Big(\frac{v}{2}\Big)\sin v, u\sin\Big(\frac{v}{2}\Big)\Big) \in \mathbb{R}^3 : \\ (u,v) \in [-1,1] \times [0,2\pi]\Big\}.$$

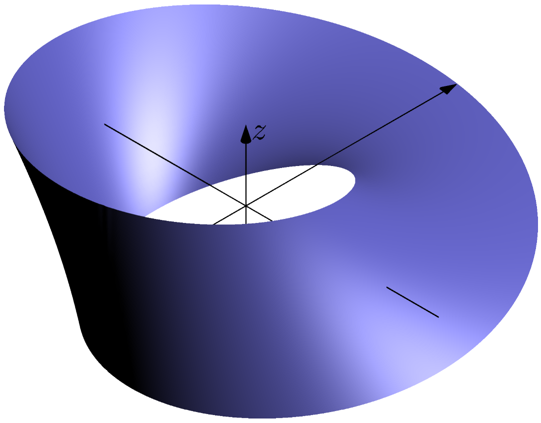

Figure 5.1: Möbius strip

iii) Every closed piecewise smooth surface is orientable.

Definition 5.25. Let $S \subset \mathbb{R}^3$ denote a piecewise smooth surface oriented by its `oriented partitioning` $\psi_1, \psi_2, \dots, \psi_n$[15] and let a vector-field $f : \mathbb{R}^3 \to \mathbb{R}^3$ be continuous on S.

The surface integral of the second kind of f over an oriented piecewise smooth surface S[16] is defined by the equality

$$\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma := \sum_{i=1}^{n} \iint_{(\psi_i)} f(x,y,z)\,\mathrm{d}\sigma.$$

Remark 5.26. It can be shown that the above definition is correct; it does not depend on choice of oriented partitioning of S (chosen in accordance to the orientation of S).

Moreover, if $\psi_1, \dots, \psi_n$ and $\psi_1^*, \dots, \psi_m^*$ are oriented partitionings of S that define opposite orientations of S, it holds that

$$\sum_{i=1}^{n} \iint_{(\psi_i)} f(x,y,z)\,\mathrm{d}\sigma = -\sum_{j=1}^{m} \iint_{(\psi_j^*)} f(x,y,z)\,\mathrm{d}\sigma.$$

Example 5.27. Evaluate the surface integral of the second kind $\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma$ if

a) $f(x,y,z) := (x^2, y^2, z^2)$ and S is the boundary of the cube $[0,6] \times [0,6] \times [0,6]$ oriented by the 'exterior' normal vectors;

[15] This means that the vector field $n : \mathbb{R}^3 \to \mathbb{R}^3$ defining the orientation of surface S is in every regular point $p = \psi_i(u,v)$ defined by the equality

$$n(p) := \frac{\frac{\partial \psi_i}{\partial u}(u,v) \times \frac{\partial \psi_i}{\partial v}(u,v)}{\left\| \frac{\partial \psi_i}{\partial u}(u,v) \times \frac{\partial \psi_i}{\partial v}(u,v) \right\|}.$$

[16] Sometimes we speak of the flow of a vector-field f through an oriented surface S.

b) $f(x,y,z) := (2y-z, 6z-2x, 3x-y)$ and the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 :\ 2x+y+2z=6 \wedge x \geq 0 \wedge y \geq 0 \wedge z \geq 0\}$$

is oriented by the vector field $n(x,y,z) := \frac{1}{3}(2,1,2)$;

c) $f(x,y,z) := (x,y,xyz)$ and the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 :\ z = xy \wedge x^2+y^2 \leq 5\}$$

is oriented by normal vectors such that the angles between the normal vectors and the vector $(0,0,1)$ are acute.

Solution.

a) Firstly parametrize the sides of the cube by

$$\begin{aligned}
\psi_1(u,v) &:= (u,v,0), \quad (u,v) \in [0,6]\times[0,6],\\
\psi_2(u,v) &:= (u,v,6), \quad (u,v) \in [0,6]\times[0,6],\\
\psi_3(u,v) &:= (u,0,v), \quad (u,v) \in [0,6]\times[0,6],\\
\psi_4(u,v) &:= (u,6,v), \quad (u,v) \in [0,6]\times[0,6],\\
\psi_5(u,v) &:= (0,u,v), \quad (u,v) \in [0,6]\times[0,6],\\
\psi_6(u,v) &:= (6,u,v), \quad (u,v) \in [0,6]\times[0,6],
\end{aligned}$$

and compute the corresponding normal vectors:

$$\begin{aligned}
\frac{\partial\psi_1}{\partial u}(u,v) \times \frac{\partial\psi_1}{\partial v}(u,v) &= (0,0,1),\\
\frac{\partial\psi_2}{\partial u}(u,v) \times \frac{\partial\psi_2}{\partial v}(u,v) &= (0,0,1),\\
&\vdots
\end{aligned}$$

Now notice (!!!) that the prescribed orientation of the side $[0,6]\times[0,6]\times\{0\}$ is `opposite` to the orientation of the chosen parametrization ψ_1 and that the orientation of ψ_2 and the corresponding side $[0,6]\times[0,6]\times\{6\}$ are the same. Making use of the symmetry of the vector field

$$f(x,y,z) := (x^2,y^2,z^2)$$

and the surface S leads to

$$\begin{aligned}
\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma &= -3\iint_{(\psi_1)} f(x,y,z)\,\mathrm{d}\sigma + 3\iint_{(\psi_2)} f(x,y,z)\,\mathrm{d}\sigma\\
&= -3\int_0^6\Big(\int_0^6 (u^2,v^2,0)\cdot(0,0,1)\,\mathrm{d}u\Big)\,\mathrm{d}v\\
&\quad + 3\int_0^6\Big(\int_0^6 (u^2,v^2,36)\cdot(0,0,1)\,\mathrm{d}u\Big)\,\mathrm{d}v = \underline{\underline{3888}}\,.
\end{aligned}$$

b) Clearly

$$S = \left\{ \left(x, y, 3 - x - \frac{y}{2} \right) \in \mathbb{R}^3 : \ x \geq 0 \ \wedge \ y \geq 0 \ \wedge \ 3 - x - \frac{y}{2} \geq 0 \right\} = \langle \psi \rangle,$$

where

$$\psi(u, v) := \left(u, v, 3 - u - \frac{v}{2} \right), \quad \mathrm{D}\psi = \{(u, v) \in \mathbb{R}^2 : \ u \in [0, 3] \ \wedge \ v \in [0, 6 - 2u]\}.$$

Moreover it holds that

$$\frac{\partial \psi}{\partial u} = (1, 0, -1), \ \frac{\partial \psi}{\partial v} = \left(0, 1, -\frac{1}{2} \right), \ \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} = \left(1, \frac{1}{2}, 1 \right)$$

(thus `the orientation agrees`), and so

$$\iint_{(S)} f(x, y, z) \, \mathrm{d}\sigma = \int_0^3 \left(\int_0^{6-2u} \left(-3 + u + \frac{5v}{2}, 18 - 8u - 3v, 3u - v \right) \cdot \left(1, \frac{1}{2}, 1 \right) \mathrm{d}v \right) \mathrm{d}u$$
$$= \int_0^3 6(6 - 2u) \, \mathrm{d}u = \underline{\underline{54}} \, .$$

c) One of the possible parametrizations of $S = \langle \psi \rangle$ is

$$\psi(r, t) := (r \cos t, r \sin t, r^2 \cos t \sin t) = \left(r \cos t, r \sin t, \frac{r^2}{2} \sin(2t) \right),$$
$$\mathrm{D}\psi = \{(r, t) \in \mathbb{R}^2 : \ r \in [0, \sqrt{5}] \ \wedge \ t \in [-\pi, \pi]\},$$

for which we have

$$\frac{\partial \psi}{\partial r} = (\cos t, \sin t, r \sin(2t)) \, , \quad \frac{\partial \psi}{\partial t} = (-r \sin t, r \cos t, r^2 \cos(2t)) \, ,$$
$$\frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial t} = (r^2 (\cos^2 t - \sin^2 t) \sin t - 2r^2 \cos^2 t \sin t,$$
$$- r^2 (\cos^2 t - \sin^2 t) \cos t - 2r^2 \cos t \sin^2 t, r)$$

(the orientation `agrees!`), and we can conclude that

$$\iint_{(S)} f(x, y, z) \, \mathrm{d}\sigma = \int_0^{\sqrt{5}} \left(\int_{-\pi}^{\pi} -2r^3 (\cos^3 t \sin t + \sin^3 t \cos t) + r^5 \cos^2 t \sin^2 t \, \mathrm{d}t \right) \mathrm{d}r$$
$$= \left(\left[\frac{r^6}{6} \right]_0^{\sqrt{5}} \right) \left(\frac{1}{4} \int_{-\pi}^{\pi} \frac{1 - \cos(4t)}{2} \, \mathrm{d}t \right) = \underline{\underline{\frac{125}{24} \pi}} \, .$$

We used the fact that the functions $\cos^3 t \sin t$ and $\sin^3 t \cos t$ are odd.

5.6 The Gauss – Ostrogradsky theorem

Definition 5.28. Let the vector field $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 in an open set $M \subset \mathbb{R}^3$.[17] Divergence of the vector field f (in M) is defined (in M) by the equality

$$\mathrm{div} f(x,y,z) := \frac{\partial f_1}{\partial x}(x,y,z) + \frac{\partial f_2}{\partial y}(x,y,z) + \frac{\partial f_3}{\partial z}(x,y,z).$$

Definition 5.29. `A bounded` domain $\Omega \subset \mathbb{R}^3$ is a regular domain, if its boundary $\partial\Omega$ is a `closed` piecewise smooth surface.

Theorem 5.30 (Gauss – Ostrogradsky).
Let the vector field $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 in an open set $M \subset \mathbb{R}^3$, let $\Omega \subset \mathbb{R}^3$ denote a regular domain such that

$$\Omega \subset \overline{\Omega} = \Omega \cup \partial\Omega \subset M,$$

and let $\partial\Omega$ be oriented by its 'exterior' normal vectors (such orientation is called positive $\partial\Omega$). Then it holds that

$$\iint_{(\partial\Omega)} f(x,y,z)\,\mathrm{d}\sigma = \iiint_{\Omega} \mathrm{div} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Proof. We prove the theorem for a special case when $\overline{\Omega}$ is a cuboid, i.e.

$$\overline{\Omega} = [a,b] \times [c,d] \times [e,g].$$

(Notice the analogy to the proof of the Green theorem 3.28 in a rectangle.)

Let us choose

$$\begin{aligned}
\psi_1(u,v) &:= (a,u,v), \quad (u,v) \in [c,d] \times [e,g],\\
\psi_2(u,v) &:= (b,u,v), \quad (u,v) \in [c,d] \times [e,g],\\
\psi_3(u,v) &:= (u,c,v), \quad (u,v) \in [a,b] \times [e,g],\\
\psi_4(u,v) &:= (u,d,v), \quad (u,v) \in [a,b] \times [e,g],\\
\psi_5(u,v) &:= (u,v,e), \quad (u,v) \in [a,b] \times [c,d],\\
\psi_6(u,v) &:= (u,v,g), \quad (u,v) \in [a,b] \times [c,d],
\end{aligned}$$

and compute the corresponding normal vectors

$$\frac{\partial\psi_1}{\partial u}(u,v) \times \frac{\partial\psi_1}{\partial v}(u,v) = (1,0,0) = \frac{\partial\psi_2}{\partial u}(u,v) \times \frac{\partial\psi_2}{\partial v}(u,v), \ \ldots$$

[17] I.e., $f_1, f_2, f_3 \in C^1(M)$.

Similarly as in Example 5.27, after 'taking care' of the orientation we obtain

$$\begin{aligned}\iint_{(\partial\Omega)} f(x,y,z)\,\mathrm{d}\sigma &= -\iint_{(\psi_1)} f(x,y,z)\,\mathrm{d}\sigma + \iint_{(\psi_2)} f(x,y,z)\,\mathrm{d}\sigma + \iint_{(\psi_3)} f(x,y,z)\,\mathrm{d}\sigma \\ &\quad - \iint_{(\psi_4)} f(x,y,z)\,\mathrm{d}\sigma - \iint_{(\psi_5)} f(x,y,z)\,\mathrm{d}\sigma + \iint_{(\psi_6)} f(x,y,z)\,\mathrm{d}\sigma \\ &= -\iint_{[c,d]\times[e,g]} f(\psi_1(u,v))\cdot(1,0,0)\,\mathrm{d}u\,\mathrm{d}v + \iint_{[c,d]\times[e,g]} f(\psi_2(u,v))\cdot(1,0,0)\,\mathrm{d}u\,\mathrm{d}v + \ \dots \\ &= \underline{\iint_{[c,d]\times[e,g]} (f_1(b,u,v) - f_1(a,u,v))\,\mathrm{d}u\,\mathrm{d}v + \ \dots}\end{aligned}$$

Now (using the Fubini theorem) we transform the integral on the right-hand side of the equality

$$\begin{aligned}&\iiint_{\overline{\Omega}} \operatorname{div} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iiint_{\overline{\Omega}} \left(\frac{\partial f_1}{\partial x}(x,y,z) + \frac{\partial f_2}{\partial y}(x,y,z) + \frac{\partial f_3}{\partial z}(x,y,z)\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iint_{[c,d]\times[e,g]} \left(\int_a^b \frac{\partial f_1}{\partial x}(x,y,z)\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}z + \ \dots \ = \iint_{[c,d]\times[e,g]} [f_1(x,y,z)]_{x=a}^{b}\,\mathrm{d}y\,\mathrm{d}z + \ \dots \\ &= \underline{\iint_{[c,d]\times[e,g]} (f_1(b,y,z) - f_1(a,y,z))\,\mathrm{d}y\,\mathrm{d}z + \ \dots}\end{aligned}$$

The reader who notices that the underlined numbers are equal can consider this proof finished. □

Remark 5.31 (to the physical interpretation of the Gauss – Ostrogradsky theorem). If we interpret f as a (stationary) velocity field of incompressible fluid, then

$$\iint_{(\partial\Omega)} f(x,y,z)\,\mathrm{d}\sigma$$

determines the amount of fluid, which flows through the surface $\partial\Omega$ in the direction of the normal vector in a time unit.

If

$$\iint_{(\partial\Omega)} f(x,y,z)\,\mathrm{d}\sigma = 0,$$

the inflow and outflow through $\overline{\Omega}$ is the same.

If

$$\iint_{(\partial\Omega)} f(x,y,z)\,\mathrm{d}\sigma \neq 0,$$

there have to exist points in $\overline{\Omega}$ which are <u>sources</u>; i.e. points adding the fluid to the system or <u>sinks</u> where the fluid drains.

It can be shown that for $p \in \overline{\Omega}$ it holds that

$$\operatorname{div} f(p) = \lim_{\varepsilon \to 0+} \frac{\iint\limits_{(\partial U(p,\varepsilon))} f(x,y,z)\,\mathrm{d}\sigma}{\lambda(U(p,\varepsilon))} ,$$

where $\partial U(p,\varepsilon)$ denotes a positively oriented sphere of the radius ε centered in p. Thus, the number $\operatorname{div} f(p)$ describes `the yield of the source` in point p ($\operatorname{div} f(p) > 0$ then p is a source; $\operatorname{div} f(p) < 0$ then p is a sink). If $\operatorname{div} f$ vanishes (in M), the vector field f is <u>divergence-free</u> (or <u>solenoidal</u>) (in M).

Example 5.32. Evaluate

$$\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma,$$

using the Gauss – Ostrogradsky theorem, if

a) $f(x,y,z) := (x^2, y^2, z^2)$ and S is a positively oriented sphere

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : \ (x-1)^2 + (y-1)^2 + (z-1)^2 \le 1\};$$

b) $f(x,y,z) := (x-y+z, y-z+x, z-y+x)$ and S is a negatively oriented boundary of the octahedron

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : \ |x| + |y| + |z| \le 3\}.$$

Solution.

a) Due to the Gauss – Ostrogradsky theorem and the substitution

$$x = \varrho \cos u \cos v + 1, \quad y = \varrho \sin u \cos v + 1, \quad z = \varrho \sin v + 1,$$

where

$$\varrho \in [0,1], \quad u \in [0,2\pi], \quad v \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big],$$

it holds that [18]

$$\begin{aligned}\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma &= \iiint_{\Omega} (2x+2y+2z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= 2\int_0^1 \Big(\int_0^{2\pi} \Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\varrho\cos u\cos v + \varrho \sin u \cos v + \varrho \sin v + 3)\,\varrho^2 \cos v\,\mathrm{d}v\Big)\,\mathrm{d}u\Big)\,\mathrm{d}\varrho = \underline{\underline{8\pi}}\,.\end{aligned}$$

b) Using the Gauss – Ostrogradsky theorem (be careful about the negative orientation of S) yields

$$\begin{aligned}\iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma &= -\iiint_{\Omega} (1+1+1)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \Big(= -3\lambda(\Omega)\Big) \\ &= -24\int_0^3 \Big(\int_0^{3-x}\Big(\int_0^{3-x-y} \mathrm{d}z\Big)\,\mathrm{d}y\Big)\,\mathrm{d}x = -24\int_0^3 \Big(\int_0^{3-x} 3-x-y\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &= -24\int_0^3 (3-x)^2 - \frac{(3-x)^2}{2}\,\mathrm{d}x = -12\Big[\frac{(3-x)^3}{-3}\Big]_0^3 = 4(0-27) = \underline{\underline{-108}}\,.\end{aligned}$$

[18] Do not forget the Jacobian!

Exercise 5.33. Evaluate

$$\iint_{(S)} (x^3 - yz, y^3 - xz, z^3 - xy)\,\mathrm{d}\sigma,$$

using the Gauss – Ostrogradsky theorem, if S is a positively oriented sphere

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leq 18z\}.$$

5.7 Stokes theorem

Definition 5.34. Let the vector field $f = (f_1, f_2, f_3) : \ \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 in an open set $M \subset \mathbb{R}^3$. Curl of the vector field f (in M) is given by the vector field defined (in M) by the equality

$$\operatorname{curl} f(x,y,z) := \left(\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right)(x,y,z), \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right)(x,y,z), \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)(x,y,z) \right).$$

Remark 5.35. To compute curl f we can make use of the formal equality[19]

$$\text{` } \operatorname{curl} f = \begin{vmatrix} e_1, & e_2, & e_3 \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \\ f_1, & f_2, & f_3 \end{vmatrix} \text{'} .$$

'Definition' 5.36. Let $S \subset \mathbb{R}^3$ denote an oriented piecewise smooth surface such that its boundary ∂S is a geometrical image of a simple closed piecewise smooth curve. Then S and ∂S are of the same orientation, if the following holds: '*when walking along ∂S in the direction of the orientation of ∂S and the head pointing in the direction of the vector field n, the surface S is on the left-hand side*'.

(Compare this 'definition' to 'Definition' 5.22.)

Theorem 5.37 (Stokes)**.** *Let the vector field $f : \ \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 in an open set $M \subset \mathbb{R}^3$ and let $S \subset M$ denote a piecewise smooth surface of the same orientation as ∂S. Then*

$$\int_{(\partial S)} f(x,y,z)\,\mathrm{d}s = \iint_{(S)} \operatorname{curl} f(x,y,z)\,\mathrm{d}\sigma.$$

Remark 5.38 (to the physical interpretation of curl $f(x,y,z)$).
We interpret again the vector field f as the velocity field of a stationary flow of incompressible fluid. It can be shown that rot $f(x,y,z)$ corresponds (roughly speaking) to the direction vector of the line passing through the point (x,y,z) around which the fluid rotates in a 'small' neihgbourhood of (x,y,z). The norm of the vector curl $f(x,y,z)$ corresponds (in a certain sense) to the angular velocity of this rotation.

If curl f vanishes (in M), the vector-field f is irrotational (in M).

[19] e_1, e_2, and e_3 denote 'the coordinate' vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, respectively.

Example 5.39. Evaluate $\int_{(k)} f(x,y,z)\,\mathrm{d}s$ using the Stokes theorem, if

a) $f(x,y,z) := (y^2, z^2, x^2)$ and (k) is the boundary of the triangle given by the vertices $(3,0,0)$, $(0,0,3)$, and $(0,3,0)$ in this order;

b) $f(x,y,z) := (z,x,y)$,

$$(k) = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 = 4 \ \wedge\ \frac{x}{2} + \frac{z}{3} = 1\right\}$$

and orientation of (k) is given by the order of the vertices $(2,0,0)$, $(0,2,3)$, and $(-2,0,6)$;

c) $f(x,y,z) := (-y,x,0)$ and (k) (including its orientation) is given by the parametrization

$$\varphi(t) := (\sin t, \cos t, 0), \quad \mathrm{D}\varphi = [0, 2\pi].$$

Solution.

a) We choose

$$\psi(u,v) := (u, v, 3-u-v),$$
$$D\psi = \Omega = \left\{(u,v) \in \mathbb{R}^2 :\ u+v \le 3 \ \wedge\ u \ge 0 \ \wedge\ v \ge 0\right\}.$$

Since

$$\frac{\partial \psi}{\partial u}(u,v) \times \frac{\partial \psi}{\partial v}(u,v) = (1,1,1),$$

we have (see the Stokes theorem)

$$\begin{aligned}
\int_{(k)} y^2\,\mathrm{d}x + z^2\,\mathrm{d}y + x^2\,\mathrm{d}z &= -\iint_{(\psi)} \operatorname{rot}(y^2, z^2, x^2)\,\mathrm{d}\sigma \\
&= -\iint_{(\psi)} (-2z, -2x, -2y)\,\mathrm{d}\sigma = -(-2)\iint_{\Omega} (3-u-v, u, v)\cdot(1,1,1)\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\int_0^3 \Big(\int_0^{3-u} 3\,\mathrm{d}v\Big)\,\mathrm{d}u = 6\int_0^3 3-u\,\mathrm{d}u = 6\big[3u - \frac{u^2}{2}\big]_0^3 = \underline{\underline{27}}\,.
\end{aligned}$$

A question to the reader:

Why is there a minus sign in front of the surface integral?

b) Since $\operatorname{curl}(z,x,y) = (1,1,1)$, we have (due to the Stokes theorem)

$$\int_{(k)} f(x,y,z)\,\mathrm{d}s = \iint_{(S)} (1,1,1)\,\mathrm{d}\sigma$$

e.g. for the surface

$$S = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 \le 4 \ \wedge\ \frac{x}{2} + \frac{z}{3} = 1\right\}$$

of the same orientation as (k). We parametrize $S = \langle\psi\rangle$, where

$$\psi(u,v) := \left(u, v, 3\left(1 - \frac{u}{2}\right)\right), \quad \mathrm{D}\psi = K = \{(u,v) \in \mathbb{R}^2 : \ u^2 + v^2 \le 4\}.$$

Then

$$\frac{\partial\psi}{\partial u} = \left(1, 0, -\frac{3}{2}\right), \quad \frac{\partial\psi}{\partial v} = (0,1,0), \quad \frac{\partial\psi}{\partial u} \times \frac{\partial\psi}{\partial v} = \left(\frac{3}{2}, 0, 1\right)$$

(the orientation agrees), and thus

$$\int_{(k)} f(x,y,z)\,\mathrm{d}s = \iint_K (1,1,1) \cdot \left(\frac{3}{2}, 0, 1\right) \mathrm{d}u\,\mathrm{d}v = \frac{5}{2}\lambda(K) = \frac{5}{2} \cdot \pi \cdot 4 = \underline{\underline{10\pi}}\,.$$

c) We present two approaches to the evaluation of the integral. First, we replace the line integral by a surface integral of the second kind over the disk $K = \langle\psi\rangle$, where

$$\psi(r,t) := (r\cos t, r\sin t, 0), \quad \mathrm{D}\psi = \{(r,t) \in \mathbb{R}^2 : \ r \in [0,1] \wedge t \in [0,2\pi]\}.$$

Then

$$\frac{\partial\psi}{\partial r} = (\cos t, \sin t, 0), \quad \frac{\partial\psi}{\partial t} = (-r\sin t, r\cos t, 0), \quad \frac{\partial\psi}{\partial r} \times \frac{\partial\psi}{\partial t} = (0,0,r),$$

$$\operatorname{curl}(-y, x, 0) = (0,0,2),$$

and thus (be careful about 'the opposite orientation' of ψ and φ)

$$\int_{(k)} f(x,y,z)\,\mathrm{d}s = -\int_0^{2\pi} \Big(\int_0^1 (0,0,2) \cdot (0,0,r)\,\mathrm{d}r\Big)\,\mathrm{d}t = \underline{\underline{-2\pi}}\,.$$

Now we present how to replace the given integral by an integral over 'the upper hemisphere' $S = \langle\tilde{\psi}\rangle$, where

$$\tilde{\psi}(u,v) := (\cos u \cos v, \sin u \cos v, \sin v), \quad \mathrm{D}\tilde{\psi} = [0, 2\pi] \times \Big[0, \frac{\pi}{2}\Big].$$

For the parametrization $\tilde{\psi}$ it holds that

$$\frac{\partial\tilde{\psi}}{\partial u} = (-\sin u \cos v, \cos u \cos v, 0), \quad \frac{\partial\tilde{\psi}}{\partial v} = (-\cos u \sin v, -\sin u \sin v, \cos v),$$

$$\frac{\partial\tilde{\psi}}{\partial u} \times \frac{\partial\tilde{\psi}}{\partial v} = (\ldots, \ldots, \sin v \cos v).$$

Notice that

$$\Big(\frac{\partial\tilde{\psi}}{\partial u} \times \frac{\partial\tilde{\psi}}{\partial v}\Big)\Big(0, \frac{\pi}{4}\Big) = \Big(\ldots, \ldots, \frac{1}{2}\Big),$$

and we thus again obtain 'opposite orientations' of $\tilde{\psi}$ and φ. Then it easily follows that

$$\int_{(k)} f(x,y,z)\,\mathrm{d}s = -\int_0^{2\pi} \Big(\int_0^{\frac{\pi}{2}} 2\sin v \cos v\,\mathrm{d}v\Big)\,\mathrm{d}u = -2\pi\Big[-\frac{\cos(2v)}{2}\Big]_0^{\frac{\pi}{2}} = \underline{\underline{-2\pi}}\,.$$

Exercise 5.40. Evaluate

$$\int_{(\varphi)} x\,\mathrm{d}x + (x+y)\,\mathrm{d}y + (x+y+z)\,\mathrm{d}z,$$

using the Stokes theorem if $\varphi(t) := (3\cos t, 3\sin t, 3(\cos t + \sin t))$, $\mathrm{D}\varphi = [0, 2\pi]$.

Remark 5.41. The definition of the differential operators of the first order including `gradient`, `divergence`, and `curl` can be memorized by using 'the nabla operator'

$$\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

and formal equalities

$$\begin{aligned} &\text{“}\operatorname{grad} f = \nabla f\text{”} && (f:\ \mathbb{R}^3 \to \mathbb{R}),\\ &\text{“}\operatorname{div} f = \nabla \cdot f\text{”} && (f:\ \mathbb{R}^3 \to \mathbb{R}^3),\\ &\text{“}\operatorname{curl} f = \nabla \times f\text{”} && (f:\ \mathbb{R}^3 \to \mathbb{R}^3). \end{aligned}$$

Exercise 5.42. Let the vector field $f = (f_1, f_2, f_3):\ \mathbb{R}^3 \to \mathbb{R}^3$ be of class C^1 and assume it is `conservative` in a domain $M \subset \mathbb{R}^3$. Prove that f is `irrotational` in M.

5.8 Applications of the surface integral of the second kind

a) **Flow of a vector field through an oriented surface.**
Let the vector field $f:\ \mathbb{R}^3 \mapsto \mathbb{R}^3$ be continuous on an oriented smooth surface S. Flow of the vector field f through the oriented surface S is defined, as has been mentioned above, by
$$\tau(S) := \iint_{(S)} f(x,y,z)\,\mathrm{d}\sigma.$$

b) **Volume of a body (more precisely measure of a set).**
Let $\Omega \subset \mathbb{R}^3$ denote a regular domain. Then it holds that (see the Gauss – Ostrogradsky theorem)[20]
$$\lambda(\overline{\Omega}) = \frac{1}{3}\iint_{(\partial\Omega)} (x,y,z)\,\mathrm{d}\sigma = \iint_{(\partial\Omega)} (x,0,0)\,\mathrm{d}\sigma = \dots.$$

Example 5.43. Compute the flow of the vector field $f(x,y,z) := (x^2, y^2, z^2)$ through a positively oriented sphere with the radius of 1 centered in $(1,1,1)$.

Solution. We denote

$$\begin{aligned} S &= \{(x,y,z) \in \mathbb{R}^3:\ (x-1)^2 + (y-1)^2 + (z-1)^2 = 1\},\\ \Omega &= \{(x,y,z) \in \mathbb{R}^3:\ (x-1)^2 + (y-1)^2 + (z-1)^2 \le 1\}. \end{aligned}$$

[20] We obviously assume that the closed piecewise smooth surface $\partial\Omega$ is positively oriented.

From the Gauss – Ostrogradsky theorem it directly follows that (S is positively oriented)

$$\begin{aligned}
\tau(S) &= \iint_{(S)} (x^2, y^2, z^2)\,\mathrm{d}\sigma = \iiint_{\Omega} 2x + 2y + 2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= 2\int_0^{2\pi}\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\int_0^1 (3 + r\cos u\cos v + r\sin u\cos v + r\sin v)\,r^2\cos v\,\mathrm{d}r\Big)\,\mathrm{d}u\Big)\,\mathrm{d}v \\
&= 6\cdot 2\pi\Big(\int_0^1 r^2\,\mathrm{d}r\Big)\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos v\,\mathrm{d}v\Big) = 12\pi\cdot\frac{1}{3}\cdot 2 = \underline{\underline{8\pi}}\,.
\end{aligned}$$

We used the substitution

$$\begin{gathered}
x = r\cos u\cos v, \quad y = r\sin u\cos v, \quad z = r\sin v, \\
r\in[0,1], \quad u\in[0,2\pi], \quad v\in\Big[-\frac{\pi}{2},\frac{\pi}{2}\Big], \quad J = r^2\cos v
\end{gathered}$$

and obvious equalities

$$\int_0^{2\pi}\cos u\,\mathrm{d}u = \int_0^{2\pi}\sin u\,\mathrm{d}u = 0, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin v\cos v\,\mathrm{d}v = 0\,.$$

Exercise 5.44. Let a, b denote real numbers satisfying $a > b > 0$. Compute the volume of the body Ω (torus) bounded by the surface ψ, if

$$\begin{gathered}
\psi(u,v) := ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v), \\
\mathrm{D}\psi = [0,2\pi]\times[0,2\pi].
\end{gathered}$$

References

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