STEKLOV–POINCARÉ OPERÁTOR

- $H^1(\Omega)$, $H^{1/2}(\partial\Omega)$, $H^{-1/2}(\partial\Omega)$, $H^1_\Delta(\Omega)$

- $u \in H^1_\Delta(\Omega) \rightsquigarrow$ $Tu =: g \in H^{1/2}(\partial\Omega)$, $-\Delta u =: f \in L^2(\Omega)$ $\rightsquigarrow$ u řeší $-\Delta u = f$ v Ω, $u = g$ na $\partial\Omega$

$$\frac{\partial u}{\partial n} \in H^{-1/2}(\partial\Omega): \quad \boxed{\left\langle \frac{\partial u}{\partial n}, \varphi \right\rangle = \int_\Omega \nabla u \, \nabla(E\varphi)\, dx - \int_\Omega f (E\varphi)\, dx}$$

$$\boxed{\frac{\partial}{\partial n} : H^1_\Delta(\Omega) \longrightarrow H^{-1/2}(\partial\Omega)}$$

- lineární (tj. korektní)

- omezení:

$$\left\| \frac{\partial u}{\partial n} \right\|_{H^{-1/2}} = \sup_{\varphi \neq 0} \frac{\left| \left\langle \frac{\partial u}{\partial n}, \varphi \right\rangle \right|}{\|\varphi\|_{1/2}} \leq 2c \cdot \|u\|_{H^1_\Delta}$$

dk.

$$\left| \left\langle \frac{\partial u}{\partial n}, \varphi \right\rangle \right| \leq \underbrace{\sqrt{\int_\Omega |\nabla u|^2}}_{\leq \|u\|_{H^1_\Delta}} \cdot \underbrace{\sqrt{\int_\Omega |\nabla E\varphi|^2}}_{\leq \|E\varphi\|_{H^1}} + \underbrace{\sqrt{\int_\Omega f^2}}_{\leq \|u\|_{H^1_\Delta}} \cdot \underbrace{\sqrt{\int_\Omega (E\varphi)^2}}_{\leq \|E\varphi\|_{H^1}} \leq$$

$$\leq 2 \|u\|_{H^1_\Delta} \cdot \|E\varphi\|_{H^1} \leq 2 \|u\|_{H^1_\Delta} \cdot c \cdot \|\varphi\|_{H^{1/2}}$$

- na:

Buď $\mu \in H^{-1/2}(\partial\Omega)$ a buď $u \in H^1(\Omega)$ slabé řešení okrajové úlohy:

$$\begin{cases} -\Delta u + u = 0, \\ \dfrac{\partial u}{\partial n} = \mu, \end{cases}$$

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Řeš. $\forall v \in H^1(\Omega): \int \nabla u \nabla v + \int u v = \langle \mu, T v \rangle$

$\exists !\, u \in H^1(\Omega) \cap H^1_\Delta(\Omega)$

$\forall \varphi \in H^{1/2}(\partial\Omega): \quad \langle \frac{\partial u}{\partial n}, \varphi \rangle = \int \nabla u \nabla (\varepsilon\varphi) - \int \underset{\substack{\| \\ -\Delta u \\ \| \\ -u}}{f} (\varepsilon\varphi) =$

$= \int \nabla u \nabla \underbrace{(\varepsilon\varphi)}_{=v} + \int u \underbrace{(\varepsilon\varphi)}_{=v} = \langle \mu, T(\varepsilon\varphi) \rangle =$

$= \langle \mu, \varphi \rangle .$

Odtud $\boxed{\frac{\partial u}{\partial n} = \mu}$ cbd.

$\boxed{\begin{aligned} -\Delta u &= f \in L^2(\Omega) \\ u &= g \in H^{-1/2}(\partial\Omega) \end{aligned}}$ $\leadsto u = u_1 + u_2$: $\boxed{\begin{aligned} -\Delta u_1 &= 0, \\ u_1 &= g, \end{aligned}}$ $\boxed{\begin{aligned} -\Delta u_2 &= f, \\ u_2 &= 0 \end{aligned}}$

$$\boxed{\frac{\partial u}{\partial n} = \underset{\substack{\| \\ Sg}}{\frac{\partial u_1}{\partial n}} + \underset{\substack{\| \\ -Nf}}{\frac{\partial u_2}{\partial n}}}$$

$$\boxed{S : g \to \frac{\partial u_1}{\partial n} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)}$$

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$$\boxed{-N : f \to \frac{\partial u_2}{\partial n} : L^2(\Omega) \longrightarrow H^{-1/2}(\partial\Omega)}$$

... NEWTONŮV POTENCIÁL

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VLASTNOSTI STEKLOV-POINCARÉHO OPERÁTORU

- $S \in \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$, tzn. S … lineární
  … spojitý

$$g \xrightarrow[\in \mathcal{L}(H^{1/2}, H^1_\Delta)]{} u_1 \xrightarrow[\in \mathcal{L}(H^1_\Delta, H^{-1/2})]{} \frac{\partial u_1}{\partial n} \implies \text{…cbd.}$$

- $Sg = 0 \iff g = $ konstantní

$$\left.\begin{aligned} -\Delta u &= 0 \\ u &= g \\ \frac{\partial u}{\partial n} &= 0 \end{aligned}\right. \leadsto \text{u je slabým řešením } \left\{\begin{aligned} -\Delta u &= 0 \\ \frac{\partial u}{\partial n} &= 0 \end{aligned}\right. \leadsto u = 0 + c, \text{ kde } c \in \mathbb{R}$$

- S je pozitivně semidefinitní, tzn. $\forall g \in H^{1/2}(\partial\Omega): \langle Sg, g\rangle \geq 0$

$$\langle Sg, g\rangle = \int_\Omega \nabla u \, \nabla(\varepsilon g)\, dx = \int_\Omega |\nabla u|^2 \geq 0$$

$\int \nabla u \, \nabla(u - \varepsilon g) = 0$, protože u je sl. řeš. $-\Delta u = 0$, $u = g$

- S je symetrický, tzn. $\forall g, h \in H^{1/2}(\partial\Omega): \langle Sg, h\rangle = \langle Sh, g\rangle$

$$\langle Sg, h\rangle \overset{?}{=} \langle Sh, g\rangle$$

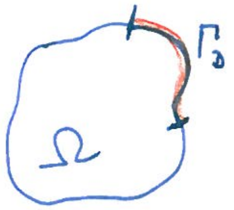
$$\langle Sg, h\rangle \overset{\left\{\begin{smallmatrix} -\Delta u_g = 0 \\ u_g = g \end{smallmatrix}\right.}{=} \int \nabla u_g \, \nabla(\varepsilon h) \overset{\varepsilon h - u_h \in H^1_0(\Omega)}{=} \int \nabla u_g \, \nabla u_h$$

$$\langle Sh, g\rangle \overset{\left\{\begin{smallmatrix} -\Delta u_h = 0 \\ u_h = h \end{smallmatrix}\right.}{=} \int \nabla u_h \, \nabla(\varepsilon g) \overset{\varepsilon g - u_g \in H^1_0(\Omega)}{=} \int \nabla u_h \, \nabla u_g$$

$$\int \nabla u_g \, \nabla u_h \overset{!}{=} \int \nabla u_h \, \nabla u_g$$

• S je eliptický na $H^{1/2}_0(\partial\Omega, \Gamma_D) := \{v \in H^{1/2}(\partial\Omega) : v = 0 \text{ s.v. na } \Gamma_D\}$ 23/5 20/5

(„meas Γ_D" > 0)



$\forall g \in H^{1/2}_0(\partial\Omega, \Gamma_D)$:

$$\langle Sg, g\rangle = \int \nabla u \cdot \nabla(Eg) = \int |\nabla u|^2 \geq k \cdot \|u\|^2_{H^1} \geq c\,\|Tu\|^2_{H^{1/2}} = c\,\|g\|^2_{H^{1/2}}$$

Friedrichsova nerovnost ($Tu = g = 0$ na Γ_D)