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Integral Calculus of Multivariate Functions

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Contents

1 Double Integral over Intervals

We will proceed in a similar manner which led to the definition of a definite integral. Let us recall, definite integral was constructed such that for a function $f : \mathbb{R} \to \mathbb{R}$, non-negative and continuous over interval $\langle a, b\rangle$, where $a, b \in \mathbb{R},\ a < b$, it equals to the **area** of set

$$\left\{(x, y) \in \mathbb{R}^2 :\ a \le x \le b\ \wedge\ 0 \le y \le f(x)\right\}.$$

Similarly: For a function $f : \mathbb{R}^2 \to \mathbb{R}$, non-negative and continuous over (two-dimensional) interval $\langle a, b\rangle \times \langle c, d\rangle$, where $a, b, c, d \in \mathbb{R},\ a < b,\ c < d$, double integral of function f over this interval (rectangle) will equal to the **volume** of a solid

$$\left\{(x, y, z) \in \mathbb{R}^3 :\ a \le x \le b\ \wedge\ c \le y \le d\ \wedge\ 0 \le z \le f(x, y)\right\}.$$

Let us remember, for every function $f : \mathbb{R} \to \mathbb{R}$, continuous but not necessarily non-negative over interval $\langle a, b\rangle$, we defined $\int_a^b f(x)\,\mathrm{d}x$ as a limit of sequence (s_n) defined as follows:

$$s_n := \sum_{j=0}^{n-1} f(x_j) \cdot \frac{b-a}{n}, \qquad x_j = a + j \cdot \frac{b-a}{n}.$$

Note that it can be shown that $\lim s_n$ exists. And now, let us make the following definition of double integral - also here, we will demand only continuity of f.

Remark 1.1.

Definition 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous over interval*

$$J = \langle a, b\rangle \times \langle c, d\rangle, \qquad a, b, c, d \in \mathbb{R}, a < b, c < d.$$

We define

$$\iint_J f(x, y)\ \mathrm{d}x\,\mathrm{d}y := \lim s_n,$$

where

$$s_n := \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(x_j, y_k) \cdot \frac{b-a}{n} \cdot \frac{d-c}{n},$$

$$x_j = a + j \cdot \frac{b-a}{n}, \qquad y_k = c + k \cdot \frac{d-c}{n}.$$

Theorem 1.3 (Fubini)**.** *If $f:\mathbb{R}^2\to\mathbb{R}$ is continuous over $J=\langle a,b\rangle\times\langle c,d\rangle$, then*

$$\iint_J f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b\left(\int_c^d f(x,y)\,\mathrm{d}y\right)\mathrm{d}x=\int_c^d\left(\int_a^b f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

The following exercise demonstrates application of the Theorem 1.3 (try it also for yourselves!):

Exercise 1.4.

1.
$$\iint_{\langle 0,1\rangle\times\langle 0,2\rangle} 1\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\left(\int_0^2 1\,\mathrm{d}y\right)\mathrm{d}x=\int_0^1[y]_{y=0}^2\,\mathrm{d}x=\int_0^1 2\,\mathrm{d}x=\underline{\underline{2}}.$$

2.
$$\iint_{\langle -2,1\rangle\times\langle 2,5\rangle}(4x-y+3)\,\mathrm{d}x\,\mathrm{d}y=\int_{-2}^1\left(\int_2^5(4x-y+3)\,\mathrm{d}y\right)\mathrm{d}x=$$

$$=\int_{-2}^1\left[4xy-\frac{y^2}{2}+3y\right]_{y=2}^5\mathrm{d}x=\int_{-2}^1\left(20x-\frac{25}{2}+15-8x+2-6\right)\mathrm{d}x=$$

$$=\int_{-2}^1\left(12x-\frac{3}{2}\right)\mathrm{d}x=\left[6x^2-\frac{3}{2}x\right]_{x=-2}^1=6-\frac{3}{2}-24-3=\underline{\underline{-\frac{45}{2}}}.$$

3.
$$\iint_{\langle 1,2\rangle\times\langle 0,1\rangle}\left(x^2+xy+y^2\right)\,\mathrm{d}x\,\mathrm{d}y=\int_1^2\left(\int_0^1\left(x^2+xy+y^2\right)\mathrm{d}y\right)\mathrm{d}x=$$

$$=\int_1^2\left[x^2y+\frac{xy^2}{2}+\frac{y^3}{3}\right]_{y=0}^1\mathrm{d}x=\int_1^2\left(x^2+\frac{x}{2}+\frac{1}{3}\right)\mathrm{d}x=\left[\frac{x^3}{3}+\frac{x^2}{4}+\frac{x}{3}\right]_1^2=$$

$$=\left(\frac{8}{3}+1+\frac{2}{3}\right)-\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{3}\right)=\frac{13}{3}-\frac{11}{12}=\underline{\underline{\frac{41}{12}}}.$$

Let us also try to calculate this particular integral by integrating in the opposite order:

$$\iint_{\langle 1,2\rangle\times\langle 0,1\rangle} \left(x^2+xy+y^2\right)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_1^2 \left(x^2+xy+y^2\right)\mathrm{d}x\right)\mathrm{d}y =$$

$$= \int_0^1 \left[\frac{x^3}{3}+\frac{x^2y}{2}+xy^2\right]_{x=1}^2 \mathrm{d}y = \int_0^1 \left(\left(\frac{8}{3}+2y+2y^2\right)-\left(\frac{1}{3}+\frac{y}{2}+y^2\right)\right)\mathrm{d}y =$$

$$= \int_0^1 \left(\frac{7}{3}+\frac{3y}{2}+y^2\right)\mathrm{d}y = \left[\frac{7y}{3}+\frac{3y^2}{4}+\frac{y^3}{3}\right]_{y=0}^1 = \frac{7}{3}+\frac{3}{4}+\frac{1}{3} = \underline{\underline{\frac{41}{12}}}\,.$$

As we can see, the difficulty of either way is more or less equal in this case.

4.

$$\iint_{\langle 0,1\rangle\times\langle 0,\frac{\pi}{2}\rangle} xy^2\sin(x^2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\left(\int_0^1 xy^2\sin(x^2y)\,\mathrm{d}x\right)\mathrm{d}y =$$

$$= \left|\begin{array}{c}\text{substitution}\\ x^2y=t\\ 2xy\ \mathrm{d}x=\mathrm{d}t\\ xy\ \mathrm{d}x=\frac{1}{2}\,\mathrm{d}t\\ 0\mapsto 0,\ 1\mapsto y\end{array}\right| = \int_0^{\frac{\pi}{2}}\left(\int_0^y \frac{1}{2}y\sin t\,\mathrm{d}t\right)\mathrm{d}y = \int_0^{\frac{\pi}{2}} \frac{y}{2}[-\cos t]_{t=0}^y\,\mathrm{d}y =$$

$$= \int_0^{\frac{\pi}{2}} \frac{y}{2}(-\cos y+1)\,\mathrm{d}y = \left[\frac{y^2}{4}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} y\cos y\ \mathrm{d}y = \left|\begin{array}{cc}\multicolumn{2}{c}{\text{by parts}}\\ u=y & v'=\cos y\\ u'=1 & v=\sin y\end{array}\right| =$$

$$= \frac{\pi^2}{16}-\frac{1}{2}\left([y\sin y]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin y\,\mathrm{d}y\right) = \frac{\pi^2}{16}-\frac{1}{2}\left(\frac{\pi}{2}-[-\cos y]_0^{\frac{\pi}{2}}\right) = \underline{\underline{\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}}}\,.$$

You may also try integrating with respect to y first, but in this case, you will very likely encounter some trouble along the calculation.

2 Double Integral over Measurable Sets

In this section, we will extend the definition of double integral. This time, we will not integrate only over intervals (rectangles) but also over more complicated sets. However, it will not be possible for any subset of $\mathbb{R}^2$ – we will restrain ourselves to so-called **measurable sets**.

Definition 2.1.

- *Let $a, b \in \mathbb{R}, a < b$. Furthermore, let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that g_1, g_2 are continuous over $\langle a, b\rangle$ and $g_1(x) \leq g_2(x)$ for every $x \in \langle a, b\rangle$. Let us define*

$$M = \left\{(x, y) \in \mathbb{R}^2 : \ a \leq x \leq b \ \wedge \ g_1(x) \leq y \leq g_2(x)\right\}.$$

- *Let $c, d \in \mathbb{R}, c < d$. Furthermore, let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that h_1, h_2 are continuous over $\langle c, d\rangle$ and $h_1(y) \leq h_2(y)$ for every $y \in \langle c, d\rangle$. Let us define*

$$N = \left\{(x, y) \in \mathbb{R}^2 : \ c \leq y \leq d \ \wedge \ h_1(y) \leq x \leq h_2(y)\right\}.$$

*Sets M, N are called **elementary measurable sets**.*

Remark 2.2. Sometimes, the sets of the type M from above are called elementary sets *of the first kind*, those of type N are called analogically – elementary sets *of the second kind*.

Definition 2.3. *We will call $M \subset \mathbb{R}^2$ **measurable**, if it can be obtained from elementary measurable sets with finite number of following set operations: set unions, set intersections, set differences.*

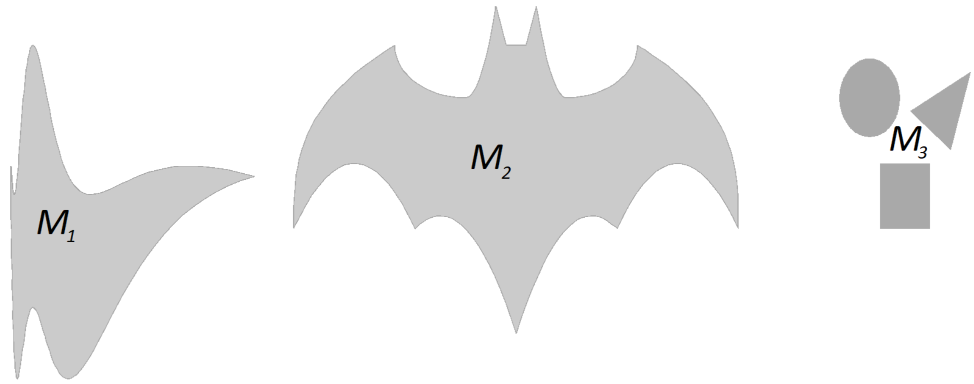
Let us recall:

- Set union: $M \cup N := \{x : \ x \in M \ \vee \ x \in N\}$
- Set intersection: $M \cap N := \{x : \ x \in M \ \wedge \ x \in N\}$
- Set difference: $M \smallsetminus N := \{x : \ x \in M \ \wedge \ x \notin N\}$

Remark 2.4. *Every measurable set is necessarily bounded.*

Some examples of measurable sets in $\mathbb{R}^2$:

- $M_0 = \varnothing$,
- M_1, M_2, M_3 are depicted in the picture below:



Some examples of non-measurable sets in $\mathbb{R}^2$:

- $N_1 = \{(x,y) \in \langle 0,1\rangle \times \langle 0,1\rangle : \ x \in \mathbb{Q} \ \wedge \ y \in \mathbb{Q}\}$,
- $N_2 = \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1 \ \wedge \ 0 \le y \le \chi(x)\}$, where χ is so-called Dirichlet function and is defined as

$$\chi(x) := \begin{cases} 0, & x \in \mathbb{R} \smallsetminus \mathbb{Q}, \\ 1, & x \in \mathbb{Q}. \end{cases}$$

Let us recall that by the term **neighbourhood** of the point $\mathbf{c} \in \mathbb{R}^2$ with radius $\varepsilon \in \mathbb{R}^+$ we mean set

$$\mathcal{U}(\mathbf{c},\varepsilon) := \left\{\mathbf{x} \in \mathbb{R}^2 : \ \|\mathbf{x} - \mathbf{c}\| < \varepsilon\right\},$$

in other words, open disk with centre $\mathbf{c}$ and radius $\varepsilon > 0$.

Definition 2.5. *Let $M \subset \mathbb{R}^2$. The set*

$$\overline{M} := \left\{\mathbf{c} \in \mathbb{R}^2 : \ \textit{there is at least one point from } M \textit{ within any } \mathcal{U}(\mathbf{c},\varepsilon)\right\}$$

is called ***closure*** *of set M.*

In other words

$$(\forall \mathbf{c} \in \mathbb{R}^2)(\forall M \subset \mathbb{R}^2): \qquad \mathbf{c} \in \overline{M} \qquad \Leftrightarrow \qquad [\forall \varepsilon > 0 : \ \mathcal{U}(\mathbf{c},\varepsilon) \cap M \neq \varnothing].$$

It can be shown that

$$\overline{M} = \left\{\mathbf{c} \in \mathbb{R}^2 : \text{ there exists a sequence } (\mathbf{x}_n) \text{ of points from } M \text{ such that } \mathbf{x}_n \to \mathbf{c}\right\}.$$

Definition 2.6. *We will call $M \subset \mathbb{R}^2$ **closed**, if $M = \overline{M}$.*

Remark 2.7. Moreover, $\overline{M}$ is the smallest closed set containing M.

Definition 2.8. *We define **indicator function** of a set $M \subset \mathbb{R}^2$ as*

$$\chi_M(x,y) := \begin{cases} 1, & (x,y) \in M, \\ 0, & (x,y) \in \mathbb{R}^2 \smallsetminus M. \end{cases}$$

Definition 2.9. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous over $\overline{M}$, where M is a measurable set. Let us denote*

$$F(x,y) := \begin{cases} f(x,y), & (x,y) \in M, \\ 0, & (x,y) \in \mathbb{R}^2 \smallsetminus M, \end{cases}$$

and for an arbitrary interval $J = \langle a,b\rangle \times \langle c,d\rangle$ such that $M \subset J$, let us also define sequence

$$s_n := \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} F(x_j, y_k) \cdot \frac{b-a}{n} \cdot \frac{d-c}{n},$$

where

$$x_j = a + j \cdot \frac{b-a}{n}, \qquad y_k = c + k \cdot \frac{d-c}{n}.$$

*Then we define **double integral of function f over set M** as*

$$\iint_M f(x,y)\, \mathrm{d}x\, \mathrm{d}y = \lim s_n.$$

Remark 2.10. In case that f from the definition above is defined on the entire plane $\mathbb{R}^2$, we can write $F(x,y) = \chi_M(x,y) f(x,y)$.

Remark 2.11. It can be shown, that $\lim s_n$ exists and is independent of the specific choice of interval J such that $M \subset J$.

Definition 2.12. *Let $M \subset \mathbb{R}^2$ be a measurable set. We define **measure** of set M as number*

$$\lambda(M) := \iint_M 1 \, \mathrm{d}x \, \mathrm{d}y.$$

Remark 2.13. Measure of a set in $\mathbb{R}^2$ is a generalization of the term area, which is clear for some specific sets, e.g. rectangle, disk etc.

Theorem 2.14 (Properties of Measure)**.**

1. *If $M \subset \mathbb{R}^2$ has at most finite number of elements, then it is measurable and $\lambda(M) = 0$.*
2. *The set $M = \langle a, b\rangle \times \langle c, d\rangle$, where $a, b, c, d \in \mathbb{R}$, $a \le b$, $c \le d$, is measurable and*

$$\lambda(M) = (b - a) \cdot (d - c).$$

3. *If $M_1, M_2, \ldots M_n \subset \mathbb{R}^2$ are measurable and pairwise disjoint (i.e. $i \neq j \;\Rightarrow\; M_i \cap M_j = \varnothing$), then*

$$\lambda\left(M_1 \cup M_2 \cup \ldots \cup M_n\right) = \lambda\left(M_1\right) + \lambda\left(M_2\right) + \ldots + \lambda\left(M_n\right).$$

4. *If $M, N \subset \mathbb{R}^2$ are measurable and $M \subset N$, then*

$$0 \le \lambda(M) \le \lambda(N).$$

Theorem 2.15 (Properties of Double Integral)**.** *Let $M \subset \mathbb{R}^2$ be a measurable set and let functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ be continuous over $\overline{M}$. Then it holds:*

1. *For every $\alpha, \beta \in \mathbb{R}$*

$$\iint_M (\alpha f(x,y) + \beta g(x,y))\ \mathrm{d}x\,\mathrm{d}y = \alpha \iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y + \beta \iint_M g(x,y)\ \mathrm{d}x\,\mathrm{d}y.$$

2. *If $M = M_1 \cup M_2$, where M_1, M_2 are measurable and $\lambda(M_1 \cap M_2) = 0$, then*

$$\iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \iint_{M_1} f(x,y)\ \mathrm{d}x\,\mathrm{d}y + \iint_{M_2} f(x,y)\ \mathrm{d}x\,\mathrm{d}y.$$

3. *If $f(x,y) \le g(x,y)$ for every $(x,y) \in M$, then*

$$\iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y \le \iint_M g(x,y)\ \mathrm{d}x\,\mathrm{d}y.$$

4.

$$\left| \iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y \right| \le \iint_M |f(x,y)|\ \mathrm{d}x\,\mathrm{d}y.$$

5. *If $\lambda(M) = 0$, then*

$$\iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y = 0.$$

Theorem 2.16 (Fubini)**.** *Let $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$.*

1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous over set*

$$M = \left\{(x,y) \in \mathbb{R}^2 : \ a \le x \le b \ \wedge \ g_1(x) \le y \le g_2(x)\right\},$$

where $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are continuous over $\langle a, b\rangle$ and $g_1(x) \le g_2(x)$ for every $x \in \langle a, b\rangle$. Then

$$\iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous over set*

$$N = \left\{(x,y) \in \mathbb{R}^2 : \ c \le y \le d \ \wedge \ h_1(y) \le x \le h_2(y)\right\},$$

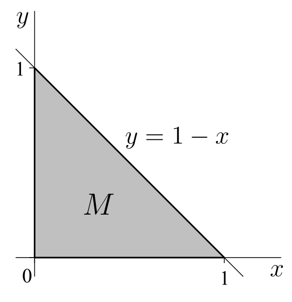
where $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are continuous over $\langle c, d\rangle$ and $h_1(y) \le h_2(y)$ for every $y \in \langle c, d\rangle$. Then

$$\iint_M f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Exercise 2.17. Evaluate the integral $I = \iint_M (x^2 + y^2 + 1)\ \mathrm{d}x\,\mathrm{d}y$, where

$$M = \left\{(x,y) \in \mathbb{R}^2 : \ x \ge 0 \wedge y \ge 0 \wedge x + y \le 1\right\}.$$

Solution:
The set M is triangle with vertices $(0,0)$, $(1,0)$ a $(0,1)$ (see picture below).



It is not hard to realize that

$$M = \left\{(x,y) \in \mathbb{R}^2 : \ x \in \langle 0, 1\rangle \wedge 0 \le y \le 1 - x\right\},$$

thus (according to Theorem 2.16) it holds

$$I = \int_0^1 \left(\int_0^{1-x} \left(x^2 + y^2 + 1 \right) \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left[x^2 y + \frac{y^3}{3} + y \right]_{y=0}^{1-x} \mathrm{d}x =$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} + (1-x) \right) \mathrm{d}x =$$

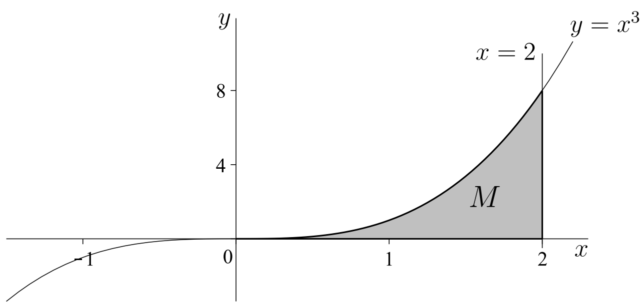
$$= \frac{1}{3} \int_0^1 \left(3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3 + 3 - 3x \right) \mathrm{d}x =$$

$$= \frac{1}{3} \int_0^1 \left(-4x^3 + 6x^2 - 6x + 4 \right) \mathrm{d}x = \frac{1}{3} \left[-x^4 + 2x^3 - 3x^2 + 4x \right]_0^1 =$$

$$= \frac{1}{3} (-1 + 2 - 3 + 4) = \underline{\underline{\frac{2}{3}}} .$$

Exercise 2.18. Evaluate the integral $I = \iint_M x^2 \mathrm{e}^{-y} \, \mathrm{d}x \, \mathrm{d}y$, where M is region bounded by curves $y = x^3, y = 0, x = 2$.

The set M is depicted in the following picture.



We can obviously write

$$M = \left\{ (x, y) \in \mathbb{R}^2 : \; x \in \langle 0, 2 \rangle \; \wedge \; 0 \le y \le x^3 \right\} .$$

And thus, according to Theorem 2.16, it holds

$$I = \int_0^2 \left(\int_0^{x^3} x^2 \mathrm{e}^{-y} \, \mathrm{d}y \right) \mathrm{d}x = \int_0^2 x^2 \left[-\mathrm{e}^{-y} \right]_{y=0}^{x^3} \mathrm{d}x = \int_0^2 x^2 \left(-\mathrm{e}^{-x^3} + 1 \right) \mathrm{d}x =$$

$$= \left| \begin{array}{c} \text{substitution} \\ -x^3 = t \\ -3x^2 \, \mathrm{d}x = \mathrm{d}y \\ 0 \mapsto 0, \, 2 \mapsto -8 \end{array} \right| = \frac{1}{3} \int_{-8}^0 (-e^t + 1) \, \mathrm{d}y = \frac{1}{3} \left[-e^t + t \right]_{-8}^0 = \underline{\underline{\frac{7 + \mathrm{e}^{-8}}{3}}} \, .$$

We could also proceed in a different manner (if we consider M as an elementary set of the second kind). That is to say it holds

$$M = \left\{ (x, y) \in \mathbb{R}^2 : \; y \in \langle 0, 8 \rangle \, \wedge \, \sqrt[3]{y} \le x \le 2 \right\},$$

from which we have

$$I = \int_0^8 \left(\int_{\sqrt[3]{y}}^2 x^2 \mathrm{e}^{-y} \, \mathrm{d}x \right) \mathrm{d}y = \int_0^8 \mathrm{e}^{-y} \left[\frac{x^3}{3} \right]_{x=\sqrt[3]{y}}^2 \mathrm{d}y = \int_0^8 \frac{\mathrm{e}^{-y}}{3} (8 - y) \, \mathrm{d}y =$$

$$= \left| \begin{array}{cc} \multicolumn{2}{c}{\text{by parts}} \\ u = 8 - y, & v' = \frac{\mathrm{e}^{-y}}{3} \\ u' = -1, & v = -\frac{1}{3}\mathrm{e}^{-y} \end{array} \right| = \left[-\frac{1}{3} \mathrm{e}^{-y} (8 - y) \right]_0^8 - \int_0^8 \left(\frac{1}{3} \mathrm{e}^{-y} \right) \mathrm{d}y =$$

$$= \frac{8}{3} + \left[\frac{1}{3} \mathrm{e}^{-y} \right]_0^8 = \frac{8}{3} + \frac{1}{3} \mathrm{e}^{-8} - \frac{1}{3} = \underline{\underline{\frac{7 + \mathrm{e}^{-8}}{3}}} \, .$$

Note that in the first approach we needed substitution method (for one-dimensional integrals), while in the second solution we needed to use integration by parts.

Exercise 2.19. Calculate following integrals using Fubini's theorem:

1. $\displaystyle\iint_M (x^2 + y) \, \mathrm{d}x \, \mathrm{d}y, \quad$ where $M = \langle 0, 2 \rangle \times \langle 1, 3 \rangle$;

2. $\displaystyle\iint_M (x + y^2) \, \mathrm{d}x \, \mathrm{d}y, \quad$ where $M = \langle 0, 2 \rangle \times \langle 1, 3 \rangle$;

3. $\iint_M (x-y)\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ y \geq 0 \wedge y \leq x \wedge x+y \leq 2\}$;

4. $\iint_M \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ 0 \leq x \leq 2 \wedge y \leq x \wedge xy \geq 1\}$;

5. $\iint_M \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0 \wedge y \geq x \wedge y \leq \pi\}$;

6. $\iint_M (3xy^2 - 2x)\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ 0 \leq x \leq 1 \wedge x^2 \leq y \leq x\}$;

7. $\iint_M \frac{x}{y}\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ x \geq 1 \wedge x \leq 2 \wedge y \leq x \wedge y \geq 1\}$;

8. $\iint_M \mathrm{e}^{2x+y}\,\mathrm{d}x\,\mathrm{d}y$, where $M = \{(x,y) \in \mathbb{R}^2 :\ x+y \leq 2 \wedge y \geq 0 \wedge y \leq 1 \wedge x \geq 0\}$;

9. $\iint_M x^2 y\,\mathrm{d}x\,\mathrm{d}y$, where M is a triangle with vertices $A = (0,0)$, $B = (3,0)$, $C = (2,1)$;

10. $\iint_M xy^2\,\mathrm{d}x\,\mathrm{d}y$, where M is a triangle with vertices $A = (0,0)$, $B = (3,0)$, $C = (2,1)$;

11. $\iint_M \frac{1}{y^2+1}\,\mathrm{d}x\,\mathrm{d}y$, where M is a triangle with vertices $A = (0,0)$, $B = (1,1)$, $C = (0,1)$.

Results:

1. $\frac{40}{3}$;

2. $\frac{64}{3}$;

3. $\frac{2}{3}$;

4. $\frac{9}{4}$;

5. -2;

6. $-\frac{11}{120}$;

7. $2\ln 2 - \frac{3}{4}$;

8. $\frac{1}{2}\left(e^4 - e^3 - e + 1\right)$;

9. $\frac{33}{20}$;

10. $\frac{9}{20}$;

11. $\frac{1}{2}\ln 2$.

3 Substitution in Double Integral

There are certain situations when we need to "bypass" the shape of integration domain, either because Fubini's theorem is not applicable at all, or because it would just be the hard way to calculate given integral. In some of these situations, substitution method can help us.

Theorem 3.1 (Substitution in Double Integral)**.**

1. *Let mapping* $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ *given by equations*

$$\left.\begin{aligned} x &= g_1(u,v) \\ y &= g_2(u,v) \end{aligned}\right\}, \quad i.e.\, \Phi(u,v) = \big(g_1(u,v), g_2(u,v)\big),$$

be injective over an open set $\Omega_{uv} \subseteq \mathbb{R}^2$*, whereas* $g_1, g_2 \in C^1(\Omega_{uv})$ *(i.e. all first order partial derivatives of* g_1 *and* g_2 *are continuous over* Ω_{uv}*) and let so-called Jacobian*

$$J(u,v) = \begin{vmatrix} \frac{\partial g_1(u,v)}{\partial u} & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2(u,v)}{\partial u} & \frac{\partial g_2(u,v)}{\partial v} \end{vmatrix} \neq 0$$

for every $(u,v) \in \Omega_{uv}$.

2. *Let* $M_{uv} \subset \Omega_{uv}$ *and let* M_{uv} *and* $M_{xy} := \Phi(M_{uv})$ *be closed measurable sets.*

3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be continuous over* M_{xy}.

Then

$$\iint_{M_{xy}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{M_{uv}} f\big(g_1(u,v), g_2(u,v)\big)|J(u,v)|\,\mathrm{d}u\,\mathrm{d}v.$$

Now, let us investigate some special cases of substitution method in double integrals.

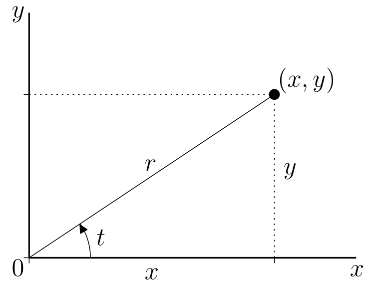
3.1 Polar coordinates

Probably the most utilized type of substitution is given by relations

$$x = r\cos t,$$

$$y = r\sin t,$$

where $r \geq 0$ and $t \in \langle 0, 2\pi\rangle$ (alternatively, $t \in \langle -\pi, \pi\rangle$ or $t \in \langle \alpha, \alpha + 2\pi\rangle$ for $\alpha \in \mathbb{R}$).



The Jacobian can be obtained by direct calculation:

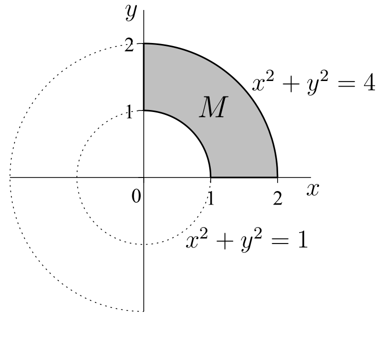
$$J(r,t) = \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r\cos^2 t + r\sin^2 t = r.$$

Remark 3.2. We typically use this kind of substitution in cases, when the boundary of the integration domain M contains parts of circle. Suitability of this method also depends on the integrated function f, of course.

Exercise 3.3. *Calculate the integral* $I = \iint_M x \,\mathrm{d}x\,\mathrm{d}y$, *where*

$$M = \left\{(x,y) \in \mathbb{R}^2 :\ x \geq 0 \wedge y \geq 0 \wedge 1 \leq x^2 + y^2 \leq 4\right\}.$$

Solution:
The set M is quarter of annulus with center in origin and with radii 1 and 2.



Thus, it is desirable to use substitution

$$x = r\cos t, \quad y = r\sin t \qquad (J(r,t) = r)\,.$$

Let us try to describe M using these new coordinates. Considering the geometric meaning of variable t, we easily obtain first constraint $0 \le t \le \frac{\pi}{2}$. Now, let us imagine that t is fixed. How can the distance from origin, i.e. variable r, vary? We can conclude from the picture that $1 \le r \le 2$. Hence,

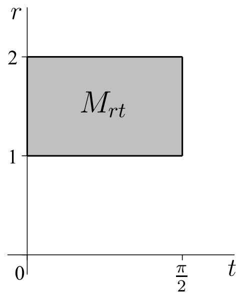
$$M = \left\{(r\cos t, r\sin t) \in \mathbb{R}^2 :\ 0 \le t \le \frac{\pi}{2} \wedge 1 \le r \le 2\right\}. \tag{1}$$

Let us set

$$\Omega_{rt} = (0, +\infty) \times (-\pi, \pi)$$

and

$$M_{rt} = \left\{(r,t) \in \mathbb{R}^2 :\ 0 \le t \le \frac{\pi}{2} \wedge 1 \le r \le 2\right\}.$$



Theorem 3.1 gives us

$$I = \int_0^{\frac{\pi}{2}} \left(\int_1^2 r\cos t \cdot \underbrace{|J(r,t)|}_{r}\, \mathrm{d}r \right) \mathrm{d}t = \int_0^{\frac{\pi}{2}} \left(\int_1^2 r^2 \cos t\, \mathrm{d}r \right) \mathrm{d}t =$$

$$= \left(\int_1^2 r^2\, \mathrm{d}r \right) \cdot \left(\int_0^{\frac{\pi}{2}} \cos t\, \mathrm{d}t \right) = \left[\frac{r^3}{3} \right]_1^2 \cdot [\sin t]_0^{\frac{\pi}{2}} = \underline{\underline{\frac{7}{3}}}\,.$$

In the calculations above, we used the following observation:

$$\int_a^b \int_c^d f(x)g(y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b f(x) \underbrace{\left(\int_c^d g(y)\, \mathrm{d}y \right)}_{\text{constant}} \mathrm{d}x = \left(\int_a^b f(x)\, \mathrm{d}x \right) \left(\int_c^d g(y)\, \mathrm{d}y \right).$$

Remark 3.4. If we would not be able to get the constraints (1) from the picture, we would have to obtain them through calculation – such that we plug transformation relations

$$x = r\cos t, \quad y = r\sin t$$

to inequalities which define set M. We should also take into consideration that $r \geq 0$ and $t \in \langle -\pi, \pi \rangle$.

In our case, we would obtain

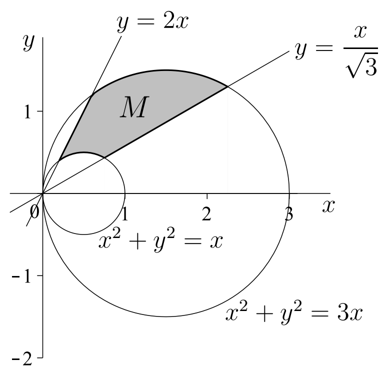
$$r\cos t \geq 0, \quad r\sin t \geq 0 \quad \text{and} \quad 1 \leq \underbrace{r^2\cos^2 t + r^2\sin^2 t}_{r^2} \leq 4.$$

The last condition (and $r \geq 0$) yields $1 \leq r \leq 2$ and the first two then give us $\cos t \geq 0$ and $\sin t \geq 0$, hence (taking in account $t \in \langle -\pi, \pi \rangle$) t lies in the first quadrant, i.e. $0 \leq t \leq \frac{\pi}{2}$.

Exercise 3.5. *Calculate integral* $I = \iint_M \frac{1}{(x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y$*, where*

$$M = \left\{(x,y) \in \mathbb{R}^2 : \frac{x}{\sqrt{3}} \leq y \leq 2x \wedge x \leq x^2 + y^2 \leq 3x\right\} \setminus \{(0,0)\}.$$

Solution:



We will perform transformation into polar coordinates. Since for every $(x,y) \in M$ holds that $x > 0$ and $y > 0$, relations $x = r\cos t$, $y = r\sin t$ (and $r \geq 0$, $t \in \langle 0, 2\pi \rangle$) give us $t \in \left(0, \frac{\pi}{2}\right)$ and $r > 0$. The condition $\frac{x}{\sqrt{3}} \leq y \leq 2x$ yields

$$\frac{r\cos t}{\sqrt{3}} \leq r\sin t \leq 2r\cos t,$$

thus

$$\frac{1}{\sqrt{3}} \le \tan t \le 2,$$

i.e. $t \in \langle \frac{\pi}{6}, \arctan 2 \rangle$. Similarly, the condition $x \le x^2 + y^2 \le 3x$ yields

$$r \cos t \le \underbrace{r^2 \cos^2 t + r^2 \sin^2 t}_{r^2} \le 3r \cos t,$$

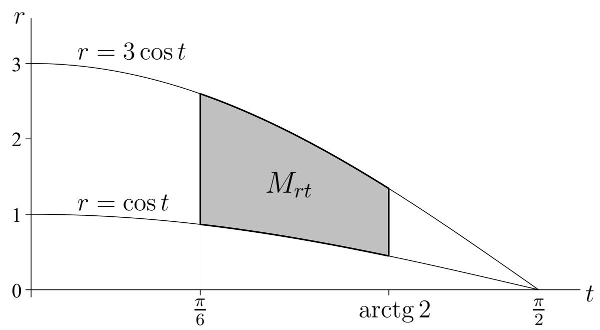
resulting in inequality $\cos t \le r \le 3 \cos t$.

Let us set

$$\Omega_{rt} = (0, +\infty) \times (0, 2\pi)$$

and

$$M_{rt} = \left\{ (r,t) \in \mathbb{R}^2 : \ t \in \langle \tfrac{\pi}{6}, \arctan 2 \rangle \wedge \cos t \le r \le 3 \cos t \right\}.$$



According to Theorem 3.1 and Fubini's theorem 2.16 we have

$$I = \int_{\frac{\pi}{6}}^{\arctan 2} \left(\int_{\cos t}^{3 \cos t} \frac{1}{(r^2 \cos^2 t + r^2 \sin^2 t)^2} \cdot r \, \mathrm{d}r \right) \mathrm{d}t = \int_{\frac{\pi}{6}}^{\arctan 2} \left(\int_{\cos t}^{3 \cos t} \frac{1}{r^3} \, \mathrm{d}r \right) \mathrm{d}t =$$

$$= \int_{\frac{\pi}{6}}^{\arctan 2} \left[-\frac{1}{2r^2} \right]_{r=\cos t}^{3 \cos t} \mathrm{d}t = \int_{\frac{\pi}{6}}^{\arctan 2} -\frac{1}{2} \left(\frac{1}{9 \cos^2 t} - \frac{1}{\cos^2 t} \right) \mathrm{d}t = \int_{\frac{\pi}{6}}^{\arctan 2} \frac{4}{9} \cdot \frac{1}{\cos^2 t} \, \mathrm{d}t =$$

$$= \frac{4}{9} \left[\tan t \right]_{\frac{\pi}{6}}^{\arctan 2} = \frac{4}{9} \left(\tan(\arctan 2) - \tan \frac{\pi}{6} \right) = \frac{4}{9} \left(2 - \frac{\sqrt{3}}{3} \right) = \underline{\underline{\frac{8}{9} - \frac{4\sqrt{3}}{27}}} .$$

Exercise 3.6. *Try to obtain the constraints*

$$t \in \langle \tfrac{\pi}{6}, \arctan 2 \rangle \quad \textit{and} \quad \cos t \le r \le 3 \cos t$$

only by geometrical means.

3.2 Generalized polar coordinates

This time, let us consider substitution

$$x = ar\cos t,$$

$$y = br\sin t,$$

where $a, b > 0$ are constants, $r \geq 0$ and $t \in \langle 0, 2\pi\rangle$ (alternatively $t \in \langle -\pi, \pi\rangle$ or $t \in \langle \alpha, \alpha + 2\pi\rangle$ for $\alpha \in \mathbb{R}$).

Direct calculation yields the Jacobian:

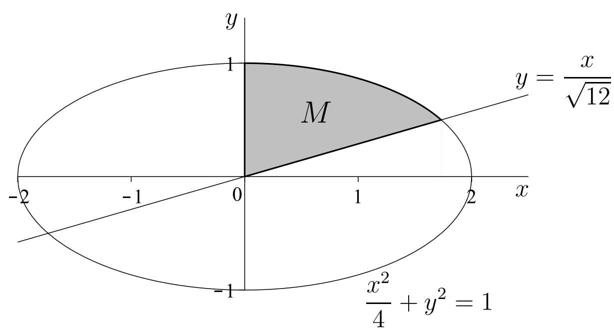
$$J(r,t) = \begin{vmatrix} a\cos t & -ar\sin t \\ b\sin t & br\cos t \end{vmatrix} = ab \cdot r\cos^2 t + ab \cdot r\sin^2 t = ab \cdot r.$$

Remark 3.7. We typically use this kind of substitution in cases, when the boundary of the integration domain M has elliptical shape (a, b are semi-axes of the ellipse in question).

Exercise 3.8. *Calculate integral* $\iint_M (x - 2y)\,\mathrm{d}x\,\mathrm{d}y$, *where*

$$M = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 \leq 1 \wedge 0 \leq x \leq \sqrt{12} \cdot y\right\}.$$

Solution:



We will utilize generalized polar coordinates.

$$x = 2r\cos t, \qquad \big(J(r,t) = 2r\big).$$
$$y = r\sin t,$$

The condition $0 \le x \le \sqrt{12} \cdot y$ then yields $0 \le 2r\cos t \le \sqrt{12} \cdot r \sin t$ and $t \in (0, \frac{\pi}{2}\rangle$. Hence

$$\cot t \le \frac{\sqrt{12}}{2} = \sqrt{3}.$$

Thus we obtain $t \in \langle \frac{\pi}{6}, \frac{\pi}{2} \rangle$. The condition $\frac{x^2}{4} + y^2 \le 1$ yields

$$\underbrace{\frac{4r^2\cos^2 t}{4} + r^2 \sin^2 t}_{r^2} \le 1,$$

thus (considering the condition $r \ge 0$) we have $0 \le r \le 1$. According to Theorem 3.1 it holds that

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_0^1 (2r\cos t - 2r \sin t) \cdot 2r \,\mathrm{d}r \right) \mathrm{d}t = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_0^1 4r^2 (\cos t - \sin t) \,\mathrm{d}r \right) \mathrm{d}t =$$

$$4 \cdot \left(\int_0^1 r^2 \,\mathrm{d}r \right) \cdot \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos t - \sin t) \,\mathrm{d}t \right) = 4 \cdot \left[\frac{r^3}{3} \right]_0^1 \cdot [\sin t + \cos t]_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$= 4 \cdot \frac{1}{3} \left(1 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) \right) = \underline{\underline{\frac{2}{3}\left(1 - \sqrt{3}\right)}}.$$

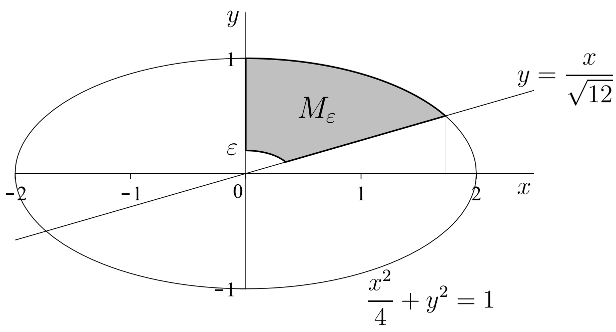
Remark 3.9. Let us note that we used Theorem 3.1 despite not all assumptions were met.

How so? The mapping $\Phi(r,t) = (2r\cos t, r \sin t)$ maps the whole line segment $\{0\} \times \langle \frac{\pi}{6}, \frac{\pi}{2} \rangle$ onto point $(0,0)$ (i.e. mapping Φ is not injective) and furthermore, $J(0,t) = 0$.

This kind of incorrectness (when the assumptions of Theorem 3.1 are not met on certain set with zero measure) is committed most often. Although, in "reasonable cases", we will obtain correct result in spite of this incorrectness.

Remark 3.10 (Glimpse of the correct solution to 3.8)**.** For $\varepsilon \in (0,1)$ let us define set

$$M_\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 : \ \varepsilon^2 \le \frac{x^2}{4} + y^2 \le 1 \ \wedge \ 0 \le x \le \sqrt{12} \cdot y \right\}.$$



It is easy to show that

$$I = \iint_M (x - 2y)\ \mathrm{d}x\,\mathrm{d}y = \lim_{\varepsilon\to 0+} \iint_{M_\varepsilon} (x - 2y)\ \mathrm{d}x\,\mathrm{d}y.$$

But, for every $\varepsilon \in (0,1)$ the integral $I_\varepsilon = \iint_{M_\varepsilon}(x-2y)\ \mathrm{d}x\,\mathrm{d}y$ can be (this time completely correctly) transformed into generalized polar coordinates. Choosing $\Omega_{rt} = (0,+\infty) \times (0,2\pi)$, $M_{rt} = \langle \varepsilon, 1\rangle \times \langle \frac{\pi}{6}, \frac{\pi}{2}\rangle$, we have met all assumptions of Theorem 3.1 and it holds

$$I_\varepsilon = \iint_{M_\varepsilon} (x - 2y)\ \mathrm{d}x\,\mathrm{d}y = \iint_{M_{rt}} (2r\cos t - 2r\sin t)\cdot 2r\ \mathrm{d}r\,\mathrm{d}t =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_\varepsilon^1 (2r\cos t - 2r\sin t)\cdot 2r\,\mathrm{d}r\right)\mathrm{d}t = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_\varepsilon^1 4r^2(\cos t - \sin t)\,\mathrm{d}r\right)\mathrm{d}t =$$

$$= 4\cdot\left(\int_\varepsilon^1 r^2\,\mathrm{d}r\right)\cdot\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\cos t - \sin t)\,\mathrm{d}t\right) = 4\cdot\left[\frac{r^3}{3}\right]_\varepsilon^1\cdot[\sin t + \cos t]_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$= 4\left(\frac{1}{3} - \frac{\varepsilon^3}{3}\right)\cdot\left(1 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right) = \frac{2}{3}\left(1 - \varepsilon^3\right)\left(1 - \sqrt{3}\right).$$

Thus

$$I = \lim_{\varepsilon\to 0+} I_\varepsilon = \lim_{\varepsilon\to 0+} \frac{2}{3}\left(1-\varepsilon^3\right)\left(1-\sqrt{3}\right) = \underline{\underline{\frac{2}{3}\left(1-\sqrt{3}\right)}}\ .$$

Exercise 3.11. *Calculate following integrals using suitable substitution:*

1. $\iint_M \sqrt{1 - x^2 - y^2}\ \mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 \le 1 \wedge x \ge 0 \wedge y \ge 0\}$*;*

2. $\iint_M x\sqrt{x^2 + y^2}\ \mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 \le 4 \wedge y \ge 0\}$*;*

3. $\displaystyle\iint_M y \,\mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 \le 3x \ \wedge\ y \ge 0\}$;

4. $\displaystyle\iint_M \arctan\frac{y}{x} \,\mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ 1 \le x^2 + y^2 \le 4 \ \wedge\ 0 \le y \le x\}$;

5. $\displaystyle\iint_M \mathrm{e}^{x^2+y^2} \,\mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ x \ge 0 \ \wedge\ x^2 + y^2 \le 1\}$;

6. $\displaystyle\iint_M xy \,\mathrm{d}x\,\mathrm{d}y,$ *where* $M = \left\{(x,y) \in \mathbb{R}^2 :\ x \ge 0 \ \wedge\ y \ge 0 \ \wedge\ \tfrac{x^2}{4} + y^2 \le 1\right\}$;

7. $\displaystyle\iint_M (2x + y) \,\mathrm{d}x\,\mathrm{d}y,$ *where* $M = \{(x,y) \in \mathbb{R}^2 :\ x^2 + 4y^2 \le 36 \ \wedge\ x \ge 0\}$.

Results:

1. $\frac{\pi}{6}$;
2. 0;
3. $\frac{9}{4}$;
4. $\frac{3\pi^2}{64}$;
5. $\frac{\pi}{2}(\mathrm{e} - 1)$;
6. $\frac{1}{2}$;
7. 144.

4 Some Applications of Double Integrals

4.1 Area of a Planar Shape

Let $M \subset \mathbb{R}^2$ is measurable set. Then the area (measure) $\lambda(M)$ of the planar shape M can be calculated as

$$\lambda(M) = \iint_M 1 \,\mathrm{d}x\,\mathrm{d}y.$$

Exercise 4.1. *For any $a > 0$ calculate the area of shape*

$$M_a = \left\{(x,y) \in \mathbb{R}^2 : \left(x^2 + y^2\right)^2 \le 2a^2xy\right\}.$$

Solution:
First of all, let us describe M_a in polar coordinates. Thus, let us substitute relations

$$x = r\cos t \qquad \text{a} \qquad y = r\sin t$$

into condition

$$\left(x^2 + y^2\right)^2 \le 2a^2xy.$$

We obtain

$$\left(r^2\cos^2 t + r^2\sin^2 t\right)^2 \le 2a^2 r\cos t \cdot r\sin t,$$

$$\left(r^2\right)^2 \le 2a^2r^2\cos t\sin t,$$

$$r^4 \le a^2r^2\sin 2t,$$

$$r^2\left(r^2 - a^2\sin 2t\right) \le 0. \tag{2}$$

Now, let $t \in \langle -\pi, \pi\rangle$ be fixed. Let us investigate what $r \ge 0$ satisfy the condition (2). If

$$t \in \left\langle -\pi, -\frac{\pi}{2}\right\rangle \cup \left\langle 0, \frac{\pi}{2}\right\rangle \qquad \text{(the first and third quadrant)},$$

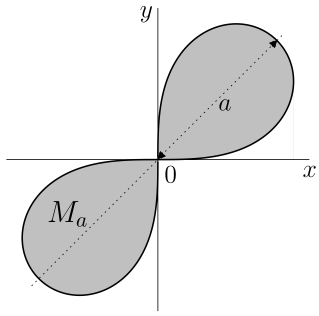
i.e. $\sin 2t \ge 0$, the condition (2) gives us

$$0 \le r \le a\sqrt{\sin 2t}.$$

If

$$t \in \left(-\frac{\pi}{2}, 0\right) \cup \left(\frac{\pi}{2}, \pi\right),$$

i.e. $\sin 2t < 0$, only $r = 0$ satisfies (2).



From Theorems 3.1 and 2.16 it follows that

$$\lambda(M_a) = \iint_{M_a} 1 \,\mathrm{d}x\,\mathrm{d}y \overset{\text{pol. coords.}}{=} \int_{-\pi}^{-\frac{\pi}{2}} \left(\int_0^{a\sqrt{\sin 2t}} r\,\mathrm{d}r \right) \mathrm{d}t + \int_0^{\frac{\pi}{2}} \left(\int_0^{a\sqrt{\sin 2t}} r\,\mathrm{d}r \right) \mathrm{d}t =$$

$$= \int_{-\pi}^{-\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\sin 2t}} \mathrm{d}t + \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\sin 2t}} \mathrm{d}t = \frac{1}{2} \int_{-\pi}^{-\frac{\pi}{2}} a^2 \sin 2t \,\mathrm{d}t + \frac{1}{2} \int_0^{\frac{\pi}{2}} a^2 \sin 2t \,\mathrm{d}t =$$

$$= -\frac{1}{4} a^2 \underbrace{\Big[\cos 2t \Big]_{-\pi}^{-\frac{\pi}{2}}}_{-2} - \frac{1}{4} a^2 \underbrace{\Big[\cos 2t \Big]_0^{\frac{\pi}{2}}}_{-2} = \frac{1}{2} a^2 + \frac{1}{2} a^2 = \underline{\underline{a^2}}.$$

Remark 4.2. Boundary of set M_a from the previous example is a curve called lemniscate.

4.2 Volume of a Cylindrical Body

Let

$$T = \{(x, y, z) \in \mathbb{R}^3 : \ (x, y) \in M \ \wedge \ 0 \le z \le f(x, y)\},$$

where $M \subset \mathbb{R}^2$ is closed measurable set and f is nonnegative continuous function on M. The volume $V(T)$ of the body T is given by formula

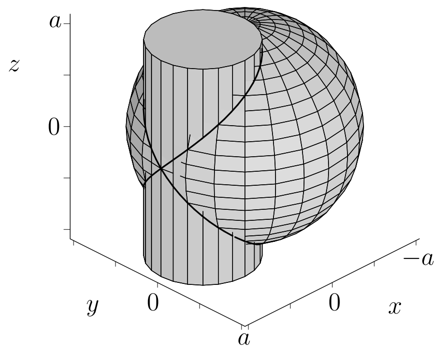
$$V(T) = \iint_M f(x, y) \,\mathrm{d}x\,\mathrm{d}y.$$

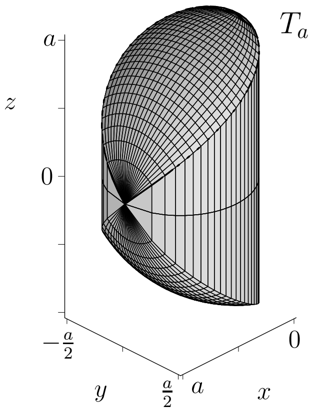
Exercise 4.3. *For any $a > 0$ calculate the volume of so called Viviani's figure*

$$T_a = \left\{ (x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \le a^2 \ \wedge \ x^2 + y^2 \le ax \right\}.$$

Solution:

The body T_a is an intersection of ball with radius a and cylinder with diameter a, while surface of the cylinder passes through the center of the ball.





Considering symmetry of T_a with respect to axis $z = 0$, it is obvious that $V(T_a) = 2V(T_a^*)$, where

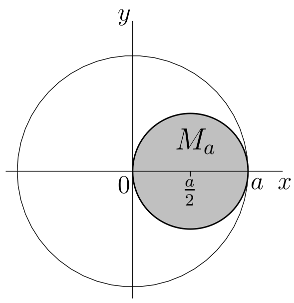
$$T_a^* = \left\{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 \le ax \wedge 0 \le z \le \sqrt{a^2 - x^2 - y^2}\right\}.$$

Furthermore,

$$V(T_a^*) = \iint_{M_a} \sqrt{a^2 - x^2 - y^2} \,\mathrm{d}x \,\mathrm{d}y,$$

where

$$M_a = \left\{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 \le ax\right\}.$$



Transforming into polar coordinates $x = r\cos t$, $y = r\sin t$, the constraint $x^2 + y^2 \le ax$ takes form (having $r \ge 0$ and $t \in \langle -\pi, \pi\rangle$ in mind)

$$r^2\cos^2 t + r^2\sin^2 t \le ar\cos t,$$

$$r^2 \le ar\cos t,$$

$$r(r - a\cos t) \le 0, \tag{3}$$

which yields $(r - a\cos t) \le 0$, i.e.

$$0 \le r \le a\cos t.$$

It follows from the last inequality (since $a\cos t > 0$ and $a > 0$) that

$$t \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle.$$

Thus it holds

$$V(T_a) = 2V(T_a^*) =$$

$$= 2 \iint_{M_a} \sqrt{a^2 - x^2 - y^2}\, \mathrm{d}x\, \mathrm{d}y \overset{\text{pol. coords.}}{=} 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{a\cos t} r\sqrt{a^2 - r^2}\, \mathrm{d}r \right) \mathrm{d}t =$$

$$= \left| \begin{array}{c} \text{substitution} \\ a^2 - r^2 = u \\ -2r\ \mathrm{d}r = \mathrm{d}u \\ r\ \mathrm{d}r = -\frac{1}{2}\,\mathrm{d}u \\ 0 \mapsto a^2,\ a\cos t \mapsto a^2 \sin^2 t \end{array} \right| = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_{a^2}^{a^2 \sin^2 t} \sqrt{u}\, \mathrm{d}u \right) \mathrm{d}t =$$

$$= \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\sqrt{u^3} \right]_{a^2 \sin^2 t}^{a^2} \mathrm{d}t = \frac{2}{3} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{(1 - |\sin t|^3)}_{\text{even function}} \mathrm{d}t = \frac{4}{3} a^3 \int_0^{\frac{\pi}{2}} (1 - \sin^3 t)\, \mathrm{d}t =$$

$$= \frac{2\pi a^3}{3} - \frac{4}{3} a^3 \int_0^{\frac{\pi}{2}} \underbrace{\left(1 - \cos^2 t\right) \sin t}_{\sin^3 t}\ \mathrm{d}t = \left| \begin{array}{c} \text{substitution} \\ \cos t = v \\ -\sin t\ \mathrm{d}t = \mathrm{d}v \\ \sin t\ \mathrm{d}t = -\mathrm{d}v \\ 0 \mapsto 1,\ \frac{\pi}{2} \mapsto 0 \end{array} \right| =$$

$$= \frac{2\pi a^3}{3} + \frac{4}{3} a^3 \int_1^0 \left(1 - v^2\right) \mathrm{d}v = \frac{2\pi a^3}{3} - \frac{4}{3} a^3 \int_0^1 \left(1 - v^2\right) \mathrm{d}v =$$

$$= \frac{2\pi a^3}{3} - \frac{4}{3} a^3 \left[v - \frac{v^3}{3} \right]_0^1 = \frac{2\pi a^3}{3} - \frac{8}{9} a^3 = \underline{\underline{a^3 \left(\frac{2\pi}{3} - \frac{8}{9} \right)}}.$$

4.3 Area of a Surface

Let $\varnothing \neq M \subset \mathbb{R}^2$ closed measurable set and let $f:\ \mathbb{R}^2 \to \mathbb{R}$ be function of class C^1 on an open set $\Omega \subset \mathbb{R}^2$ such that $M \subset \Omega$. This means that f has continuous partial derivatives

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ on Ω. Area of the surface τ, defined as graph of the function $f_{|M}$, i.e.

$$\tau = \{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in M \ \wedge \ z = f(x,y)\},$$

is given by formula

$$P(\tau) = \iint_M \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Exercise 4.4. *For any $a > 0$ calculate area of the surface*

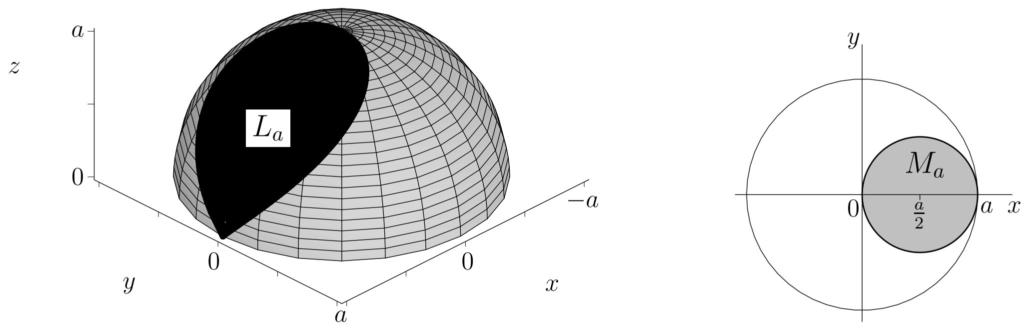
$$L_a = \left\{(x,y,z) \in \mathbb{R}^3 : \ z \geq 0 \ \wedge \ x^2 + y^2 + z^2 = a^2 \ \wedge \ x^2 + y^2 \leq ax\right\}.$$

Solution:
It clearly holds that

$$L_a = \left\{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in M_a \ \wedge \ z = \sqrt{a^2 - x^2 - y^2}\right\},$$

where

$$M_a = \left\{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq ax\right\}.$$



Let us set

$$f(x,y) = \sqrt{a^2 - x^2 - y^2}.$$

Its straightforward that

$$\frac{\partial f}{\partial x}(x,y) = -\frac{x}{\sqrt{a^2 - x^2 - y^2}} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}.$$

Thus

$$P(L_a) = \iint_{M_a} \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} \, \mathrm{d}x \, \mathrm{d}y = \iint_{M_a} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, \mathrm{d}x \, \mathrm{d}y.$$

By applying polar coordinates (see the previous exercise) we obtain

$$P(L_a) = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{a\cos t} \frac{r}{\sqrt{a^2-r^2}}\,\mathrm{d}r\right)\mathrm{d}t = \left|\begin{array}{c} \text{substitution} \\ a^2-r^2=u \\ -2r\;\mathrm{d}r = \mathrm{d}u \\ r\,\mathrm{d}r = -\frac{1}{2}\,\mathrm{d}u \\ 0\mapsto a^2,\ a\cos t\mapsto a^2\sin^2 t \end{array}\right| =$$

$$= \frac{1}{2}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{a^2\sin^2 t}^{a^2}\frac{1}{\sqrt{u}}\,\mathrm{d}u\right)\mathrm{d}t = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sqrt{u}\right]_{a^2\sin^2 t}^{a^2}\mathrm{d}t = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{(a-a|\sin t|)}_{\text{even function}}\,\mathrm{d}t =$$

$$= 2a^2\int_0^{\frac{\pi}{2}}(1-\sin t)\;\mathrm{d}t = 2a^2\left[t+\cos t\right]_0^{\frac{\pi}{2}} = 2a^2\left(\frac{\pi}{2}-1\right) = \underline{\underline{a^2(\pi-2)}}\,.$$

Remark 4.5. The above stated calculation is not correct, according to previous reading. Let us notice that partial derivatives of the function $f(x,y)=\sqrt{a^2-x^2-y^2}$ are not even defined at point $(a,0)\in M_a$.

Exercise 4.6. *Try to think through how could have the calculation above been done correctly.*

4.4 Applications in Mechanics

Let us imagine thin panel as a set $M\subset\mathbb{R}^2$ and assume that its areal density is described with function h which is continuous and nonnegative over M.

Then the following formulae can be derived:

- Mass of panel M:
$$m(M) = \iint_M h(x,y)\;\mathrm{d}x\,\mathrm{d}y.$$
- Static moments of panel M with respect to axes x and y, respectively:
$$S_x(M) = \iint_M y\,h(x,y)\;\mathrm{d}x\,\mathrm{d}y, \qquad \text{resp.} \qquad S_y(M) = \iint_M x\,h(x,y)\;\mathrm{d}x\,\mathrm{d}y.$$
- Centroid of the panel M:
$$T(M) = \left(\frac{S_y(M)}{m(M)}, \frac{S_x(M)}{m(M)}\right).$$

- Moments of inertia of panel M with respect to axes x and y, respectively:

$$I_x(M) = \iint_M y^2\, h(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad \text{resp.} \qquad I_y(M) = \iint_M x^2\, h(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Remark 4.7 (physical interpretation of the number $h(x,y)$)**.** Let $(x,y) \in M$ be given and for every $\delta > 0$ it holds that

$$\lambda\big(U((x,y),\delta) \cap M\big) \neq 0.$$

The number $h(x,y)$ is to be understood as a limit

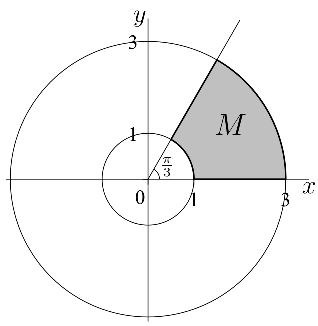
$$h(x,y) = \lim_{\delta \to 0+} \frac{m\big(U((x,y),\delta) \cap M\big)}{\lambda\big(U((x,y),\delta) \cap M\big)}.$$

Remark 4.8. If areal density h is constant over M, we call the panel M homogeneous.

Remark 4.9. Plenty of other physical problems lead to calculations of double integrals, e.g. in the theory of planar fields (gravitational, electromagnetic, heat, ...).

Exercise 4.10. *Thin homogeneous panel M has shape of sector of an annulus with radii 1 and 3 and central angle $\frac{\pi}{3}$. Determine distance d of the centroid of this panel from the center of mentioned annulus.*

Solution:
First of all, let us put the panel M into coordinate system (see the figure).



The panel M is homogeneous, thus there is a number $c > 0$ such that for every $(x,y) \in M$ it holds

$$h(x,y) = c.$$

It is convenient to use polar coordinates $x = r\cos t$, $y = r\sin t$ because M can then be described by relations

$$t \in \left\langle 0, \frac{\pi}{3} \right\rangle \qquad \text{a} \qquad r \in \langle 1, 3 \rangle.$$

For mass of the panel it holds

$$m(M) = \iint_M c \,\mathrm{d}x\,\mathrm{d}y \overset{\text{pol. coords.}}{=} c\int_0^{\frac{\pi}{3}} \left(\int_1^3 r \,\mathrm{d}r\right) \mathrm{d}t = c\cdot\frac{\pi}{3}\left[\frac{r^2}{2}\right]_1^3 = c\cdot\frac{\pi}{3}\cdot\frac{1}{2}\cdot 8 = c\cdot\frac{4\pi}{3}.$$

Now, let us calculate the static moment of M with respect to both axes:

$$S_x(M) = \iint_M cy \,\mathrm{d}x\,\mathrm{d}y \overset{\text{pol. coords.}}{=} c\int_0^{\frac{\pi}{3}} \left(\int_1^3 r^2 \sin t \,\mathrm{d}r\right) \mathrm{d}t = c\underbrace{\left[-\cos t\right]_0^{\frac{\pi}{3}}}_{\frac{1}{2}}\cdot\left[\frac{r^3}{3}\right]_1^3 = c\cdot\frac{13}{3},$$

$$S_y(M) = \iint_M cx \,\mathrm{d}x\,\mathrm{d}y \overset{\text{pol. coords.}}{=} c\int_0^{\frac{\pi}{3}} \left(\int_1^3 r^2 \cos t \,\mathrm{d}r\right) \mathrm{d}t = c\underbrace{\left[\sin t\right]_0^{\frac{\pi}{3}}}_{\frac{\sqrt{3}}{2}}\cdot\left[\frac{r^3}{3}\right]_1^3 = c\cdot\frac{13\sqrt{3}}{3}.$$

The coordinates of centroid of the panel M satisfy

$$T(M) = \left(\frac{S_y(M)}{m(M)}, \frac{S_x(M)}{m(M)}\right) = \left(\frac{13\sqrt{3}}{3}\cdot\frac{3}{4\pi}, \frac{13}{3}\cdot\frac{3}{4\pi}\right) = \left(\frac{13\sqrt{3}}{4\pi}, \frac{13}{4\pi}\right).$$

As we can see, these coordinates are for homogeneous panel independent of its density (which is not really surprising).

The wanted distance is

$$d = \|T(M)\| = \sqrt{\frac{169\cdot 3}{16\pi^2} + \frac{169}{16\pi^2}} = \underline{\underline{\frac{13}{2\pi}}}\,.$$

Exercise 4.11. *Try to come up with better placement of panel M from previous example into coordinate system in order to further simplify the calculation.*

5 Triple Integral over Intervals

Have a look at the definition of double integral over (two-dimensional) interval once again. We will introduce the triple integral over (three-dimensional) interval in an analogous manner.

Definition 5.1. *Let $f:\mathbb{R}^3 \to \mathbb{R}$ be continuous over interval*

$$J = \langle a,b\rangle \times \langle c,d\rangle \times \langle e,g\rangle, \qquad a,b,c,d,e,g \in \mathbb{R}, a<b, c<d, e<g.$$

We define

$$\iiint_J f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z := \lim s_n,$$

where

$$s_n := \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(x_j,y_k,z_l)\cdot\frac{b-a}{n}\cdot\frac{d-c}{n}\cdot\frac{g-e}{n},$$

$$x_j = a + j\cdot\frac{b-a}{n}, \qquad y_k = c + k\cdot\frac{d-c}{n}, \qquad z_l = e + l\cdot\frac{g-e}{n}.$$

Remark 5.2. Similarly as with double integrals, it can be shown, that $\lim s_n$ exists.

Theorem 5.3 (Fubini). *If $f:\mathbb{R}^3 \to \mathbb{R}$ is continuous over $J = \langle a,b\rangle \times \langle c,d\rangle \times \langle e,g\rangle$, then*

$$\iiint_J f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$$

$$= \int_a^b\left(\int_c^d\left(\int_e^g f(x,y,z)\ \mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x = \int_a^b\left(\int_e^g\left(\int_c^d f(x,y,z)\ \mathrm{d}y\right)\mathrm{d}z\right)\mathrm{d}x =$$

$$= \int_c^d\left(\int_a^b\left(\int_e^g f(x,y,z)\ \mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y = \int_e^g\left(\int_a^b\left(\int_c^d f(x,y,z)\ \mathrm{d}y\right)\mathrm{d}x\right)\mathrm{d}z =$$

$$= \int_c^d\left(\int_e^g\left(\int_a^b f(x,y,z)\ \mathrm{d}x\right)\mathrm{d}z\right)\mathrm{d}y = \int_e^g\left(\int_c^d\left(\int_a^b f(x,y,z)\ \mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z.$$

Exercise 5.4. *Compute the following integrals via Fubini theorem:*

1.

$$\iiint_J xy^2z\ \ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \ \textit{where } J = \langle 0,2\rangle \times \langle 1,3\rangle \times \langle 1,2\rangle,$$

2.

$$\iiint_J \mathrm{e}^{3x+2y+z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \ \textit{where } J = \langle 0,1\rangle \times \langle 0,1\rangle \times \langle 0,1\rangle\,,$$

3.

$$\iiint_J \frac{1}{1-x-y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \ \textit{where } J = \langle 0,1\rangle \times \langle 2,5\rangle \times \langle 2,4\rangle\,.$$

6 Triple Integral over Measurable Sets

We will now extend triple integrals in the same way we treated double integral once we wanted to extend its definition for more general sets than intervals. Let us start with definition of three-dimensional measurable sets.

Definition 6.1. *Let $M \subset \mathbb{R}^2$ be a closed two-dimensional measurable set, $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ continuous over M and*

$$\forall (s,t) \in M : \quad h_1(s,t) \le h_2(s,t).$$

The sets

- $$\Omega_1 = \left\{ (x,y,z) \in \mathbb{R}^3 : \quad (x,y) \in M \quad \wedge \quad h_1(x,y) \le z \le h_2(x,y) \right\},$$
- $$\Omega_2 = \left\{ (x,y,z) \in \mathbb{R}^3 : \quad (x,z) \in M \quad \wedge \quad h_1(x,z) \le y \le h_2(x,z) \right\},$$
- $$\Omega_3 = \left\{ (x,y,z) \in \mathbb{R}^3 : \quad (y,z) \in M \quad \wedge \quad h_1(y,z) \le x \le h_2(y,z) \right\}$$

are called ***elementary measurable sets*** *(in $\mathbb{R}^3$).*

Definition 6.2. *We will call $M \subset \mathbb{R}^3$* ***measurable****, if it can be obtained from elementary measurable sets with finite number of the following set operations: set unions, set intersections, set differences.*

Definition 6.3. *Let $M \subset \mathbb{R}^3$. The set*

$$\overline{M} := \left\{ \mathbf{c} \in \mathbb{R}^3 : \text{ there is at least one point from } M \text{ within any } \mathcal{U}(\mathbf{c}, \varepsilon) \right\}$$

is called ***closure*** *of set M.*
Furthermore, we will call $M \subset \mathbb{R}^3$ ***closed****, if $M = \overline{M}$.*

Remark 6.4. Note that our approach towards triple integral is indeed more or less "the same" as for the double integral so far. And it will not be much different with the definition of triple integral itself.

Definition 6.5. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous over $\overline{\Omega}$, where $\Omega \in \mathbb{R}^3$ is a measurable set. Let us denote*

$$F(x,y,z) := \begin{cases} f(x,y,z), & (x,y,z) \in \Omega, \\ 0, & (x,y,z) \in \mathbb{R}^3 \smallsetminus \Omega, \end{cases}$$

and for an arbitrary interval $J = \langle a,b\rangle \times \langle c,d\rangle \times \langle e,g\rangle$ such that $\Omega \subset J$, let us also define sequence

$$s_n := \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} F(x_j, y_k, z_l) \cdot \frac{b-a}{n} \cdot \frac{d-c}{n} \cdot \frac{g-e}{n},$$

$$x_j = a + j \cdot \frac{b-a}{n}, \qquad y_k = c + k \cdot \frac{d-c}{n}, \qquad z_l = e + l \cdot \frac{g-e}{n}.$$

Then we define ***triple integral of function*** f ***over set*** Ω *as*

$$\iiint_\Omega f(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \lim s_n.$$

Definition 6.6. *Let $\Omega \subset \mathbb{R}^3$ be a measurable set. We define* ***measure*** *of set Ω as number*

$$\lambda(\Omega) := \iiint_\Omega 1\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

Remark 6.7. Again, similarly as the measure of two-dimensional measurable set was generalization of the term area, the measure of three-dimensional measurable set is generalization of the term volume.

The measure in $\mathbb{R}^3$ and triple integrals have analogous properties as the ones we saw earlier for measure in $\mathbb{R}^2$ and double integrals (see Theorems 2.14 and 2.15, respectively).

Theorem 6.8 (Fubini)**.** *Let $M \subset \mathbb{R}^2$ be a measurable set and:*

1. *Let*
$$\Omega_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ (x,y) \in M \ \wedge \ h_1(x,y) \le z \le h_2(x,y)\right\},$$
where $M \subset \mathbb{R}^2$ is measurable, $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous over M and $h_1(x,y) \le h_2(x,y)$ for every $(x,y) \in M$. Also, let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous over $\overline{\Omega}_1$. Then
$$\iiint_{\Omega_1} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_M \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\ \mathrm{d}z\right) \mathrm{d}x\,\mathrm{d}y.$$
(Similar statements hold for the other two types of elementary measurable sets in $\mathbb{R}^3$)

2. *Let $e, g \in \mathbb{R}, e < g$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous over set $\overline{\Omega}$, where*
$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \ e \le z \le g \ \wedge \ (x,y) \in M_z\right\},$$
$M_z \subset \mathbb{R}^2$ is measurable for every $z \in \langle e, g\rangle$. Moreover, let function
$$z \mapsto \iint_{M_z} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y$$
be continuous over $\langle e, g\rangle$. Then
$$\iiint_{\Omega} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_e^g \left(\iint_{M_z} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\right) \mathrm{d}z.$$

Like the two-dimensional version of Fubini theorem allowed us to transform double integral into two nested simple integrals, the three-dimensional version of Fubini theorem allows us to transform triple integral into two nested integrals, where one is simple and the other one is double. The latter can then be solved via the two-dimensional version of Fubini theorem. However, note that this means that every triple integral, upon meeting necessary assumptions, can be transformed into three nested simple integrals.

For instance, if we have

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x \in \langle a, b\rangle \wedge g_1(x) \le y \le g_2(x) \wedge h_1(x,y) \le z \le h_2(x,y)\right\}$$

and the proper assumptions are met, we obtain

$$\iiint_\Omega f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x.$$

Exercise 6.9. *Calculate the integral*

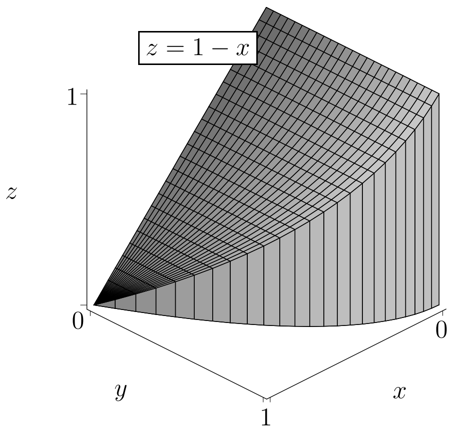
$$\iiint_\Omega 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

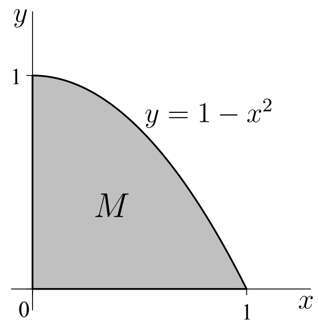
where

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : \quad x \ge 0 \wedge y \ge 0 \wedge z \ge 0 \wedge y \le 1 - x^2 \wedge z \le 1 - x \right\}.$$

Solution:

The following figure depicts the set Ω and its xy-plane projection M:





One can clearly write

$$\Omega = \left\{ (x,y,z) \in \mathbb{R}^3 : \quad x \in \langle 0,1 \rangle \wedge 0 \le y \le 1 - x^2 \wedge 0 \le z \le 1 - x \right\},$$

hence (according to the first part of Theorem 6.8 and the two-dimensional version from Theorem 2.16) it holds that

$$I = \iint_M \left(\int_0^{1-x} 1\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y = \int_0^1 \left(\int_0^{1-x^2} \left(\int_0^{1-x} 1\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x =$$

$$= \int_0^1 \left(\int_0^{1-x^2} [z]_{z=0}^{1-x}\,\mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left(\int_0^{1-x^2} (1-x)\,\mathrm{d}y \right) \mathrm{d}x = \int_0^1 [y(1-x)]_{y=0}^{1-x^2}\,\mathrm{d}x =$$

$$= \int_0^1 (1-x^2)(1-x)\,\mathrm{d}x = \int_0^1 (1 - x - x^2 + x^3)\,\mathrm{d}x =$$

$$= \left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \right]_0^1 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \underline{\underline{\frac{5}{12}}}\,.$$

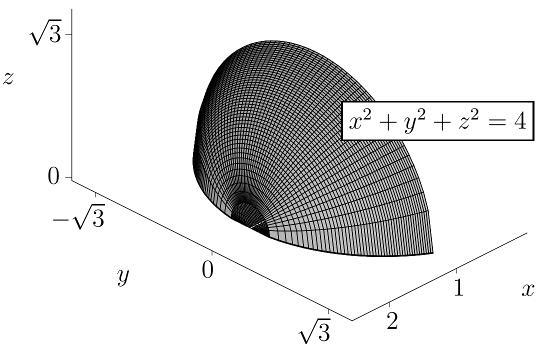
Exercise 6.10. *Calculate the integral*

$$I = \iiint_\Omega \frac{z}{(x^2+y^2+z^2)^{\frac{5}{2}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x \geq 1 \wedge z \geq 0 \wedge x^2+y^2+z^2 \leq 4\right\}.$$

Solution:
The set Ω is half of spherical cap (see Figure)

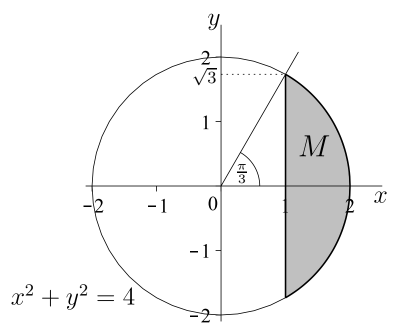


According to the first part of Theorem 6.8 we have

$$I = \iint_M \left(\int_0^{\sqrt{4-(x^2+y^2)}} \frac{z}{(x^2+y^2+z^2)^{\frac{5}{2}}}\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y,$$

where

$$M = \left\{(x,y) \in \mathbb{R}^2 : \quad x \geq 1 \wedge x^2+y^2 \leq 4\right\}.$$



If we transform set M into polar coordinates, we obtain

$$M = \left\{(r\cos t, r\sin t) \in \mathbb{R}^2 : \quad -\frac{\pi}{3} \leq t \leq \frac{\pi}{3} \wedge \frac{1}{\cos t} \leq r \leq 2\right\}.$$

Thus, since the Jacobian of applied transform is equal to r, we now have (using Theorems 6.8 and 2.16)

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_{\frac{1}{\cos t}}^{2} \left(r \cdot \int_0^{\sqrt{4-r^2}} \frac{z}{(r^2+z^2)^{\frac{5}{2}}} \, \mathrm{d}z \right) \mathrm{d}r \right) \mathrm{d}t = \left| \begin{array}{c} \text{substitution} \\ r^2 + z^2 = u \\ z \, \mathrm{d}z = \frac{1}{2} \, \mathrm{d}u \\ 0 \mapsto r^2, \ \sqrt{4-r^2} \mapsto 4 \end{array} \right| =$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_{\frac{1}{\cos t}}^{2} \left(\frac{r}{2} \cdot \int_{r^2}^{4} \frac{1}{u^{\frac{5}{2}}} \, \mathrm{d}u \right) \mathrm{d}r \right) \mathrm{d}t = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_{\frac{1}{\cos t}}^{2} -\frac{r}{3} \cdot \left[\frac{1}{\sqrt{u^3}} \right]_{u=r^2}^{4} \mathrm{d}r \right) \mathrm{d}t =$$

$$= -\frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\int_{\frac{1}{\cos t}}^{2} \underbrace{r \left(\frac{1}{8} - \frac{1}{r^3} \right)}_{\left(\frac{r}{8} - \frac{1}{r^2}\right)} \mathrm{d}r \right) \mathrm{d}t = -\frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{16} + \frac{1}{r} \right]_{r=\frac{1}{\cos t}}^{2} \mathrm{d}t =$$

$$= -\frac{1}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \underbrace{\left(\frac{1}{4} + \frac{1}{2} - \frac{1}{16\cos^2 t} - \cos t \right)}_{\text{even function}} \mathrm{d}t = -\frac{2}{3} \int_0^{\frac{\pi}{3}} \left(\frac{3}{4} - \frac{1}{16\cos^2 t} - \cos t \right) \mathrm{d}t =$$

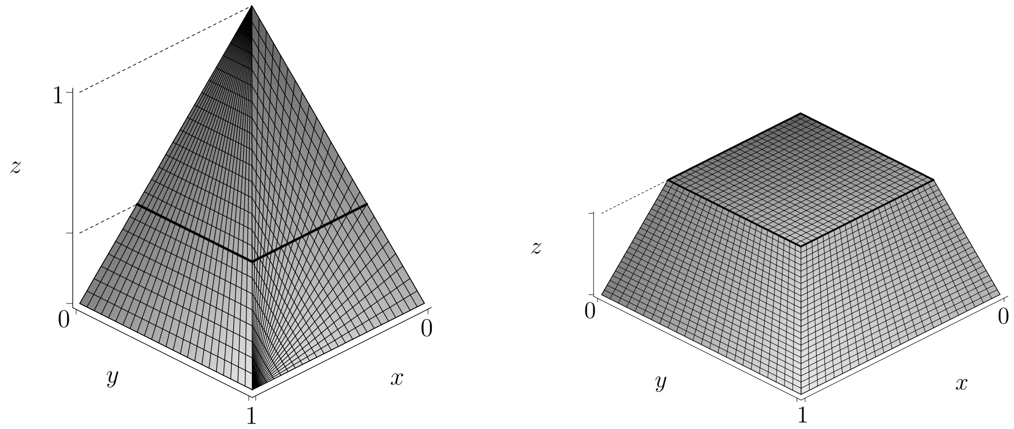
$$= -\frac{2}{3} \left[\frac{3}{4} t - \frac{1}{16} \tan t - \sin t \right]_0^{\frac{\pi}{3}} = -\frac{2}{3} \left(\frac{\pi}{4} - \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{2} \right) = \underline{\underline{-\frac{\pi}{6} + \frac{3}{8}\sqrt{3}}} \, .$$

Exercise 6.11. *Calculate the integral*

$$I = \iiint_\Omega z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

where Ω *is a square pyramid with vertices* $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$ *a* $(0,0,1)$.

Solution:
The Figure below depicts our pyramid and its cut through plane parallel to plane $z = 0$.

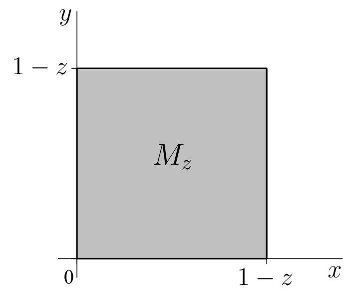


It is not hard to realize that this kind of cut will always be a square. For solving the exercise, we will use the second part of Theorem 6.8, thus obtaining

$$I = \int_0^1 \left(\iint_{M_z} z \, \mathrm{d}x \, \mathrm{d}y \right) \mathrm{d}z,$$

where

$$M_z = \left\{ (x, y) \in \mathbb{R}^2 : \quad 0 \le x \le 1 - z \wedge 0 \le y \le 1 - z \right\}.$$



Hence, the following calculations hold:

$$I = \int_0^1 \left(\int_0^{1-z} \left(\int_0^{1-z} z \, \mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z = \int_0^1 \left(\int_0^{1-z} [zy]_{y=0}^{1-z} \, \mathrm{d}x \right) \mathrm{d}z =$$

$$= \int_0^1 \left(\int_0^{1-z} (z - z^2) \, \mathrm{d}x \right) \mathrm{d}z = \int_0^1 \left[(z - z^2) x \right]_{x=0}^{1-z} \mathrm{d}z = \int_0^1 (z - z^2)(1 - z) \, \mathrm{d}z =$$

$$= \int_0^1 (z - 2z^2 + z^3) \, \mathrm{d}z = \left[\frac{z^2}{2} - \frac{2z^3}{3} + \frac{z^4}{4} \right]_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \underline{\underline{\frac{1}{12}}}.$$

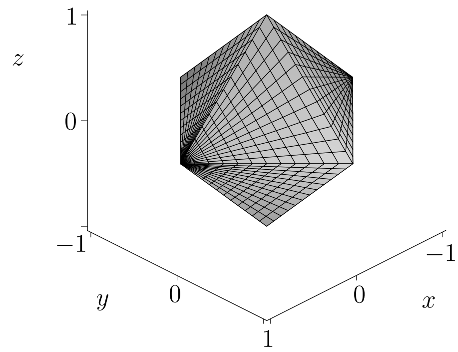
Exercise 6.12. *Calculate the integral*

$$I = \iiint_\Omega x^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

where

$$\Omega = \Big\{(x,y,z) \in \mathbb{R}^3 : \quad |x| + |y| + |z| \le 1\Big\}.$$

Solution:
The set Ω is (think about it) a regular octahedron.

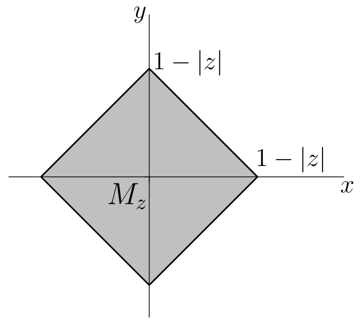


According to the second part of Theorem 6.8 we have

$$I = \int_{-1}^{1} \left(\iint_{M_z} x^2 \,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}z,$$

where

$$M_z = \Big\{(x,y) \in \mathbb{R}^2 : \quad |x| + |y| \le 1 - |z|\Big\} \qquad (z \in \langle -1, 1 \rangle).$$



One can easily realize that

$$M_z = \{(x,y) \in \mathbb{R}^2 : \quad -(1-|z|) \le x \le 1-|z| \wedge -(1-|z|-|x|) \le y \le 1-|z|-|x|\},$$

henceforth obtaining

$$I = \int_{-1}^{1} \left(\int_{-(1-|z|)}^{1-|z|} \left(\int_{-(1-|z|-|x|)}^{1-|z|-|x|} x^2 \,\mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z = \int_{-1}^{1} \left(\int_{-(1-|z|)}^{1-|z|} 2x^2(1-|z|-|x|)\,\mathrm{d}x \right) \mathrm{d}z =$$

$$= 2\int_{-1}^{1}\left(\int_0^{1-|z|} 2x^2(1-|z|-|x|)\,\mathrm{d}x\right)\mathrm{d}z =$$

$$= 2\cdot 2\int_{0}^{1}\left(\int_0^{1-|z|} 2x^2(1-|z|-|x|)\,\mathrm{d}x\right)\mathrm{d}z = 8\int_0^1\left(\int_0^{1-z} x^2(1-z-x)\,\mathrm{d}x\right)\mathrm{d}z =$$

$$= 8\int_0^1\left(\int_0^{1-z}\left(x^2(1-z)-x^3\right)\mathrm{d}x\right)\mathrm{d}z = 8\int_0^1\left[\frac{1}{3}x^3(1-z)-\frac{1}{4}x^4\right]_{x=0}^{1-z}\mathrm{d}z =$$

$$= \frac{2}{3}\int_0^1(1-z)^4\,\mathrm{d}z = \left|\begin{array}{c}\text{substitution}\\ 1-z=u\\ -\mathrm{d}z=\mathrm{d}u\\ 0\mapsto 1,\ 1\mapsto 0\end{array}\right| = -\frac{2}{3}\int_1^0 u^4\,\mathrm{d}u = \frac{2}{3}\int_0^1 u^4\,\mathrm{d}u =$$

$$= \frac{2}{3}\left[\frac{u^5}{5}\right]_0^1 = \underline{\underline{\frac{2}{15}}}\,.$$

Remark 6.13. For solving some of the previous exercises, we used the fact that every even function f integrable over $\langle 0,a\rangle$ satisfies

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x.$$

Exercise 6.14. *Calculate following integrals using Fubini's theorem:*

1. $\iiint_M xy^2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad$ *where* $M = \langle 0,2\rangle\times\langle 1,3\rangle\times\langle 1,2\rangle$;

2. $\iiint_M \mathrm{e}^{3x+2y+z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad$ *where* $M = \langle 0,1\rangle\times\langle 0,1\rangle\times\langle 0,1\rangle$;

3. $\iiint_M \frac{1}{1-x-y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad$ *where* $M = \langle 0,1\rangle\times\langle 2,5\rangle\times\langle 2,4\rangle$;

4. $\iiint_M xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad$ *where*
   $M = \{(x,y,z)\in\mathbb{R}^3:\ x\ge 0\wedge y\ge 0\wedge x+y\le 1\wedge 0\le z\le x^2+y^2+1\}$;

5. $\iiint_M \frac{1}{x+y+1}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,\quad$ *where*
   $M = \{(x,y,z)\in\mathbb{R}^3:\ x\ge 0\wedge y\ge 0\wedge z\ge 0\wedge x+y+z\le 1\}$;

6. $\iiint_M x^2yz^3 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$, *where*
$M = \{(x, y, z) \in \mathbb{R}^3 : \ z \leq xy \ \wedge \ y \geq x \geq 0 \ \wedge \ y \leq 1 \ \wedge \ z \geq 0\}$.

Results:

1. 26;
2. $\frac{1}{6}(\mathrm{e} - 1)(\mathrm{e}^2 - 1)(\mathrm{e}^3 - 1)$;
3. $20 \ln 2 - 10 \ln 5$;
4. $\frac{7}{120}$;
5. $\frac{3}{2} - 2 \ln 2$;
6. $\frac{1}{364}$.

7 Substitution in Triple Integral

Same as with some double integrals, some triple integrals might be tricky to calculate only with Fubini's theorem. Let us have a look on substitution method for triple integrals:

Theorem 7.1 (Substitution in triple integral)**.**

1. *Let the mapping* $\Phi: \mathbb{R}^3 \to \mathbb{R}^3$ *given by equations*

$$x = g_1(u,v,w),$$

$$y = g_2(u,v,w),$$

$$z = g_3(u,v,w)$$

   be injective over open set $\Omega_{uvw} \subset \mathbb{R}^3$*, furthermore, let functions* g_1, g_2, g_3 *belong to class* C^1 *over* Ω_{uvw} *and let the so called Jacobian be*

$$J(u,v,w) = \begin{vmatrix} \frac{\partial g_1}{\partial u}(u,v,w) & \frac{\partial g_1}{\partial v}(u,v,w) & \frac{\partial g_1}{\partial w}(u,v,w) \\ \frac{\partial g_2}{\partial u}(u,v,w) & \frac{\partial g_2}{\partial v}(u,v,w) & \frac{\partial g_2}{\partial w}(u,v,w) \\ \frac{\partial g_3}{\partial u}(u,v,w) & \frac{\partial g_3}{\partial v}(u,v,w) & \frac{\partial g_3}{\partial w}(u,v,w) \end{vmatrix} \neq 0$$

   for every $(u,v,w) \in \Omega_{uvw}$*.*

2. *Let* $M_{uvw} \subset \Omega_{uvw}$ *and let* M_{uvw} *and* $M_{xyz} = \Phi(M_{uvw})$ *be closed measurable sets.*

3. *Let the function* f *be continuous over* M_{xyz}*.*

Then

$$\iiint_{M_{xyz}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$$

$$= \iiint_{M_{uvw}} f\big(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)\big) \cdot |J(u,v,w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Remark 7.2. As analogous to double integrals, we will not commit a mistake by using the theorem above, despite some of its assumptions (non-zero Jacobian, simplicity of Φ, etc.) are not met in some set of zero measure (in $\mathbb{R}^3$).

7.1 Cylindrical coordinates

Now, we will speak about specific substitution

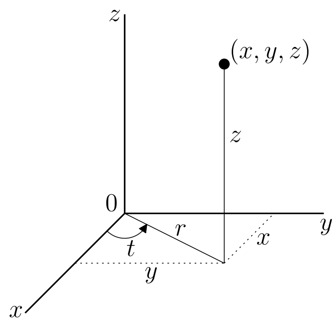
$$x = r\cos t,$$

$$y = r\sin t,$$

$$z = z^*(= z),$$

where

$$r \ge 0, \quad t \in \langle 0, 2\pi\rangle \quad (\text{or } t \in \langle -\pi, \pi\rangle \text{ or } t \in \langle \alpha, \alpha + 2\pi\rangle \text{ for } \alpha \in \mathbb{R}) \text{ and } \quad z^* \in \mathbb{R}.$$



The Jacobian of this mapping is equal to

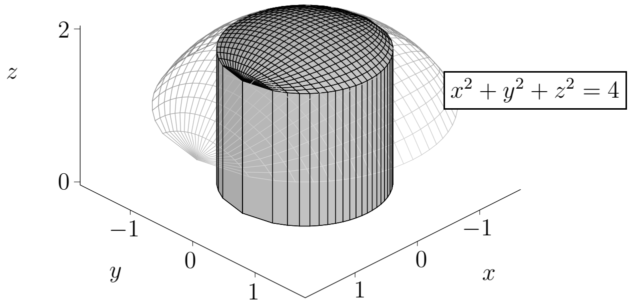
$$J(r,t,z^*) = \begin{vmatrix} \cos t & -r\sin t & 0 \\ \sin t & r\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2 t + \sin^2 t\right) = r.$$

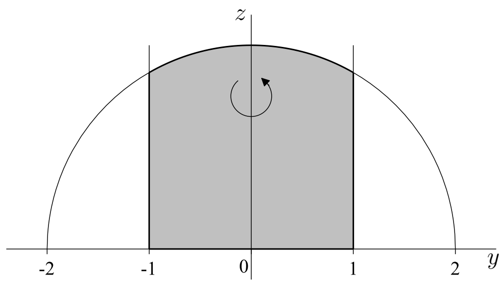
Remark 7.3. We typically use this kind of substitution in cases, when the ground projection of the integration domain Ω contains parts of circle. Suitability of this method also depends on the integrated function f, of course, which is necessary thing to be considered in any case, regardless the specific substitution method.

Exercise 7.4. *Calculate the integral* $I = \iiint_\Omega (x^4 + y^4)\, z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$*, where*

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : \quad x^2 + y^2 \le 1 \wedge z \ge 0 \wedge x^2 + y^2 + z^2 \le 4\right\}.$$

Solution:
The set Ω is cylinder "trimmed" from above by sphere.





By applying cylindrical coordinates

$$x = r\cos t,$$

$$y = r\sin t,$$

$$z = z,$$

we obtain constraint

$$0 \le r \le 1, \quad 0 \le t \le 2\pi, \quad 0 \le z \le \sqrt{4 - r^2}.$$

thus, according to Theorem 7.1 it holds that

$$I = \iiint_{M_{rtz}} \left(r^4 \cos^4 t + r^4 \sin^4 t\right) z \cdot r \,\mathrm{d}r \,\mathrm{d}t \,\mathrm{d}z,$$

where

$$M_{rtz} = \left\{(r, t, z) \in \mathbb{R}^3 : \quad 0 \le r \le 1 \,\wedge\, 0 \le t \le 2\pi \,\wedge\, 0 \le z \le \sqrt{4 - r^2}\right\}.$$

Now, using Theorems (6.8 and 2.16), we obtain

$$I = \int_0^1 \left(\int_0^{2\pi} \left(\int_0^{\sqrt{4-r^2}} \left(r^4 \cos^4 t + r^4 \sin^4 t \right) z \cdot r \, \mathrm{d}z \right) \mathrm{d}t \right) \mathrm{d}r =$$

$$= \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t \cdot \int_0^1 \left(r^5 \left(\int_0^{\sqrt{4-r^2}} z \, \mathrm{d}z \right) \right) = \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t \cdot$$

$$\cdot \int_0^1 \left(r^5 \left[\frac{z^2}{2} \right]_{z=0}^{z=\sqrt{4-r^2}} \right) = \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t \cdot \frac{1}{2} \int_0^1 \left(4r^5 - r^7 \right) =$$

$$= \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t \cdot \frac{1}{2} \left[\frac{2r^6}{3} - \frac{r^8}{8} \right]_0^1 = \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t \cdot \frac{1}{2} \cdot \frac{13}{24} =$$

$$= \frac{13}{48} \int_0^{2\pi} \left(\cos^4 t + \sin^4 t \right) \mathrm{d}t = \frac{13}{48} \int_0^{2\pi} \left(\left(\cos^2 t + \sin^2 t \right)^2 - 2 \sin^2 t \cos^2 t \right) \mathrm{d}t =$$

$$= \frac{13}{48} \int_0^{2\pi} \left(1 - \frac{1}{2} \sin^2 2t \right) \mathrm{d}t = \frac{13}{48} \int_0^{2\pi} \left(1 - \frac{1 - \cos 4t}{4} \right) \mathrm{d}t =$$

$$= \frac{13}{48} \int_0^{2\pi} \left(\frac{3}{4} + \frac{\cos 4t}{4} \right) \mathrm{d}t = \frac{13}{48} \left[\frac{3}{4} t + \frac{\sin 4t}{16} \right]_0^{2\pi} = \frac{13}{48} \cdot \frac{3}{4} \cdot 2\pi = \underline{\underline{\frac{13}{32}\pi}} \, .$$

Remark 7.5. Try to think over the fact, that using substitution to cylindrical coordinates and Fubini theorem gives the same result as using Fubini theorem and substitution to polar coordinates afterwards.

7.2 Spherical coordinates

Let us consider the following mapping:

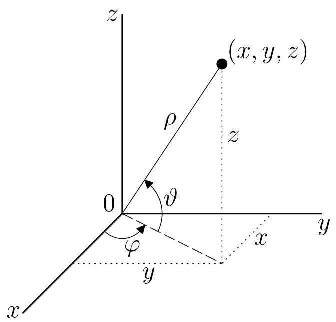
$$x = \rho \cos \varphi \cos \vartheta,$$

$$y = \rho \sin \varphi \cos \vartheta,$$

$$z = \rho \sin \vartheta,$$

where

$$\rho \geq 0, \quad \varphi \in \langle 0, 2\pi \rangle \quad (\text{or } \varphi \in \langle -\pi, \pi \rangle \text{ or } \varphi \in \langle \alpha, \alpha + 2\pi \rangle \text{ for } \alpha \in \mathbb{R}), \ \vartheta \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2} \rangle.$$



Now, let us determine the Jacobian of this mapping by direct calculation (e.g. by last row expansion). We obtain

$$J(\rho,\varphi,\vartheta) = \begin{vmatrix} \cos\varphi\cos\vartheta & -\rho\sin\varphi\cos\vartheta & -\rho\cos\varphi\sin\vartheta \\ \sin\varphi\cos\vartheta & \rho\cos\varphi\cos\vartheta & -\rho\sin\varphi\sin\vartheta \\ \sin\vartheta & 0 & \rho\cos\vartheta \end{vmatrix} =$$

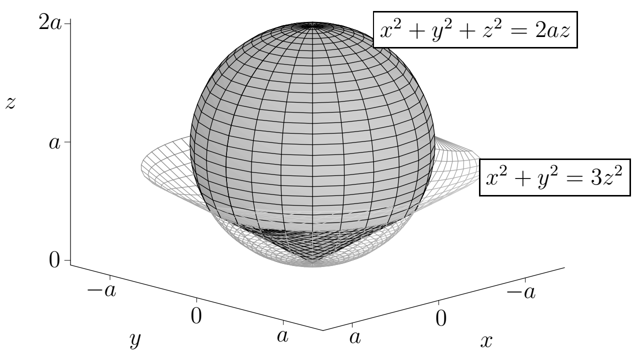
$$= \sin\vartheta\left(\rho^2\sin\vartheta\cos\vartheta\right) + \rho\cos\vartheta\left(\rho\cos^2\vartheta\right) = \rho^2\cos\vartheta\left(\sin^2\vartheta+\cos^2\vartheta\right) = \rho^2\cos\vartheta.$$

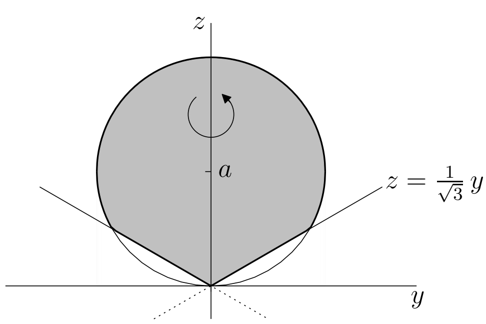
Remark 7.6. We typically use this kind of substitution in cases, where the boundary of the integration domain Ω contains parts of spheres.

Exercise 7.7. *For $a>0$ calculate the integral $I_a = \iiint_{\Omega_a}\left(x^2+y^2+z^2\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where*

$$\Omega_a = \left\{(x,y,z)\in\mathbb{R}^3 : \quad x^2+y^2+z^2 \le 2az \wedge x^2+y^2 \le 3z^2\right\}.$$

Solution:
The condition $x^2+y^2+z^2 \le 2az$ can be equivalently rewritten as $x^2+y^2+(z-a)^2 \le a^2$. Hence, the set Ω_a is intersection of a ball (with center at $(0,0,a)$ and radius a) and a cone (see Figure below).





Let us apply spherical coordinates

$$x = \rho \cos\varphi \cos\vartheta,$$

$$y = \rho \sin\varphi \cos\vartheta,$$

$$z = \rho \sin\vartheta.$$

Once again, we should realize that

$$\rho \geq 0, \quad \varphi \in \langle 0, 2\pi \rangle \quad \text{and} \quad \vartheta \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2} \rangle.$$

Applying the transformation relations to conditions

$$x^2 + y^2 + z^2 \leq 2az \quad \text{and} \quad x^2 + y^2 \leq 3z^2$$

we obtain

$$\rho^2 \leq 2a\rho \sin\vartheta \wedge \rho^2 \cos^2\vartheta \leq 3\rho^2 \sin^2\vartheta,$$

resulting in ($\rho \geq 0$)

$$\rho \leq 2a \sin\vartheta \wedge \cos^2\vartheta \leq 3 \sin^2\vartheta.$$

The first inequality gives us $\vartheta \in \langle 0, \frac{\pi}{2} \rangle$, while the second yields $\cos\vartheta \leq \sqrt{3} \sin\vartheta$. All in all, we have

$$\varphi \in \langle 0, 2\pi \rangle, \quad \vartheta \in \langle \tfrac{\pi}{6}, \tfrac{\pi}{2} \rangle, \quad \rho \in \langle 0, 2a \sin\vartheta \rangle.$$

From Theorems 7.1, 6.8 and 2.16 it follows that

$$I_a = \int_0^{2\pi} \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\int_0^{2a\sin\vartheta} \rho^2 \cdot \rho^2 \cos\vartheta \, \mathrm{d}\rho \right) \mathrm{d}\vartheta \right) \mathrm{d}\varphi = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\vartheta \left[\frac{\rho^5}{5} \right]_{\rho=0}^{2a\sin\vartheta} \mathrm{d}\vartheta =$$

$$= \frac{2\pi}{5} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\vartheta \cdot 2^5 \cdot a^5 \cdot \sin^5\vartheta \, \mathrm{d}\vartheta = \left| \begin{array}{c} \text{substitution} \\ \sin\vartheta = u \\ \cos\vartheta \, \mathrm{d}\vartheta = \mathrm{d}u \\ \frac{\pi}{6} \mapsto \frac{1}{2}, \ \frac{\pi}{2} \mapsto 1 \end{array} \right| = \frac{2\pi}{5} \cdot 32a^5 \int_{\frac{1}{2}}^{1} u^5 \, \mathrm{d}u =$$

$$= \frac{64}{5}\pi a^5 \left[\frac{u^6}{6} \right]_{\frac{1}{2}}^{1} = \frac{64}{5}\pi a^5 \left(\frac{1}{6} - \frac{1}{6} \cdot \frac{1}{64} \right) = \underline{\underline{\frac{21}{10}\pi a^5}}\ .$$

7.3 Generalized spherical coordinates

Let $a, b, c > 0$ be given numbers. Let us consider the mapping

$$x = a\rho \cos\varphi \cos\vartheta,$$

$$y = b\rho \sin\varphi \cos\vartheta,$$

$$z = c\rho \sin\vartheta,$$

where

$$\rho \ge 0, \quad \varphi \in \langle 0, 2\pi\rangle \quad (\text{or} \varphi \in \langle -\pi, \pi\rangle \text{ or } \varphi \in \langle \alpha, \alpha + 2\pi\rangle \text{ for } \alpha \in \mathbb{R}), \ \vartheta \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2}\rangle.$$

We can obtain the Jacobian of this mapping in analogous manner to "classic" spherical coordinates and we obtain

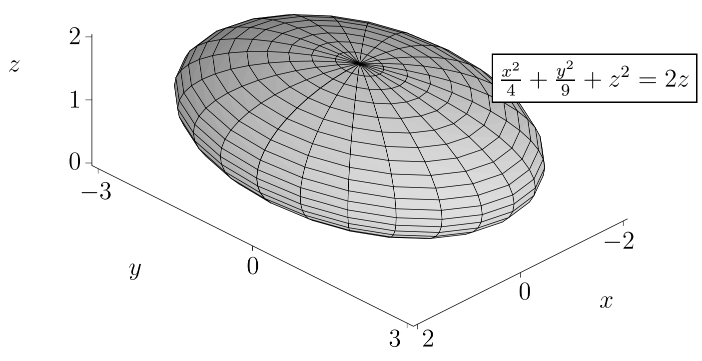
$$J(\rho, \varphi, \vartheta) = abc \cdot \rho^2 \cos\vartheta.$$

Remark 7.8. We typically use this kind of substitution if the integration domain Ω has shape of an ellipsoid.

Exercise 7.9. *Calculate the integral* $I = \iiint_\Omega z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where*

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 : \quad \frac{x^2}{4} + \frac{y^2}{9} + z^2 \le 2z\right\}.$$

Solution:
The condition $\frac{x^2}{4} + \frac{y^2}{9} + z^2 \le 2z$ can be equivalently rewritten as $\frac{x^2}{4} + \frac{y^2}{9} + (z-1)^2 \le 1$. Hence it follows that Ω is an ellipsoid with center at $(0, 0, 1)$ with semi-axes 2, 3 and 1.



Let us use the generalized spherical coordinates:

$$x = 2\rho \cos\varphi \cos\vartheta,$$

$$y = 3\rho \sin\varphi \cos\vartheta,$$

$$z = \rho \sin\vartheta.$$

The Jacobian of this mapping is clearly $J = 6\rho^2 \cos\vartheta$. Applying substitution relations into condition $\frac{x^2}{4} + \frac{y^2}{9} + z^2 \le 2z$ yields $\rho^2 \le 2\rho \sin\vartheta$. One can easily realize that ellipsoid Ω is given by constraints

$$\varphi \in \langle 0, 2\pi \rangle, \quad \vartheta \in \langle 0, \tfrac{\pi}{2} \rangle, \quad \rho \in \langle 0, 2\sin\vartheta \rangle.$$

Thus

$$I = \iiint_M \rho \sin\vartheta \cdot |J| \, d\rho \, d\varphi \, d\vartheta = \iiint_M 6\rho^3 \sin\vartheta \cos\vartheta \, d\rho \, d\varphi \, d\vartheta,$$

where

$$M = \left\{ (\rho, \varphi, \vartheta) \in \mathbb{R}^3 : \quad 0 \le \varphi \le 2\pi \wedge 0 \le \vartheta \le \frac{\pi}{2} \wedge 0 \le \rho \le 2\sin\vartheta \right\}.$$

Thus, according to Theorems 6.8 and 2.16, we have

$$I = \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} \left(\int_0^{2\sin\vartheta} 6\rho^3 \sin\vartheta \cos\vartheta \, d\rho \, d\vartheta \right) \right) d\varphi =$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\vartheta \cos\vartheta \left[\frac{6\rho^4}{4} \right]_{\rho=0}^{2\sin\vartheta} d\vartheta = 3\pi \int_0^{\frac{\pi}{2}} 16 \sin^5\vartheta \cos\vartheta \, d\vartheta = \left| \begin{array}{c} \text{substitution} \\ \sin\vartheta = u \\ \cos\vartheta \, d\vartheta = du \\ 0 \mapsto 0, \ \frac{\pi}{2} \mapsto 1 \end{array} \right| =$$

$$= 48\pi \int_0^1 u^5 \, du = 48\pi \left[\frac{u^6}{6} \right]_0^1 = 48\pi \cdot \frac{1}{6} = \underline{\underline{8\pi}}.$$

Other way to calculate this integral is to use "shifted" generalized spherical coordinates

$$x = 2\rho \cos\varphi \cos\vartheta,$$

$$y = 3\rho \sin\varphi \cos\vartheta,$$

$$z = 1 + \rho \sin\vartheta.$$

Once again, the Jacobian of this mapping is $J = 6\rho^2 \cos\vartheta$ and by applying the substitution relations to condition $\frac{x^2}{4}+\frac{y^2}{9}+z^2 \le 2z$, which is equivalent with inequality $\frac{x^2}{4}+\frac{y^2}{9}+(z-1)^2 \le 1$, we obtain $\rho^2 \le 1$, i.e. $0 \le \rho \le 1$. Hence (in manner similar to the first way of calculation) we obtain

$$I = \iiint_N (1+\rho\sin\vartheta)\cdot|J|\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\vartheta = \iiint_N (1+\rho\sin\vartheta)\cdot 6\rho^2\cos\vartheta\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\vartheta,$$

where

$$N = \left\{(\rho,\varphi,\vartheta)\in\mathbb{R}^3:\quad 0\le\varphi\le 2\pi \wedge -\frac{\pi}{2}\le\vartheta\le\frac{\pi}{2} \wedge 0\le\rho\le 1\right\}.$$

According to Theorems 6.8 and 2.16, the integral I satisfies

$$I = \int_0^1 \left(\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\rho\sin\vartheta)\cdot 6\rho^2\cos\vartheta\right)\mathrm{d}\varphi\right)\mathrm{d}\rho =$$

$$= 2\pi\int_0^1\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(6\rho^2\cos\vartheta + 6\rho^3\sin\vartheta\cos\vartheta)\right)\mathrm{d}\rho =$$

$$= 2\pi\int_0^1\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(6\rho^2\cos\vartheta + 3\rho^3\sin 2\vartheta)\right)\mathrm{d}\rho =$$

$$= 2\pi\int_0^1\left[6\rho^2\sin\vartheta - \frac{3}{2}\rho^3\cos 2\vartheta\right]_{\vartheta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\rho = 2\pi\int_0^1 12\rho^2\,\mathrm{d}\rho = 24\pi\left[\frac{\rho^3}{3}\right]_0^1 =$$

$$= 24\pi\cdot\frac{1}{3} = \underline{\underline{8\pi}}.$$

Exercise 7.10. *Calculate following integrals using suitable substitution:*

1. $\iiint_M z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where* $M = \{(x,y,z)\in\mathbb{R}^3:\ z\ge\sqrt{x^2+y^2} \wedge z\le 1\}$;

2. $\iiint_M z^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where* $M = \{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2\le z\le 2-x^2-y^2\}$;

3. $\iiint_M z\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, *where*
   $M = \{(x,y,z)\in\mathbb{R}^3:\ x^2+y^2\le 9 \wedge y\ge 0 \wedge 0\le z\le 2\}$;

4. $\iiint_M (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where* $M = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2 \le 2z \ \wedge\ z \le 2\}$;

5. $\iiint_M z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where*
$M = \{(x,y,z) \in \mathbb{R}^3 :\ x \ge 0 \ \wedge\ \sqrt{x^2+y^2}-1 \le z \le 1-\sqrt{x^2+y^2}\}$;

6. $\iiint_M (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where*
$M = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 \le 1 \ \wedge\ x^2+y^2+z^2 \le 2z\}$;

7. $\iiint_M xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where*
$M = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 \le 1 \ \wedge\ x \ge 0 \ \wedge\ y \ge 0 \ \wedge\ z \ge 0\}$;

8. $\iiint_M \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where*
$M = \{(x,y,z) \in \mathbb{R}^3 :\ 0 \le x \le y \ \wedge\ z \ge 0 \ \wedge\ x^2+y^2+z^2 \le 1\}$;

9. $\iiint_M 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where* $M = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 \le 4 \ \wedge\ z \ge \sqrt{x^2+y^2}\right\}$;

10. $\iiint_M z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$ *where* $M = \left\{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 \le 4 \ \wedge\ z \ge \sqrt{x^2+y^2}\right\}$.

Results:

1. $\frac{\pi}{4}$;

2. $\frac{7\pi}{6}$;

3. 18π;

4. $\frac{16\pi}{3}$;

5. 0;

6. $\frac{53\pi}{480}$;

7. $\frac{1}{15}$;

8. $\frac{\pi}{16}$;

9. $\frac{8\pi}{3}\left(2-\sqrt{2}\right)$;

10. 2π.

8 Some Applications of Triple Integrals

8.1 Volume of a body

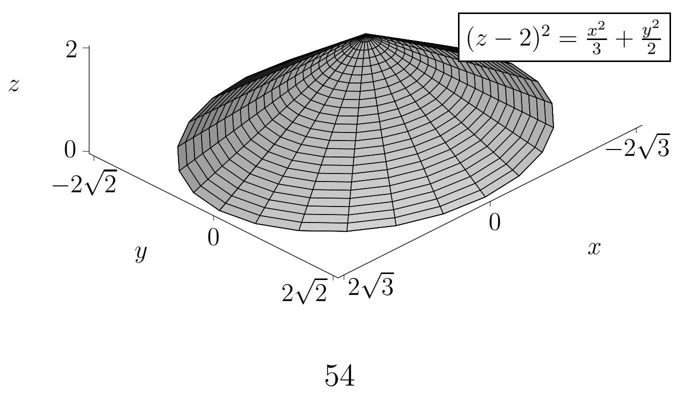
Let $M \subset \mathbb{R}^3$ be measurable set. Then we can define the volume of the body M as

$$\lambda(M) = \iiint_M 1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Exercise 8.1. *Calculate the volume of body $M \subset \mathbb{R}^3$ constrained by surfaces*

$$(z-2)^2 = \frac{x^2}{3} + \frac{y^2}{2}, \quad z = 0.$$

Solution:
Our body M is part of an elliptical cone.



Its intersection with plane $z = z_0$, where $0 \le z_0 \le 2$, is set

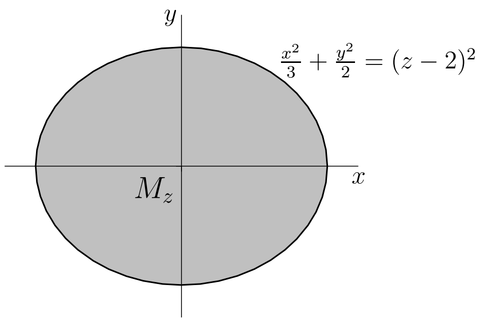
$$\left\{(x, y, z_0) \in \mathbb{R}^3 : \quad \frac{x^2}{3} + \frac{y^2}{2} \le (z_0 - 2)^2\right\} \qquad \text{(i.e. ellipse)}.$$

It is not hard to realize that volume of M satisfies (see Theorem 6.8)

$$\lambda(M) = \iiint_M 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^2 \left(\iint_{M_z} 1 \, \mathrm{d}x \, \mathrm{d}y\right) \mathrm{d}z,$$

where

$$M_z = \left\{(x, y) \in \mathbb{R}^2 : \quad \frac{x^2}{3} + \frac{y^2}{2} \le (z - 2)^2\right\}.$$



To calculate the inner integral $\iint_{M_z} 1 \, \mathrm{d}x \, \mathrm{d}y$ we can use generalized polar coordinates

$$x = \sqrt{3} \cdot r \cos t,$$

$$y = \sqrt{2} \cdot r \sin t.$$

The Jacobian is clearly $J = \sqrt{6} \cdot r$. Furthermore, the set M_z (for $z \in \langle 0, 2\rangle$) can be, using these coordinates, described by inequalities

$$0 \le t \le 2\pi, \quad 0 \le r \le 2 - z.$$

This and Theorems 3.1 and 2.16 give us that

$$\lambda(M) = \iiint_M 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^2 \left(\iint_{M_z} 1 \, \mathrm{d}x \, \mathrm{d}y\right) \mathrm{d}z =$$

$$= \int_0^2 \left(\int_0^{2\pi} \left(\int_0^{2-z} r\sqrt{6} \, \mathrm{d}r\right) \mathrm{d}t\right) \mathrm{d}z = 2\pi \int_0^2 \left[\sqrt{6}\frac{r^2}{2}\right]_{r=0}^{2-z} \mathrm{d}z =$$

$$= 2\pi \int_0^2 \sqrt{6} \cdot \frac{(2-z)^2}{2} \, \mathrm{d}z = \pi\sqrt{6}\left[-\frac{(2-z)^3}{3}\right]_0^2 = \underline{\underline{\frac{8\pi\sqrt{6}}{3}}}.$$

Remark 8.2. The previous example could also be solved in other way. For suitable a, b, we could use so called generalized cylindrical coordinates, i.e.

$$x = ar\cos t,$$

$$y = br\sin t,$$

$$z = z^*(= z),$$

where

$$r \geq 0, \quad t \in \langle 0, 2\pi \rangle \quad (\text{or } t \in \langle -\pi, \pi \rangle \text{ or } t \in \langle \alpha, \alpha + 2\pi \rangle \text{ for } \alpha \in \mathbb{R}) \text{ and } \quad z^* \in \mathbb{R}.$$

Jacobian of this mapping satisfies

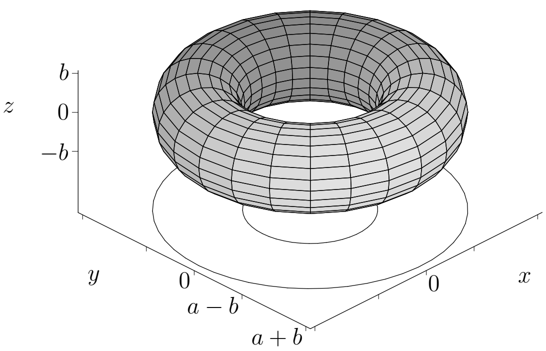
$$J(r, t, z^*) = \begin{vmatrix} a\cos t & -ar\sin t & 0 \\ b\sin t & br\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = ab \cdot r\left(\cos^2 t + \sin^2 t\right) = ab \cdot r.$$

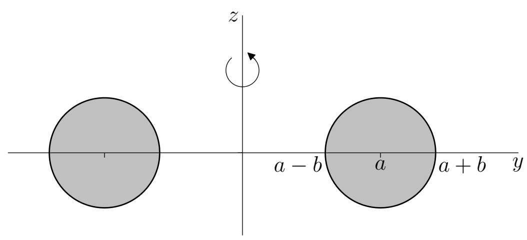
Exercise 8.3. *Try solving Exercise 8.1 with generalized cylindrical coordinates (see Remark 8.2).*

Exercise 8.4. *Calculate the volume of body M bounded by surface (torus)*

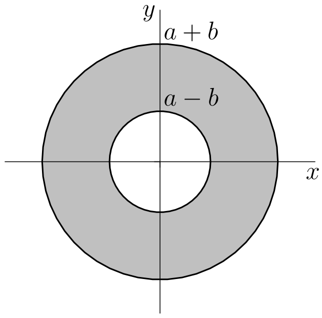
$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = b^2 \quad (0 < b < a).$$

Solution:





We already know that the volume of M can be calculated as $\lambda(M) = \iiint_M 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$. If we cut M by plane $z = 0$ we obtain annulus as can bee seen in the Figure below:



Thus, the most suitable choice for evaluating $\lambda(M)$ seems to be using cylindrical coordinates in respective integral:

$$x = r\cos t,$$

$$y = r\sin t,$$

$$z = z.$$

The Jacobian of this mapping is $J = r$. The body M can be now described by conditions (think it through!)

$$t \in \langle 0, 2\pi\rangle, \quad r \in \langle a-b, a+b\rangle, \quad z \in \left\langle -\sqrt{b^2-(r-a)^2}, \sqrt{b^2-(r-a)^2}\right\rangle.$$

Thus (according to Theorems 2.16, 6.8 a 7.1) the volume of M satisfies

$$\lambda(M) = \iiint_M 1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi} \left(\int_{a-b}^{a+b} \left(\int_{-\sqrt{b^2-(r-a)^2}}^{\sqrt{b^2-(r-a)^2}} r \,\mathrm{d}z \right) \mathrm{d}r \right) \mathrm{d}t =$$

$$= 2\pi \cdot 2 \cdot \int_{a-b}^{a+b} r\sqrt{b^2-(r-a)^2}\,\mathrm{d}r = \left| \begin{array}{c} \text{substitution} \\ r-a=u \\ \mathrm{d}r = \mathrm{d}u \\ a-b \mapsto -b \\ a+b \mapsto b \end{array} \right| = 4\pi \int_{-b}^{b} (u+a)\sqrt{b^2-u^2}\,\mathrm{d}u =$$

$$= 4\pi \int_{-b}^{b} \underbrace{u\sqrt{b^2-u^2}}_{\text{odd for } u} \,\mathrm{d}u + 4\pi a \int_{-b}^{b} \underbrace{\sqrt{b^2-u^2}}_{\text{even for } u} \,\mathrm{d}u = 8\pi a \int_0^b \sqrt{b^2-u^2}\,\mathrm{d}u =$$

$$= \left| \begin{array}{c} \text{substitution} \\ u = b\sin s \\ \mathrm{d}u = b\cos s\,\mathrm{d}s \\ 0 \mapsto 0,\ b \mapsto \frac{\pi}{2} \end{array} \right| = 8\pi a \int_0^{\frac{\pi}{2}} \sqrt{b^2 - b^2\sin^2 s}\, b\cos s\,\mathrm{d}s = 8\pi a b^2 \int_0^{\frac{\pi}{2}} \cos^2 s\,\mathrm{d}s =$$

$$= 8\pi a b^2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2s}{2}\,\mathrm{d}s = 8\pi a b^2 \left(\frac{1}{2}\cdot\frac{\pi}{2} + \underbrace{\left[\frac{\sin 2s}{4}\right]_0^{\frac{\pi}{2}}}_{=0} \right) = \underline{\underline{2\pi^2 a b^2}}.$$

Other way to calculate the volume of our body M is given by so called Guldinus rule (also known as Pappus's centroid theorem):

"Given profile of area P moves such that its plane stays perpendicular to space curve, which its centroid curves along. Then the volume of resulting body is $P \cdot D$, D is length of the scope, traveled by profile P."

According to Guldinus rule it holds

$$\lambda(M) = P \cdot D = \left(\pi b^2\right) \cdot (2\pi a) = \underline{\underline{2\pi^2 a b^2}}.$$

8.2 Applications in Mechanics

Let the body M be closed measurable set in $\mathbb{R}^3$ let its volume density be represented by a continuous non-negative function h

$$\left(h(x,y,z) = \lim_{\delta \to 0+} \frac{m\big(U((x,y,z),\delta) \cap M\big)}{\lambda\big(U((x,y,z),\delta) \cap M\big)}\right).$$

Then the following formulae hold:

- Mass of the body M:

$$m(M) = \iiint_M h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

- Static moments of the body M with respect to coordinate planes xy, yz, xz, respectively:

$$S_{xy}(M) = \iiint_M z\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad S_{yz}(M) = \iiint_M x\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$S_{xz}(M) = \iiint_M y\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

- Coordinates of the centroid T of the body M:

$$T(M) = \left(\frac{S_{yz}(M)}{m(M)}, \frac{S_{xz}(M)}{m(M)}, \frac{S_{xy}(M)}{m(M)}\right).$$

- Moments of inertia of the body M with respect to axes x, y, z, respectively:

$$I_x(M) = \iiint_M (y^2 + z^2)\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$I_y(M) = \iiint_M (x^2 + z^2)\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$I_z(M) = \iiint_M (x^2 + y^2)\,h(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Remark 8.5. The body M is called homogeneous if its volume density h is constant over M.

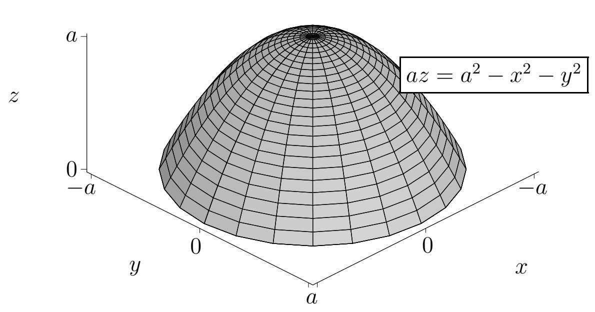
Exercise 8.6. *Calculate the mass of body M_a, $a > 0$, bounded by surfaces*

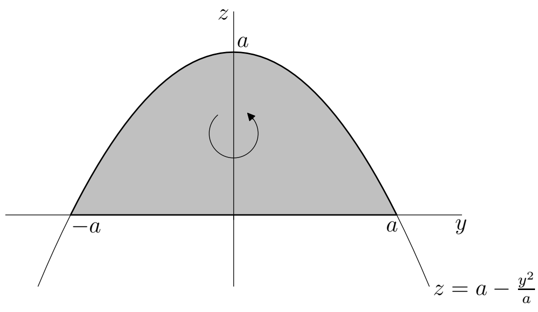
$$z = 0, \quad az = a^2 - x^2 - y^2,$$

considering that its volume density h is defined as $h(x,y,z) = z^3$.

Solution:

The body M_a is bounded by plane $z = 0$ and paraboloid $az = a^2 - x^2 - y^2$.





The mass of M_a satisfies

$$m(M_a) = \iiint_{M_a} z^3 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Moreover, the last integral can be calculated using e.g. cylindrical coordinates

$$x = r\cos t,$$

$$y = r\sin t,$$

$$z = z.$$

Jacobian of this mapping is $J = r$. The body M_a can now be described by conditions

$$t \in \langle 0, 2\pi \rangle, \quad r \in \langle 0, a \rangle, \quad z \in \left\langle 0, a - \tfrac{r^2}{a} \right\rangle.$$

Thus it follows that

$$m(M_a) = \iiint_{M_a} z^3 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^{2\pi} \left(\int_0^a \left(\int_0^{a - \frac{r^2}{a}} z^3 r \, \mathrm{d}z \right) \mathrm{d}r \right) \mathrm{d}t =$$

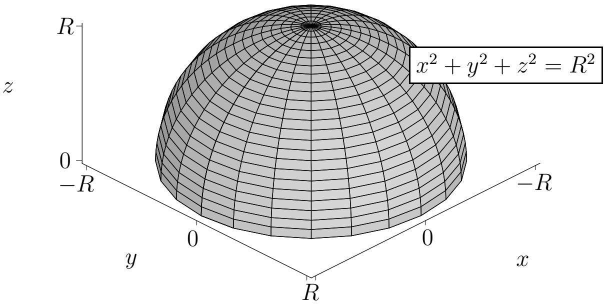
$$= 2\pi \int_0^a \frac{1}{4} r \left(a - \frac{r^2}{a} \right)^4 \mathrm{d}r = \left| \begin{array}{c} \text{substitution} \\ a - \frac{r^2}{a} = u \\ -\frac{2r}{a}\,\mathrm{d}r = \mathrm{d}u \\ r\,\mathrm{d}r = -\frac{a}{2}\,\mathrm{d}u \\ 0 \mapsto a,\ a \mapsto 0 \end{array} \right| = \frac{\pi}{2} \cdot \left(-\frac{a}{2} \right) \cdot \int_a^0 u^4 \, \mathrm{d}u =$$

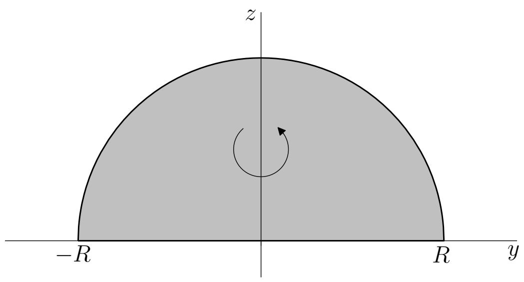
$$= \frac{\pi a}{4} \left[\frac{u^5}{5} \right]_0^a = \underline{\underline{\frac{\pi a^6}{20}}}.$$

Exercise 8.7. *For $R > 0$ calculate the coordinates of centroid T_R of homogeneous half-ball*

$$\Omega_R = \left\{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 \le R^2 \wedge z \ge 0\right\}.$$

Solution:





It is obvious that

$$T_R = \left(0, 0, \frac{S_{xy}(\Omega_R)}{m(\Omega_R)}\right),$$

where (considering volume density $k > 0$)

$$S_{xy}(\Omega_R) = \iiint_{\Omega_R} z \cdot k \;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \qquad m(\Omega_R) = \iiint_{\Omega_R} k \;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

To simplify calculation of these two integrals we use substitution to spherical coordinates

$$x = \rho\cos\varphi\cos\vartheta,$$

$$y = \rho\sin\varphi\cos\vartheta,$$

$$z = \rho\sin\vartheta.$$

Jacobian of this mapping is $J = \rho^2\cos\vartheta$. In these coordinates, we can describe Ω_R rather easily:

$$\varphi \in \langle 0, 2\pi\rangle, \quad \vartheta \in \langle 0, \tfrac{\pi}{2}\rangle, \quad \rho \in \langle 0, R\rangle.$$

Thus it holds that

$$S_{xy}(\Omega_R) = k\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^R \rho\sin\vartheta\cdot\rho^2\cos\vartheta\;\mathrm{d}\rho\right)\mathrm{d}\vartheta\right)\mathrm{d}\varphi =$$

$$= k2\pi\cdot\left[\frac{\sin^2\vartheta}{2}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{\rho^4}{4}\right]_0^R = k\cdot 2\pi\cdot\frac{1}{2}\cdot\frac{R^4}{4} = \frac{k\pi R^4}{4}$$

and

$$m(\Omega_R) = k \int_0^{2\pi} \left(\int_0^{\frac{\pi}{2}} \left(\int_0^R \rho^2 \cos\vartheta \, \mathrm{d}\rho \right) \mathrm{d}\vartheta \right) \mathrm{d}\varphi =$$

$$= k \cdot 2\pi \cdot [\sin\vartheta]_0^{\frac{\pi}{2}} \cdot \left[\frac{\rho^3}{3} \right]_0^R = k \cdot 2\pi \cdot 1 \cdot \frac{R^3}{3} = \frac{2k\pi R^3}{3}.$$

We actually did not have to calculate the last integral, since in our case it holds $m(\Omega_R) = k \cdot \lambda(\Omega_R)$. Furthermore, let us realize that Ω_R is a half-ball, so we can evaluate its measure (volume) without necessarily using integrals:

$$\lambda(\Omega_R) = \frac{1}{2} \cdot \frac{4}{3} \pi R^3 = \frac{2}{3} \pi R^3.$$

So in the end, the coordinates of centroid of our half-ball satisfy

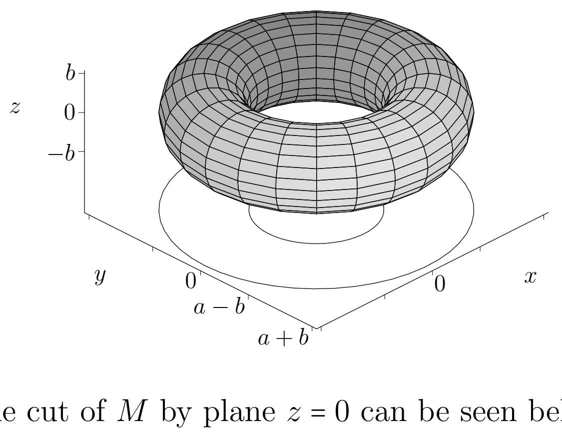
$$T_R = \left(0, 0, \frac{\frac{k\pi R^4}{4}}{\frac{2k\pi R^3}{3}} \right) = \underline{\underline{\left(0, 0, \frac{3}{8} R \right)}}.$$

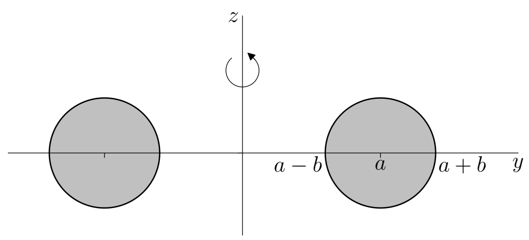
Exercise 8.8. *Calculate the moment of inertia with respect to axis z of the body M bounded by torus*

$$\left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = b^2 \quad (0 < b < a),$$

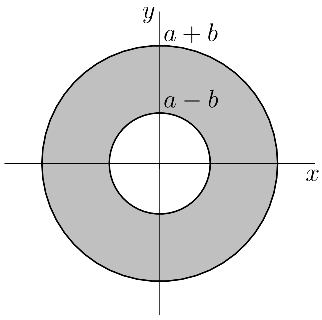
considering that its volume density is equal to 1 everywhere.

We have already met torus in Exercise 8.4. Just for the sake of assurance, let us remind what the body M looks like (see Figures below).





The cut of M by plane $z = 0$ can be seen below.



It holds that

$$I_z(M) = \iiint_M \left(x^2+y^2\right)\cdot 1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_M \left(x^2+y^2\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

To simplify our calculations, let us use transformation into cylindrical coordinates (see Exercise 8.4). Then

$$I_z(M) = \int_0^{2\pi}\left(\int_{a-b}^{a+b}\left(\int_{-\sqrt{b^2-(r-a)^2}}^{\sqrt{b^2-(r-a)^2}} r^2\cdot r\,\mathrm{d}z\right)\mathrm{d}r\right)\mathrm{d}t =$$

$$= 2\pi\cdot 2\cdot\int_{a-b}^{a+b} r^3\sqrt{b^2-(r-a)^2}\,\mathrm{d}r = \left|\begin{array}{c}\text{substitution}\\ r-a=u\\ \mathrm{d}r=\mathrm{d}u\\ a-b\mapsto -b\\ a+b\mapsto b\end{array}\right| = 4\pi\int_{-b}^{b}(u+a)^3\sqrt{b^2-u^2}\,\mathrm{d}u =$$

$$= 4\pi\left(\int_{-b}^{b}\left(\underbrace{u^3\sqrt{b^2-u^2}}_{\text{odd in } u}+\underbrace{3u^2a\sqrt{b^2-u^2}}_{\text{even in } u}+\underbrace{3ua^2\sqrt{b^2-u^2}}_{\text{odd in } u}+\underbrace{a^3\sqrt{b^2-u^2}}_{\text{even in } u}\right)\mathrm{d}u\right) =$$

$$= 8\pi\int_0^b\left(3u^2a\sqrt{b^2-u^2}+a^3\sqrt{b^2-u^2}\right)\mathrm{d}u = \left|\begin{array}{c}\text{substitution}\\ u=b\sin s\\ \mathrm{d}u=b\cos s\,\mathrm{d}s\\ 0\mapsto 0,\ b\mapsto \frac{\pi}{2}\end{array}\right| =$$

$$= 8\pi \int_0^{\frac{\pi}{2}} \left(3ab^2 \sin^2 s \cdot b \cos s + a^3 \cdot b \cos s\right) \cdot b \cos s \, \mathrm{d}s =$$

$$= 8\pi \int_0^{\frac{\pi}{2}} \left(3ab^4 \underbrace{\sin^2 s \cos^2 s}_{\frac{1}{4}\sin^2 2s} + a^3 b^2 \cos^2 s \right) \mathrm{d}s =$$

$$= 8\pi \int_0^{\frac{\pi}{2}} \left(3ab^4 \cdot \frac{1 - \cos 4s}{8} + a^3 b^2 \cdot \frac{1 + \cos 2s}{2}\right) \mathrm{d}s =$$

$$= 8\pi \left(3ab^4 \cdot \frac{1}{8} \cdot \frac{\pi}{2} + 0 + a^3 b^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 0\right) = \underline{\underline{\frac{\pi^2 ab^2}{2}\left(3b^2 + 4a^2\right)}} \, .$$

References

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[3] W. Rudin: *Principles of Mathematical Analysis*, McGraw-Hill, Inc., 1976.