Complex analysis
Exercises with solutions

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The author of the painting *Imaginární džungle* on the cover page is Jiří Bouchala (and it is owned by Ondřej Bouchala).

Preface

This text contains the solutions to all of the practice problems in the 10th chapter of the lecture notes “An Introduction to Complex Analysis” [1]. It is a translation of the Czech text [3].

The typesetting and all of the pictures are the work of my son Ondřej. He also helped to improve the text in several places with his comments.

It is not possible that we caught all of the mistakes during the proof-reading. We are grateful for your leniency and for letting us know about any and all remarks.[1]

We enjoyed working on this text. We wish the same to the reader.

In Orlová, 2022

Jiří Bouchala
(and Ondřej Bouchala)

[1]Please send all of the remarks (notes, recommendations, threats and gifts) to my e-mail address `jiri.bouchala@vsb.cz`.

EXERCISE 1.

Find the real and imaginary part of the complex number

a) $z = (1+i)(3-2i)$;

b) $z = \frac{2-3i}{3+4i}$;

c) $z = \frac{1+i}{1-i}$;

d) $z = 2i - \overline{\frac{2-4i}{2}}$.

Solution:

a) $z = (3+2)+i;\ \underline{\operatorname{Re} z = 5,\ \operatorname{Im} z = 1}.$

b) $z = \frac{2-3i}{3+4i} = \frac{(2-3i)(3-4i)}{9+16} = \frac{6-12-9i-8i}{25};\ \underline{\operatorname{Re} z = -\frac{6}{25},\ \operatorname{Im} z = -\frac{17}{25}}.$

c) $z = \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{1+2i-1}{2};\ \underline{\operatorname{Re} z = 0,\ \operatorname{Im} z = 1}.$

d) $z = 2i - \overline{\frac{2-4i}{2}} = 2i - \frac{2+4i}{2} = -1;\ \underline{\operatorname{Re} z = -1,\ \operatorname{Im} z = 0}.$

EXERCISE 2.

Write the given complex number in the trigonometric form

a) $z = -1 + \sqrt{3}i$;

b) $z = i$;

c) $z = -8$;

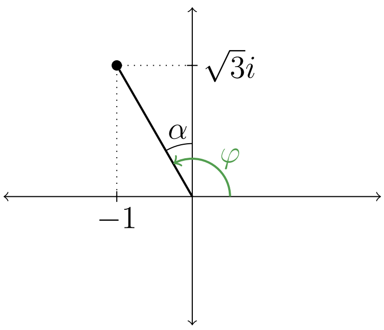
d) $z = -1 - \sqrt{3}i$;

e) $z = \frac{2+i}{3-2i}$;

f) $z = \frac{3-i}{2+i}$.

Solution:

a)



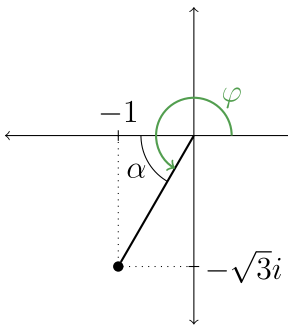
$$\cos\alpha = \frac{\sqrt{3}}{2},\ \alpha = \frac{\pi}{6},\ \varphi = \frac{\pi}{2} + \alpha = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2}{3}\pi;$$

$$\underline{z =} -1 + \sqrt{3}i = \sqrt{1+3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \underline{2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}.$$

b) $\underline{z =}\ i = \underline{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}.$

c) $\underline{z =} -8 = \underline{8(\cos\pi + i\sin\pi)}.$

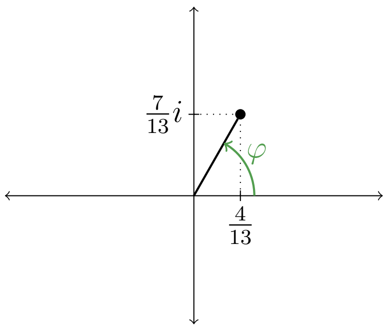
d)



$$\sin\alpha = \frac{\sqrt{3}}{2},\ \alpha = \frac{\pi}{3},\ \varphi = \pi + \alpha = \frac{4}{3}\pi;$$

$$\underline{z =} -1 - \sqrt{3}i = \underline{2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right).}$$

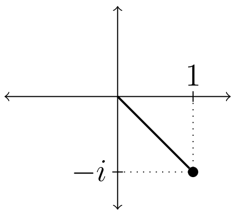
e) $z = \frac{2+i}{3-2i} = \frac{(2+i)(3+2i)}{9+4} = \frac{4}{13} + \frac{7}{13}i,$



$$|z| = \frac{1}{13}\sqrt{16+49} = \frac{\sqrt{65}}{13},\ \tan\varphi = \frac{\frac{7}{13}}{\frac{4}{13}} = \frac{7}{4},\ \varphi = \arctan\frac{7}{4};$$

$$\underline{z = \frac{\sqrt{65}}{13}\left(\cos\left(\arctan\frac{7}{4}\right) + i\sin\left(\arctan\frac{7}{4}\right)\right).}$$

f) $z = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{5} = \frac{5-5i}{5} = 1 - i,$



$$\underline{z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).}$$

EXERCISE 3.
Prove the *de Moivre's theorem*

$$(\forall n \in \mathbb{N})\,(\forall \varphi \in \mathbb{R}):\ \big(\cos\varphi + i\sin\varphi\big)^n = \cos(n\varphi) + i\sin(n\varphi)$$

using mathematical induction.

Solution:

1) We start by checking that the formula holds for $n = 1$:

$$(\cos\varphi + i\sin\varphi)^1 = \cos(1\cdot\varphi) + i\sin(1\cdot\varphi)\,.$$

2) Now we prove the implication $(\cos\varphi + i\sin\varphi)^n = \cos(n\varphi) + i\sin(n\varphi) \overset{?}{\Rightarrow}$ $\overset{?}{\Rightarrow} (\cos\varphi + i\sin\varphi)^{n+1} = \cos((n+1)\varphi) + i\sin((n+1)\varphi)$:

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^{n+1} &\overset{\text{i.p.}}{=} (\cos(n\varphi) + i\sin(n\varphi))\,(\cos\varphi + i\sin\varphi) = \\ &= (\cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi) + i\,(\sin(n\varphi)\cos\varphi + \cos(n\varphi)\sin\varphi)\,,\end{aligned}$$

and now it suffices to apply the known “trigonometric identities”:

$$\begin{aligned}\cos(n\varphi)\cos\varphi - \sin(n\varphi)\sin\varphi &= \cos(n\varphi + \varphi) = \cos((n+1)\varphi),\\ \sin(n\varphi)\cos\varphi + \cos(n\varphi)\sin\varphi &= \sin(n\varphi + \varphi) = \sin((n+1)\varphi).\end{aligned}$$

EXERCISE 4.
Let $\varphi \in \mathbb{R}$. Express $\sin(4\varphi)$ and $\cos(4\varphi)$ using $\sin\varphi$ and $\cos\varphi$.

Solution:

$$\begin{aligned}\cos(4\varphi) + i\sin(4\varphi) &= (\cos\varphi + i\sin\varphi)^4 = \\ &= \big(\cos^2\varphi + 2i\sin\varphi\cos\varphi - \sin^2\varphi\big)^2 = \\ &= \cos^4\varphi - 4\sin^2\varphi\cos^2\varphi + \sin^4\varphi + \\ &\quad + 4i\sin\varphi\cos^3\varphi - 2\cos^2\varphi\sin^2\varphi - 4i\sin^3\varphi\cos\varphi = \\ &= \cos^4\varphi - 6\sin^2\varphi\cos^2\varphi + \sin^4\varphi + i\left(4\sin\varphi\cos^3\varphi - 4\sin^3\varphi\cos\varphi\right),\end{aligned}$$

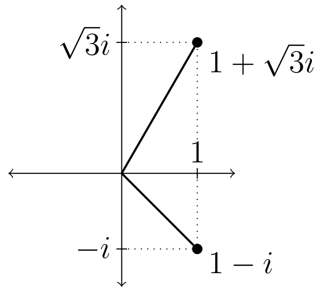
and therefore (it is enough to compare the real and imaginary parts)

$$\underline{\cos(4\varphi) = \cos^4\varphi - 6\sin^2\varphi\cos^2\varphi + \sin^4\varphi},$$

$$\underline{\sin(4\varphi) = 4\sin\varphi\cos^3\varphi - 4\sin^3\varphi\cos\varphi}.$$

Exercise 5.
Find $\operatorname{Re} z$ and $\operatorname{Im} z$ for $z = \left(\frac{1-i}{1+\sqrt{3}i}\right)^{24}$.

Solution:



$$\frac{1-i}{1+\sqrt{3}i} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(-\frac{7}{12}\pi\right)\right),$$

$$z = \frac{1}{2^{12}}\left(\cos\left(-\frac{24\cdot 7\pi}{12}\right) + i\sin\left(-\frac{24\cdot 7\pi}{12}\right)\right) = \frac{1}{2^{12}};$$

$$\underline{\operatorname{Re} z = \frac{1}{2^{12}},\ \operatorname{Im} z = 0.}$$

Exercise 6.
Find $\operatorname{Arg} z$ and $\arg z$ for

a) $z = \left(\sqrt{3}+i\right)^{126}$;

b) $z = (1+i)^{137}$;

c) $z = -1-5i$.

Solution:

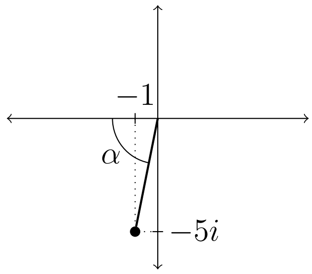
a) $z = (\sqrt{3}+i)^{126} = \left(2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\right)^{126} = 2^{126}\left(\cos(21\pi) + i\sin(21\pi)\right) = -2^{126}$;

$$\underline{\operatorname{Arg} z = \{\pi + 2k\pi\colon\ k \in \mathbb{Z}\},\ \arg z = \pi.}$$

b) $z = 2^{\frac{137}{2}}\left(\cos\left(137\frac{\pi}{4}\right) + i\sin\left(137\frac{\pi}{4}\right)\right) = 2^{\frac{137}{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$;

$$\underline{\operatorname{Arg} z = \left\{\frac{\pi}{4} + 2k\pi\colon\ k \in \mathbb{Z}\right\},\ \arg z = \frac{\pi}{4}.}$$

c)



$\tan\alpha = \frac{5}{1},\ \alpha = \arctan 5;$

$$\underline{\operatorname{Arg} z = \{-\pi + \arctan 5 + 2k\pi\colon\ k \in \mathbb{Z}\},\ \arg z = -\pi + \arctan 5.}$$

<u>EXERCISE 7</u>.

Draw in the complex plane the set

a) $\{z \in \mathbb{C}\colon\ \operatorname{Re} z \leq 1\};$

b) $\{z \in \mathbb{C}\colon\ \operatorname{Re}(z^2) = 2\};$

c) $\{z \in \mathbb{C}\colon\ \operatorname{Im} \frac{1}{z} = \frac{1}{4}\};$

d) $\{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < 1\};$

e) $\{z \in \mathbb{C}\colon\ |z| = \operatorname{Re} z + 1\};$

f) $\{z \in \mathbb{C}\colon\ |z-2| = |1-2\overline{z}|\};$

g) $\{z \in \mathbb{C}\colon\ \left|\frac{z-2}{z-3}\right| = 1\};$

h) $\{z \in \mathbb{C}\colon\ |1+z| < |1-z|\};$

i) $\{z \in \mathbb{C}\colon\ |z+1| = 2|z-1|\};$

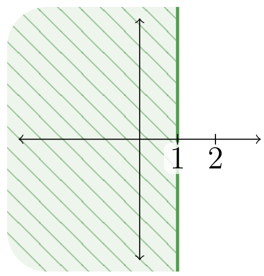
j) $\{z \in \mathbb{C}\colon\ 2 < |z+2-3i| < 4\};$

k) $\{z \in \mathbb{C}\colon\ \frac{\pi}{4} \leq \arg(z+2i) \leq \frac{\pi}{2}\};$

l) $\{z \in \mathbb{C}\colon\ |z| + \operatorname{Re} z \leq 1\ \wedge$
$\wedge -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{4}\}.$

<u>Solution:</u>

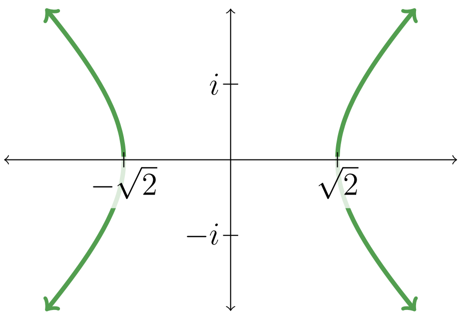
a) $\{z \in \mathbb{C}\colon\ \operatorname{Re} z \leq 1\}$:



b)

$$\{z \in \mathbb{C}\colon\ \operatorname{Re}(z^2) = 2\} = \{x + iy\colon\ \operatorname{Re}(x^2 + 2ixy - y^2) = 2\} =$$

$$= \{x + iy\colon\ x^2 - y^2 = 2\}\ :$$



c)

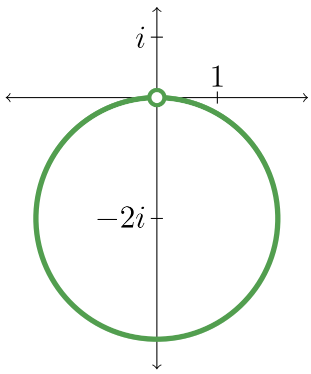
$$\left\{z \in \mathbb{C}\colon\ \operatorname{Im}\frac{1}{z} = \frac{1}{4}\right\} = \left\{x + iy \in \mathbb{C}\colon\ \operatorname{Im}\frac{1}{x + iy} = \frac{1}{4}\right\} =$$

$$= \left\{x + iy \in \mathbb{C}\colon\ \operatorname{Im}\frac{x - iy}{x^2 + y^2} = \frac{1}{4}\right\} =$$

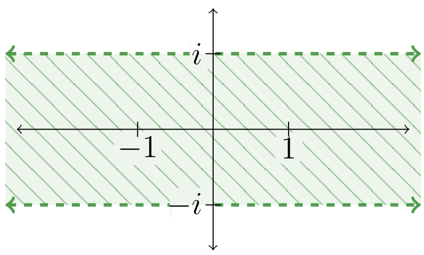
$$= \left\{x + iy \in \mathbb{C}\colon\ -\frac{y}{x^2 + y^2} = \frac{1}{4}\right\} =$$

$$= \{x + iy \in \mathbb{C}\colon\ x^2 + y^2 = -4y \wedge x^2 + y^2 \neq 0\} =$$

$$= \{x + iy \in \mathbb{C}\colon\ x^2 + (y + 2)^2 = 4\} \setminus \{0 + 0i\}\ :$$

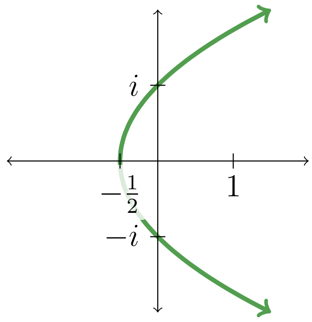


d) $\{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < 1\}$:



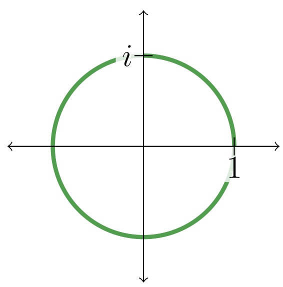
e)

$$\{z \in \mathbb{C}\colon\ |z| = \operatorname{Re} z + 1\} = \left\{x+iy\colon\ \sqrt{x^2+y^2} = x+1\right\} =$$
$$= \left\{x+iy\colon\ x^2+y^2 = x^2+2x+1\right\} =$$
$$= \left\{x+iy\colon\ y^2 = 2x+1\right\} =$$
$$= \left\{x+iy\colon\ x = \frac{y^2-1}{2}\right\}:$$

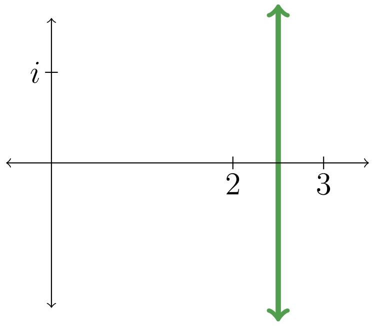


f)

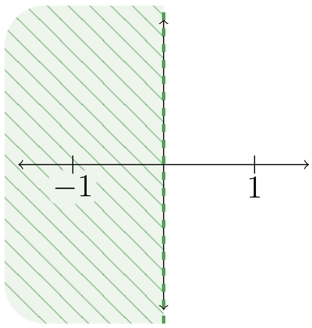
$$\{z \in \mathbb{C}\colon\ |z-2| = |1-2\overline{z}|\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ \sqrt{(x-2)^2+y^2} = |1-2(x-iy)|\right\} =$$
$$= \left\{x+iy\colon\ (x-2)^2+y^2 = (1-2x)^2+4y^2\right\} =$$
$$= \left\{x+iy\colon\ x^2-4x+4+y^2 = 1-4x+4x^2+4y^2\right\} =$$
$$= \left\{x+iy\colon\ 3x^2+3y^2 = 3\right\} =$$
$$= \left\{x+iy\colon\ x^2+y^2 = 1\right\}:$$



g) $\left\{z \in \mathbb{C}\colon\ \left|\frac{z-2}{z-3}\right| = 1\right\} = \{z \in \mathbb{C}\colon\ |z-2| = |z-3|\}$:

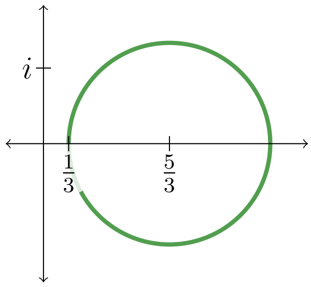


h) $\{z \in \mathbb{C}\colon\ |1+z| < |1-z|\} = \{z \in \mathbb{C}\colon\ |z-(-1)| < |z-1|\}$:

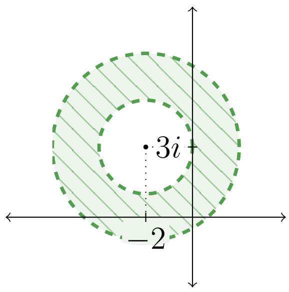


i)

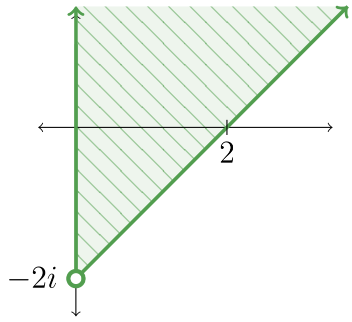
$$\begin{aligned}
&\{z \in \mathbb{C}\colon\ |z+1| = 2|z-1|\} = \\
&\qquad = \left\{x+iy \in \mathbb{C}\colon\ (x+1)^2 + y^2 = 4((x-1)^2+y^2)\right\} = \\
&\qquad = \left\{x+iy\colon\ x^2+2x+1+y^2 = 4x^2-8x+4+4y^2\right\} = \\
&\qquad = \left\{x+iy\colon\ 3x^2+3y^2-10x = -3\right\} = \\
&\qquad = \left\{x+iy\colon\ \left(x-\frac{5}{3}\right)^2 + y^2 = \frac{16}{9}\right\}:
\end{aligned}$$



j) $\{z \in \mathbb{C}\colon\ 2 < |z+2-3i| < 4\} = \{z \in \mathbb{C}\colon\ 2 < |z-(-2+3i)| < 4\}$:

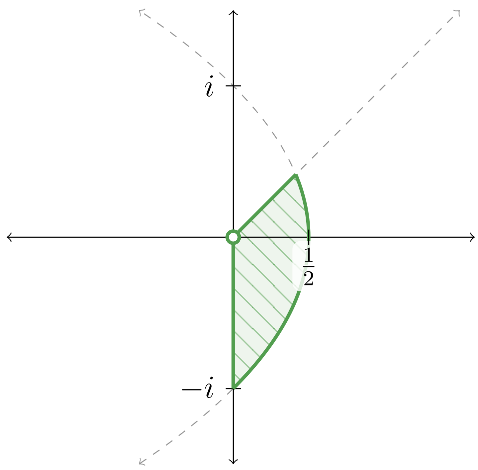


k) $\{z \in \mathbb{C}\colon\ \frac{\pi}{4} \le \arg(z+2i) \le \frac{\pi}{2}\}$:



l)

$$\left\{z \in \mathbb{C}\colon\ |z| + \operatorname{Re} z \le 1 \wedge -\frac{\pi}{2} \le \arg z \le \frac{\pi}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ \sqrt{x^2+y^2} \le 1-x \wedge -\frac{\pi}{2} \le \arg(x+iy) \le \frac{\pi}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ x^2+y^2 \le (1-x)^2 \wedge 1-x \ge 0 \wedge \right.$$
$$\left. \wedge -\frac{\pi}{2} \le \arg(x+iy) \le \frac{\pi}{4}\right\} =$$
$$= \left\{x+iy \in \mathbb{C}\colon\ y^2 \le 1-2x \wedge -\frac{\pi}{2} \le \arg(x+iy) \le \frac{\pi}{4}\right\}:$$



Exercise 8.

Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Prove the following implications:

a) $\left.\begin{matrix} \varphi_1 \in \operatorname{Arg} z_1 \\ \varphi_2 \in \operatorname{Arg} z_2 \end{matrix}\right\} \ \Rightarrow\ \varphi_1 + \varphi_2 \in \operatorname{Arg}(z_1 z_2);$

b) $\left.\begin{matrix} \varphi_1 \in \operatorname{Arg} z_1 \\ \varphi_2 \in \operatorname{Arg} z_2 \end{matrix}\right\} \ \Rightarrow\ \varphi_1 - \varphi_2 \in \operatorname{Arg}\left(\frac{z_1}{z_2}\right).$

Solution:

a)

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot (\cos\varphi_1 + i\sin\varphi_1) \cdot (\cos\varphi_2 + i\sin\varphi_2) =$$
$$= |z_1| \cdot |z_2| \cdot (\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)).$$

b)
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \cdot \frac{\cos\varphi_1 + i\sin\varphi_1}{\cos\varphi_2 + i\sin\varphi_2} = \left|\frac{z_1}{z_2}\right| (\cos\varphi_1 + i\sin\varphi_1) \cdot (\cos\varphi_2 - i\sin\varphi_2) =$$
$$= \left|\frac{z_1}{z_2}\right| \cdot (\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) .$$

Exercise 9.

Decide if the given limit exists, and if it does, compute it

a) $\lim(3-4i)^n$;

b) $\lim\left((-1)^n + \frac{i}{n}\right)$;

c) $\lim\left(\frac{1+i}{\sqrt{2}}\right)^n$;

d) $\lim\left(\frac{1-\sqrt{3}i}{2}\right)^{6n}$.

Solution:

a) $\underline{\lim(3-4i)^n = \infty}$, because $|(3-4i)^n| = \left(\sqrt{9+16}\right)^n = 5^n \to \infty$.

b) $\underline{\lim \underbrace{\left((-1)^n + \frac{i}{n}\right)}_{=:z_n} \text{ does not exist}}$, because

$$z_{2n} = (-1)^{2n} + \frac{i}{2n} = 1 + \frac{i}{2n} \to 1$$

and at the same time

$$z_{2n+1} = (-1)^{2n+1} + \frac{i}{2n+1} = -1 + \frac{i}{2n+1} \to -1.$$

c) $\underline{\lim \underbrace{\left(\frac{1+i}{\sqrt{2}}\right)^n}_{=:z_n} \text{does not exist}}$, beacause

$$z_n = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n = \cos\left(n\frac{\pi}{4}\right) + i\sin\left(n\frac{\pi}{4}\right),$$

and so

$$z_{8n} \to 1 \wedge z_{8n+2} \to i.$$

d) $\underline{\lim\left(\frac{1-\sqrt{3}i}{2}\right)^{6n} = 1}$, because

$$\left(\frac{1-\sqrt{3}i}{2}\right)^{6n} = \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^{6n} =$$
$$= \cos(-2\pi n) + i\sin(-2\pi n) \to 1.$$

Exercise 10.
Let (z_n) be a sequence of complex numbers, $r \in \mathbb{R}^+$ and $\varphi \in \mathbb{R}$. Proof the following propositions:

a) $z_n \to 0 \;\Leftrightarrow\; \frac{1}{z_n} \to \infty$;

b) $\left.\begin{array}{r} |z_n| \to r \\ \arg z_n \to \varphi \end{array}\right\} \;\Rightarrow\; z_n \to r\big(\cos\varphi + i\sin\varphi\big)$;

and show that the implication in the proposition b) cannot be reversed.

Solution:

a) It is enough to rewrite both sides of the equivalence using the definition of the limit.

- Left side:

$$z_n \to 0$$
$$\Updownarrow$$
$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ z_n \in U(0,\varepsilon)$$
$$\Updownarrow$$
$$\underline{(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ |z_n| < \varepsilon;}$$

- Right side:

$$\frac{1}{z_n} \to \infty$$
$$\Updownarrow$$
$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ \frac{1}{z_n} \in U(\infty,\varepsilon)$$
$$\Updownarrow$$
$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ \left(\left|\frac{1}{z_n}\right| > \frac{1}{\varepsilon} \vee \frac{1}{z_n} = \infty\right)$$
$$\Updownarrow$$
$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ (\varepsilon > |z_n| \vee z_n = 0)$$
$$\Updownarrow$$
$$\underline{(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n \in \mathbb{N},\ n > n_0):\ |z_n| < \varepsilon.}$$

b) From the assumtions it follows that for all sufficienlty large n we have that

$$z_n = |z_n| \left(\cos\left(\arg z_n\right) + i \sin\left(\arg z_n\right)\right),$$

and the claim follows directly from the continuity of cosine and sine and the theorem of the limit of a product.

As a counterexample disproving the reverse inequality we can use the sequence

$$z_n := \cos\left(\pi + \frac{(-1)^n}{n}\right) + i \sin\left(\pi + \frac{(-1)^n}{n}\right)$$

and the choice

$$r = 1,\ \varphi = \pi.$$

Exercise 11.

Find all $z \in \mathbb{C}$ such that

a) $z^3 = 1$;

b) $z^2 = i$;

c) $z^2 = 24i - 7$;

d) $\left(\frac{z-1}{z+1}\right)^2 = 2i$;

e) $z^4 = -1$;

f) $z^3 = i - 1$;

g) $z^5 = 1$;

h) $z^2 = -11 + 60i$;

i) $z^2 = 3 + 4i$.

Solution:

a) $z = |z| \left(\cos\varphi + i \sin\varphi\right),\ 1 = \cos 0 + i \sin 0.$

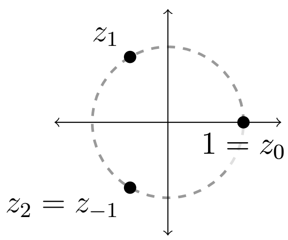
$$\begin{gathered} z^3 = |z|^3 \left(\cos\left(3\varphi\right) + i \sin\left(3\varphi\right)\right) = 1 \left(\cos 0 + i \sin 0\right) \\ \Updownarrow \\ \left(|z|^3 = 1\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ 3\varphi = 0 + 2k\pi\right) \\ \Updownarrow \\ \left(|z| = 1\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ \varphi = k\tfrac{2\pi}{3}\right), \end{gathered}$$

and therefore

$$z = z_k = \cos\left(k\frac{2\pi}{3}\right) + i \sin\left(k\frac{2\pi}{3}\right) = \begin{cases} 1, & k \in \{3l\colon\ l \in \mathbb{Z}\}, \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2}, & k \in \{3l+1\colon\ l \in \mathbb{Z}\}, \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}, & k \in \{3l+2\colon\ l \in \mathbb{Z}\}, \end{cases}$$

so

$$z^3 = 1 \Leftrightarrow \underline{z \in \left\{1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\}}.$$

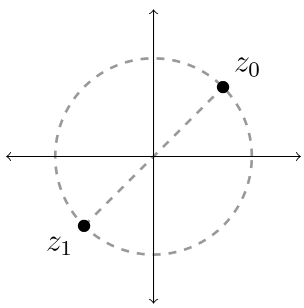


b) $z = |z|\left(\cos\varphi + i\sin\varphi\right),\ i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$,

$$z^2 = |z|^2\left(\cos(2\varphi) + i\sin(2\varphi)\right) = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}$$
$$\Updownarrow$$
$$\left(|z|^2 = 1\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ 2\varphi = \tfrac{\pi}{2} + 2k\pi\right),$$

and therefore

$$z = z_k = \cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right) =$$
$$= \begin{cases} \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, & k \in \{2l : l \in \mathbb{Z}\}, \\ -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, & k \in \{2l+1\colon\ l \in \mathbb{Z}\}. \end{cases}$$



$$z^2 = i \Leftrightarrow \underline{z \in \left\{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right\}}.$$

c) Let $z = x + iy$. Then

$$z^2 = x^2 + 2ixy - y^2 = 24i - 7 \Leftrightarrow \begin{pmatrix} x^2 - y^2 & = & -7 \\ 2xy & = & 24 \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^2 - y^2 & = & -7 \\ y & = & \frac{12}{x} \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^2 - \frac{144}{x^2} & = & -7 \\ y & = & \frac{12}{x} \end{pmatrix} \Leftrightarrow$$
$$\Leftrightarrow \begin{pmatrix} x^4 + 7x^2 - 144 & = & 0 \\ y & = & \frac{12}{x} \end{pmatrix},$$

which holds if and only if $\underline{z = x + iy = 3 + 4i \text{ or } z = -3 - 4i}$.

d) After the change of variables $\frac{z-1}{z+1} =: u = |u|\,(\cos\varphi + i\sin\varphi)$ we firstly solve the equation $u^2 = 2i$, that is

$$|u|^2\,(\cos(2\varphi) + i\sin(2\varphi)) = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right).$$

The solution is

$$u = \pm\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \pm(1+i),$$

and then easily $\underline{\frac{z-1}{z+1} = 1+i}$ if and only if $(z = x + iy)$

$$\begin{aligned} x + iy - 1 &= (1+i)(x+iy+1), \text{ that is} \\ (x-1) + iy &= (x-y+1) + i(x+y+1), \text{ and therefore} \\ (x-1 = x-y+1) &\wedge (y = x+y+1), \text{ that is} \\ y &= 2 \wedge x = -1, \end{aligned}$$

and similarly $\underline{\frac{z-1}{z+1} = -1-i}$ if and only if

$$\begin{aligned} x + iy - 1 &= -(1+i)(x+iy+1), \\ (x-1 = -x+y-1) &\wedge (2y = -x-1), \text{ and therefore} \\ y = -\frac{2}{5} &\wedge x = -\frac{1}{5}. \end{aligned}$$

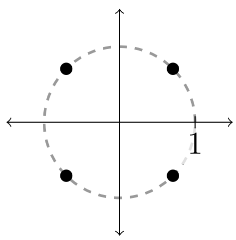
Summary:

$$\underline{\left(\frac{z-1}{z+1}\right)^2 = 2i \Leftrightarrow \left(z = -1 + 2i \vee z = -\frac{1}{5} - \frac{2}{5}i\right).}$$

e) $|z|^4\,(\cos(4\varphi) + i\sin(4\varphi)) = \cos\pi + i\sin\pi$ if and only if

$$z = z_k = \cos\left(\frac{\pi}{4} + k\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + k\frac{\pi}{2}\right), \;\; k \in \mathbb{Z}, \text{ that is}$$

$$z^4 = -1 \Leftrightarrow \underline{z \in \left\{\frac{1+i}{\sqrt{2}},\ \frac{-1+i}{\sqrt{2}},\ \frac{-1-i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}}\right\}.}$$



f)

$$|z|^3\,(\cos(3\varphi) + i\sin(3\varphi)) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

if and only if

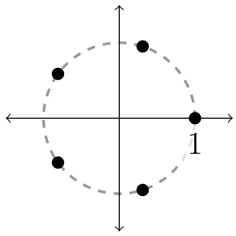
$$\left(|z| = \sqrt[3]{\sqrt{2}}\right) \wedge \left(3\varphi = \frac{3\pi}{4} + 2k\pi,\ k \in \mathbb{Z}\right).$$

From this it easily follows that $z^3 = i - 1$ if and only if

$$\underline{z \in \left\{\sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right):\ k \in \{0,\ 1,\ 2\}\right\}.}$$

g)

$$\underline{z = \cos\left(\frac{2\pi}{5}k\right) + i\sin\left(\frac{2\pi}{5}k\right),\ k \in \{0,\ 1,\ 2,\ 3,\ 4\}}.$$



h)

$$\begin{array}{rcl} z^2 = (x+iy)^2 & = & -11+60i \\ & \Updownarrow & \\ x^2 + 2ixy - y^2 & = & 11+60i \\ & \Updownarrow & \\ x^2 - y^2 = -11 & \wedge & 2xy = 60 \\ & \Updownarrow & \\ x^2 - \frac{900}{x^2} = -11 & \wedge & y = \frac{30}{x} \\ & \Updownarrow & \end{array}$$

$$y = \tfrac{30}{x} \wedge x^2 = \frac{-11 \pm \sqrt{121+3600}}{2} = \begin{cases} \dfrac{-11-\sqrt{3721}}{2} & \ldots \text{ not possible,} \\ \dfrac{-11+\sqrt{3721}}{2} = \dfrac{-11+61}{2} = 25, & \end{cases}$$

and therefore

$$z^2 = -11 + 60i \Leftrightarrow \underline{z = \pm(5+6i)}.$$

i) Let $z = x + iy$. Then

$$\begin{array}{rcl} z^2 = (x+iy)^2 & = & 3+4i \\ & \Updownarrow & \\ x^2 - y^2 = 3 & \wedge & 2xy = 4 \\ & \Updownarrow & \\ x^2 - \frac{4}{x^2} = 3 & \wedge & y = \frac{2}{x} \\ & \Updownarrow & \end{array}$$

$$y = \tfrac{2}{x} \wedge x^2 = \frac{3 \pm \sqrt{9+16}}{2} = \begin{cases} \frac{3-5}{2} & \ldots \text{ not possible,} \\ 4, & \end{cases}$$

and therefore

$$z^2 = 3 + 4i \Leftrightarrow \underline{z = \pm(2+i)}.$$

<u>Exercise 12.</u>
Find and draw the set $M = \left\{ \frac{1}{z} \colon\ z \in \Omega \right\}$, if

a) $\Omega = \{ z \in \mathbb{C} \colon\ \arg z = \alpha \},\ \alpha \in (-\pi, \pi\rangle$;

b) $\Omega = \{ z \in \mathbb{C} \colon\ |z - 1| = 1 \}$;

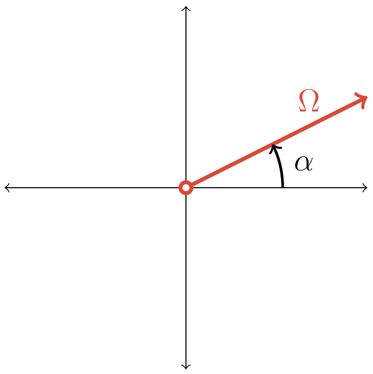
c) $\Omega = \{ z \in \mathbb{C} \colon\ \operatorname{Re} z = \operatorname{Im} z \}$;

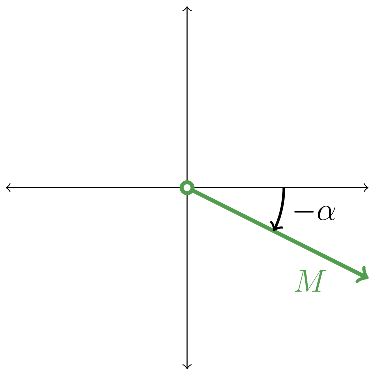
d) $\Omega = \{ x + iy \in \mathbb{C} \colon\ x = 1 \}$;

e) $\Omega = \{ x + iy \in \mathbb{C} \colon\ y = 0 \}$.

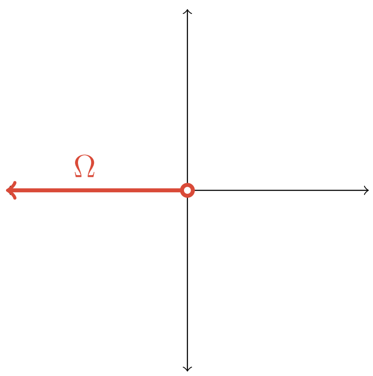
<u>Solution:</u>

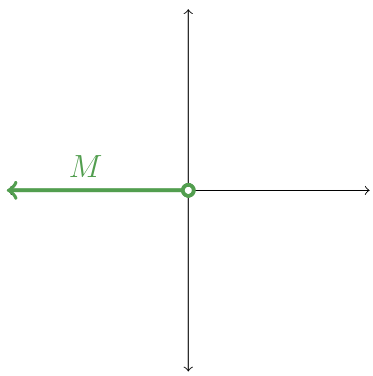
a) $\underline{\alpha \in (-\pi, \pi) \Rightarrow M = \{ z \in \mathbb{C} \colon\ \arg z = -\alpha \}}$;





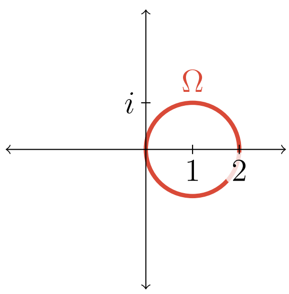
$\underline{\alpha = \pi \Rightarrow M = \Omega = \{ z \in \mathbb{C} \colon\ \arg z = \pi \}}$.

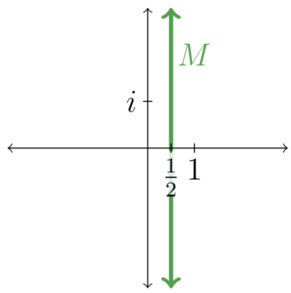




b)

$$
\begin{aligned}
\underline{M =} \left\{u+iv\colon \ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} = \\
= \left\{u+iv\colon \ \left|\frac{1}{u+iv} - 1\right| = 1\right\} \cup \{\infty\} = \\
= \{u+iv\colon \ |1-u-iv| = |u+iv|\} \cup \{\infty\} = \\
= \left\{u+iv\colon \ (1-u)^2+v^2 = u^2+v^2\right\} \cup \{\infty\} = \\
= \{u+iv\colon \ 1-2u = 0\} \cup \{\infty\} = \\
= \underline{\left\{u+iv\colon \ u = \frac{1}{2}\right\} \cup \{\infty\}}.
\end{aligned}
$$



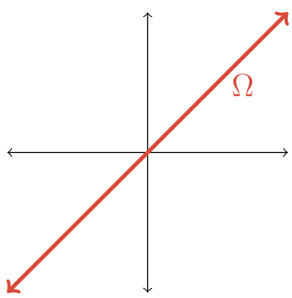


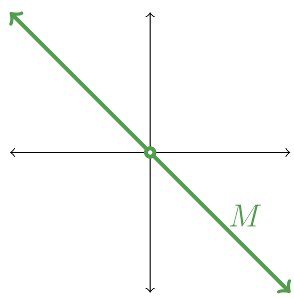
c)

$$
\begin{aligned}
M &= \left\{u+iv\colon \ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} = \\
&= \left\{u+iv\colon \ \frac{u-iv}{u^2+v^2} \in \Omega\right\} \cup \{\infty\},
\end{aligned}
$$

and because $\frac{u}{u^2+v^2} = \frac{-v}{u^2+v^2} \Leftrightarrow (u \neq 0 \wedge u = -v)$, we have that

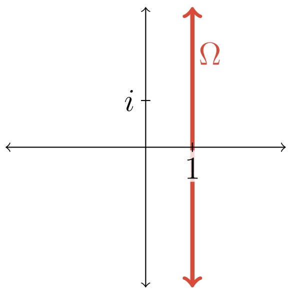
$$
\underline{M = \{u+iv\colon \ u \neq 0 \wedge u = -v\} \cup \{\infty\}}.
$$

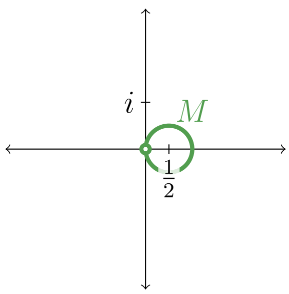




d)

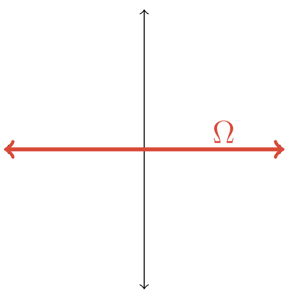
$$\begin{aligned}\underline{M =} \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} &= \\ = \left\{u+iv\colon\ \frac{u}{u^2+v^2} = 1\right\} &= \\ = \underline{\left\{u+iv\colon\ \left(u-\frac{1}{2}\right)^2 + v^2 = \frac{1}{4}\right\} \setminus \{0\}}.\end{aligned}$$

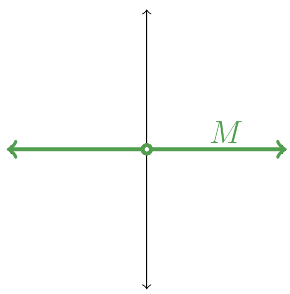




e)

$$\begin{aligned}\underline{M =} \left\{u+iv\colon\ \frac{1}{u+iv} \in \Omega\right\} \cup \{\infty\} &= \\ = \left\{u+iv\colon\ \frac{-v}{u^2+v^2} = 0\right\} \cup \{\infty\} &= \\ = \underline{\{u+iv\colon\ v = 0 \neq u\} \cup \{\infty\}}.\end{aligned}$$





EXERCISE 13.

Find and draw the set $M = \{f(z)\colon\ z \in \Omega\}$, if

a) $\Omega = \{z \in \mathbb{C}\colon\ |\arg z| \leq \frac{\pi}{6}\},\ f(z) := z^2;$

b) $\Omega = \{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < \frac{\pi}{2}\},\ f(z) := e^z;$

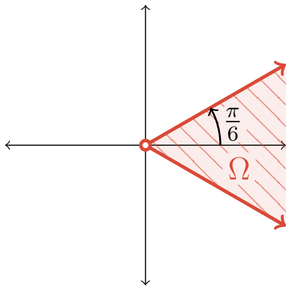
c) $\Omega = \{z \in \mathbb{C}\colon\ 0 < \operatorname{Re} z < \pi\ \wedge\ \operatorname{Im} z > 0\},\ f(z) := e^{iz};$

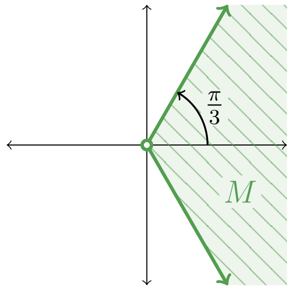
d) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Im} z = \frac{1}{2}\},\ f(z) := z^2.$

Solution:

a)

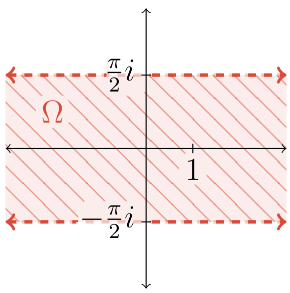
$$\underline{M = \left\{ z \in \mathbb{C}\colon\ |\arg z| \leq \frac{\pi}{6} \cdot 2 = \frac{\pi}{3} \right\}}.$$

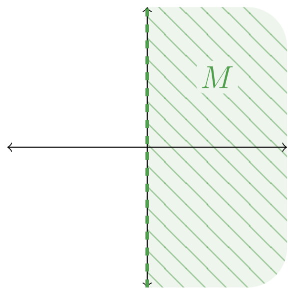




b)

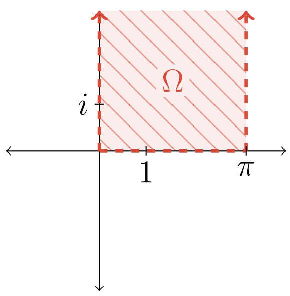
$$\underline{M =} \left\{ e^{x+iy}\colon\ |y| < \frac{\pi}{2} \right\} =$$
$$= \left\{ e^{x} \left(\cos\left(y\right) + i \sin\left(y\right)\right)\colon\ |y| < \frac{\pi}{2} \right\} =$$
$$= \underline{\left\{ z \in \mathbb{C}\colon\ \operatorname{Re} z > 0 \right\}}.$$

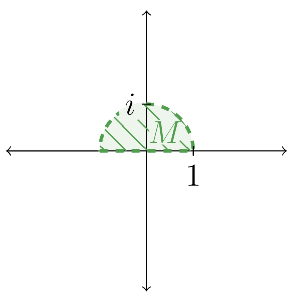




c)

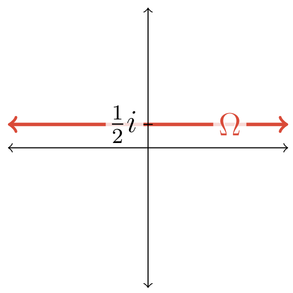
$$\underline{M =} \left\{ e^{i(x+iy)} = e^{-y} \left(\cos x + i \sin x\right)\colon\ 0 < x < \pi\ \wedge\ y > 0 \right\} =$$
$$= \underline{\left\{ z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \operatorname{Im} z > 0 \right\}}.$$

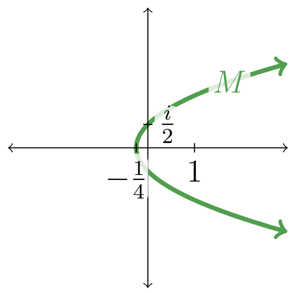




d)

$$\underline{M =} \left\{ \left(x + \frac{1}{2} i \right)^2 : \ x \in \mathbb{R} \right\} = $$
$$= \left\{ x^2 - \frac{1}{4} + xi : \ x \in \mathbb{R} \right\} = $$
$$= \underline{\left\{ y^2 - \frac{1}{4} + yi : \ y \in \mathbb{R} \right\}}.$$





<u>EXERCISE 14.</u>
Compute

a) $\sin(2-3i)$;

b) $\cos i$;

c) $\cosh i$;

d) $\mathrm{Ln}(-5+3i)$ a $\ln(-5+3i)$;

e) $\mathrm{Ln}(-4-\sqrt{3}i)$ a $\ln(-4-\sqrt{3}i)$;

f) $\mathrm{Ln}(ie^2)$.

<u>Solution:</u>

a)

$$\underline{\sin(2-3i) =} \frac{e^{i(2-3i)} - e^{-i(2-3i)}}{2i} = $$
$$= \frac{e^3\left(\cos(2) + i\sin(2)\right) - e^{-3}\left(\cos(-2) + i\sin(-2)\right)}{2i} = $$
$$= \frac{e^3 - e^{-3}}{2i} \cdot \cos 2 + \frac{i(e^3 + e^{-3}) \cdot \sin 2}{2i} = $$
$$= \underline{\cosh 3 \cdot \sin 2 - (\sinh 3 \cdot \cos 2)i} \doteq $$
$$\doteq 9.15 + 4.17i.$$

b)

$$\underline{\cos i =} \frac{e^{i \cdot i} + e^{-i \cdot i}}{2} = \underline{\cosh 1} \doteq 1.54.$$

c)

$$\underline{\cosh i =}\frac{e^i + e^{-i}}{2} = \frac{\cos 1 + i \sin 1 + \cos(-1) + i \sin(-1)}{2} = \underline{\cos 1} \doteq 0.54.$$

d)

$$-5 + 3i = \sqrt{34}\left(\cos\left(\frac{\pi}{2} + \arctan\frac{5}{3}\right) + i \sin\left(\frac{\pi}{2} + \arctan\frac{5}{3}\right)\right),$$

and therefore

$$\underline{\mathrm{Ln}(-5 + 3i) = \ln\sqrt{34} + i\left(\frac{\pi}{2} + \arctan\frac{5}{3}\right) + 2k\pi i,\ k \in \mathbb{Z};}$$

$$\underline{\ln(-5 + 3i) = \ln\sqrt{34} + i\left(\frac{\pi}{2} + \arctan\frac{5}{3}\right).}$$

e)

$$-4 - \sqrt{3}i = \sqrt{19}\left(\cos\left(-\pi + \arctan\frac{\sqrt{3}}{4}\right) + i \sin\left(-\pi + \arctan\frac{\sqrt{3}}{4}\right)\right),$$

and therefore

$$\underline{\mathrm{Ln}(-4 - \sqrt{3}i) = \ln\sqrt{19} + i\left(-\pi + \arctan\frac{\sqrt{3}}{4}\right) + 2k\pi i,\ k \in \mathbb{Z};}$$

$$\underline{\ln(-4 - \sqrt{3}i) = \ln\sqrt{19} + i\left(-\pi + \arctan\frac{\sqrt{3}}{4}\right).}$$

f)

$$\underline{\mathrm{Ln}(ie^2)} = \ln(e^2) + i\frac{\pi}{2} + 2k\pi i =$$

$$= \underline{2 + \frac{\pi}{2}i + 2k\pi i,\ k \in \mathbb{Z}.}$$

EXERCISE 15.
Find all $z \in \mathbb{C}$, for which we have that

a) $\sin z = 3$;

b) $\cos z = \frac{\sqrt{3}}{2}$;

c) $\sin z + \cos z = 2$;

d) $\sin z - \cos z = 3$;

e) $z^2 + 2z + 9 + 6i = 0$.

Solution:

a)

$$\sin z = 3$$
$$\Updownarrow$$
$$e^{iz} - e^{-iz} = 6i$$
$$\Updownarrow$$
$$e^{2iz} - 6ie^{iz} - 1 = 0$$

and from that we get for $z = x + iy$

$$e^{iz} = e^{i(x+iy)} = \frac{6i \pm \sqrt{-36+4}}{2} = (3 \pm \sqrt{8})i$$
$$\Updownarrow$$
$$e^{-y}(\cos x + i \sin x) = (3 \pm \sqrt{8})i$$
$$\Updownarrow$$
$$e^{-y} = 3 \pm \sqrt{8} \ \wedge \ \left(x = \frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}\right)$$
$$\Updownarrow$$
$$\underline{z = \frac{\pi}{2} + 2k\pi - i \ln(3 \pm \sqrt{8}), \ k \in \mathbb{Z}.}$$

b)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{\sqrt{3}}{2}$$
$$\Updownarrow$$
$$e^{2iz} - \sqrt{3}e^{iz} + 1 = 0$$
$$\Updownarrow$$
$$e^{iz} = e^{(x+iy)i} = e^{-y}(\cos x + i \sin x) = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} =$$
$$= \frac{\sqrt{3}}{2} \pm \frac{i}{2} = \begin{cases} \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \\ \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \end{cases}$$
$$\Updownarrow$$
$$e^{-y} = 1 \ \wedge \ \left(x = \pm\frac{\pi}{6} + 2k\pi, \ k \in \mathbb{Z}\right)$$
$$\Updownarrow$$
$$\underline{z = \pm\frac{\pi}{6} + 2k\pi, \ k \in \mathbb{Z}.}$$

c)

$$\frac{e^{iz}-e^{-iz}}{2i}+\frac{e^{iz}+e^{-iz}}{2}=2$$

$$\Updownarrow$$

$$e^{iz}-e^{-iz}+ie^{iz}+ie^{-iz}=4i$$

$$\Updownarrow$$

$$e^{2iz}(1+i)-4ie^{iz}+(i-1)=0$$

$$\Updownarrow$$

$$e^{iz}=\frac{4i\pm\sqrt{-16-4(1+i)(i-1)}}{2(1+i)}=\frac{4i\pm\sqrt{2}i}{2(1+i)}=$$

$$=\frac{(2\pm\sqrt{2})i}{1+i}=\frac{(2\pm\sqrt{2})(1+i)}{2},$$

and from that

$$iz=\operatorname{Ln}\frac{(2\pm\sqrt{2})(1+i)}{2}=\ln\frac{(2\pm\sqrt{2})\sqrt{2}}{2}+i\frac{\pi}{4}+2k\pi i,\ \ k\in\mathbb{Z},$$

$$\underline{z=\frac{\pi}{4}+2k\pi-i\ln(\sqrt{2}\pm 1),\ \ k\in\mathbb{Z}.}$$

d)

$$\frac{e^{iz}-e^{-iz}}{2i}-\frac{e^{iz}+e^{-iz}}{2}=3$$

$$\Updownarrow$$

$$e^{iz}-e^{-iz}-ie^{iz}-ie^{-iz}=6i$$

$$\Updownarrow$$

$$e^{2iz}(1-i)-6ie^{iz}-(1+i)=0$$

$$\Updownarrow$$

$$e^{iz}=\frac{6i\pm\sqrt{-36+4(1-i)(1+i)}}{2(1-i)}=\frac{(6\pm 2\sqrt{7})i}{2(1-i)}=$$

$$=\frac{3\pm\sqrt{7}}{2}\cdot(-1+i),$$

and therefore

$$iz=\operatorname{Ln}\left(\frac{3\pm\sqrt{7}}{2}(-1+i)\right)=\ln\left(\frac{3\pm\sqrt{7}}{2}\cdot\sqrt{2}\right)+i\frac{3}{4}\pi+2k\pi i,\ \ k\in\mathbb{Z},$$

$$\underline{z=\frac{3}{4}\pi+2k\pi-i\ln\left(\frac{3\pm\sqrt{7}}{\sqrt{2}}\right),\ \ k\in\mathbb{Z}.}$$

e)

$$\underline{z =} \frac{-2 \pm \sqrt{4 - 4(9+6i)}}{2} =$$

$$= -1 \pm \sqrt{1 - (9+6i)} = -1 \pm \sqrt{-8-6i} =$$

$$= \underline{\begin{cases} -2+3i, \\ -3i, \end{cases}}$$

because

$$\sqrt{-8-6i} = \sqrt{10\left(\cos\left(\pi + \arctan\frac{3}{4}\right) + i \sin\left(\pi + \arctan\frac{3}{4}\right)\right)} =$$

$$= \pm\sqrt{10} \cdot \left(\cos\left(\frac{\pi}{2} + \frac{1}{2}\arctan\frac{3}{4}\right) + i \sin\left(\frac{\pi}{2} + \frac{1}{2}\arctan\frac{3}{4}\right)\right) =$$

$$= \pm(-1+3i).$$

<u>EXERCISE 16</u>.
Compute

a) 2^i;

b) $(-2)^{\sqrt{2}}$;

c) $\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}$;

d) $i^{\frac{3}{4}}$;

e) $(-1)^{\sqrt{3}}$;

f) $(-\sqrt{3}i+1)^{-3}$.

<u>Solution:</u>

a)

$$\underline{2^i =} \exp(i \operatorname{Ln} 2) =$$

$$= \exp(i(\ln 2 + 2k\pi i)) = \exp(-2k\pi + i \ln 2) =$$

$$= \underline{e^{-2k\pi}\left(\cos(\ln 2) + i \sin(\ln 2)\right), \ k \in \mathbb{Z}}.$$

b)

$$\underline{(-2)^{\sqrt{2}} =} \exp\left(\sqrt{2}\operatorname{Ln}(-2)\right) = \exp\left(\sqrt{2}(\ln 2 + \pi i + 2k\pi i)\right) =$$

$$= \underline{2^{\sqrt{2}}\left(\cos\left((2k+1)\pi\sqrt{2}\right) + i \sin\left((2k+1)\pi\sqrt{2}\right)\right), \ k \in \mathbb{Z}}.$$

c)

$$\underline{\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}} = \exp\left((1+i)\operatorname{Ln}\left(\frac{1-i}{\sqrt{2}}\right)\right) =$$

$$= \exp\left((1+i)(\ln 1 - i\frac{\pi}{4} + 2k\pi i)\right) = \exp\left(\frac{\pi}{4} - 2k\pi + (2k\pi - \frac{\pi}{4})i\right) =$$

$$= e^{\frac{\pi}{4} - 2k\pi}\left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right) = e^{\frac{\pi}{4} - 2k\pi}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) =$$

$$= \underline{\frac{1-i}{\sqrt{2}} e^{\frac{\pi}{4} - 2k\pi}, \ k \in \mathbb{Z}}.$$

d)

$$\underline{i^{\frac{3}{4}}} = e^{\frac{3}{4}\operatorname{Ln} i} = e^{\frac{3}{4}(\frac{\pi}{2}i+2k\pi i)} =$$

$$= \underline{\cos\left(\frac{3}{8}\pi + \frac{3}{2}k\pi\right) + i\sin\left(\frac{3}{8}\pi + \frac{3}{2}k\pi\right),\ k \in \{0,1,2,3\}}.$$

e)

$$\underline{(-1)^{\sqrt{3}}} = e^{\sqrt{3}\operatorname{Ln}(-1)} = e^{\sqrt{3}(\pi i+2k\pi i)} =$$

$$= \underline{\cos\left(\sqrt{3}\pi + 2k\pi\sqrt{3}\right) + i\sin\left(\sqrt{3}\pi + 2k\pi\sqrt{3}\right),\ k \in \mathbb{Z}}.$$

f)

$$\underline{(-\sqrt{3}i+1)^{-3}} = \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^{-3} =$$

$$= \frac{1}{\left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^3} =$$

$$= \frac{1}{8\cos(-\pi) + i\sin(-\pi)} = \underline{-\frac{1}{8}}.$$

Differently:

$$\underline{(-\sqrt{3}i+1)^{-3}} = e^{-3\operatorname{Ln}\left(-\sqrt{3}i+1\right)} = e^{-3\left(\ln 2 - \frac{\pi}{3}i + 2k\pi i\right)} =$$

$$= \frac{1}{8}\left(\cos\pi + i\sin\pi\right) = \underline{-\frac{1}{8}}.$$

Exercise 17.

Find the real and imaginary part of the function $f\colon\ \mathbb{C} \to \mathbb{C}$ defined as

a) $f(z) := \sin z$;

b) $f(z) := z^2\cos z$;

c) $f(z) := z^3 + 5z - 1$;

d) $f(z) := |z|\,\overline{z}$;

e) $f(z) := z^2\,\overline{z}$;

f) $f(z) := \frac{1}{z}$.

Solution:

a)

$$f(z) = f(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} =$$

$$= \frac{e^{-y}\left(\cos x + i\sin x\right) - e^{y}\left(\cos x - i\sin x\right)}{2i} =$$

$$= \frac{e^{y}+e^{-y}}{2}\sin x + i\,\frac{e^{y}-e^{-y}}{2}\cos x.$$

Therefore

$$\underline{(\operatorname{Re} f)(x,y) = \cosh y\ \sin x},$$
$$\underline{(\operatorname{Im} f)(x,y) = \sinh y\ \cos x}.$$

b)

$$f(x+iy) = (x^2 - y^2 + 2xyi)\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} =$$
$$= (x^2 - y^2 + 2xyi)\frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2},$$

and therefore

$$\underline{(\operatorname{Re} f)(x,y) = (x^2 - y^2)\cosh y\, \cos x + 2xy \sinh y\, \sin x},$$
$$\underline{(\operatorname{Im} f)(x,y) = -(x^2 - y^2)\sinh y\, \sin x + 2xy \cosh y\, \cos x}.$$

c)

$$f(x+iy) = (x^2 - y^2 + 2xyi)(x+iy) + 5(x+iy) - 1 =$$
$$= (x^3 - xy^2 - 2xy^2 + 5x - 1) + i\left(2x^2y + x^2y - y^3 + 5y\right).$$

So

$$\underline{(\operatorname{Re} f)(x,y) = x^3 - 3xy^2 + 5x - 1},$$
$$\underline{(\operatorname{Im} f)(x,y) = 3x^2y - y^3 + 5y}.$$

d)

$$f(x+iy) = \sqrt{x^2+y^2}\,(x - iy) =$$
$$= x\sqrt{x^2+y^2} - iy\sqrt{x^2+y^2}.$$

We computed that

$$\underline{(\operatorname{Re} f)(x,y) = x\sqrt{x^2+y^2}},$$
$$\underline{(\operatorname{Im} f)(x,y) = -y\sqrt{x^2+y^2}}.$$

e)

$$f(x+iy) = (x^2 - y^2 + 2xyi)(x - iy) =$$
$$= x^3 - xy^2 + 2xy^2 + i(2x^2y - x^2y + y^3).$$

Therefore

$$\underline{(\operatorname{Re} f)(x,y) = x^3 + xy^2},$$
$$\underline{(\operatorname{Im} f)(x,y) = x^2y + y^3}.$$

f)

$$f(x+iy) = \frac{x - iy}{x^2+y^2},$$

therefore

$$\underline{(\operatorname{Re} f)(x,y) = \frac{x}{x^2+y^2}},$$
$$\underline{(\operatorname{Im} f)(x,y) = -\frac{y}{x^2+y^2}}.$$

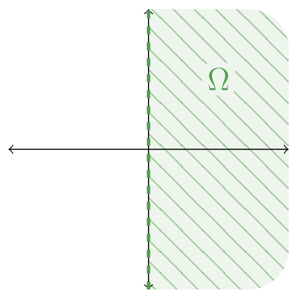
EXERCISE 18.
Decide if the function $f(z) := z^3$ is injective on the set Ω, if

a) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}$;

b) $\Omega = \{z \in \mathbb{C}\colon\ \arg z \in \langle 0, \frac{\pi}{4})\}$.

Solution:

a)



Let us choose

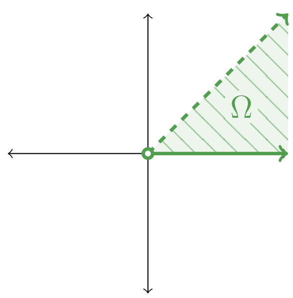
$$z_1 = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) \in \Omega,$$
$$z_2 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \in \Omega.$$

Then

$$\begin{aligned} z_1^3 &= \cos -\pi + i \sin -\pi &= -1, \\ z_2^3 &= \cos \pi + i \sin \pi &= -1, \end{aligned}$$

and therefore f is not injective on Ω.

b)



Let

$$z_1 = |z_1| \left(\cos\left(\varphi_1\right) + i \sin\left(\varphi_1\right)\right),$$
$$z_2 = |z_2| \left(\cos\left(\varphi_2\right) + i \sin\left(\varphi_2\right)\right),$$

where $\varphi_1,\ \varphi_2 \in \left\langle 0, \frac{\pi}{4}\right)$. Then

$$z_1^3 = z_2^3$$
$$\Updownarrow$$
$$|z_1|^3 \left(\cos\left(3\varphi_1\right) + i\sin\left(3\varphi_1\right)\right) = |z_2| \left(\cos\left(3\varphi_2\right) + i\sin\left(3\varphi_2\right)\right)$$
$$\Updownarrow$$
$$\left(|z_1| = |z_2|\right) \wedge \left(\exists k \in \mathbb{Z}\colon\ 3\varphi_1 = 3\varphi_2 + 2k\pi\right).$$

From that it follows that (we are using the assumption $\varphi_1,\ \varphi_2 \in \left\langle 0, \frac{\pi}{4}\right)$):

$$\begin{array}{c} z_1,\ z_2 \in \Omega \\ z_1^3 = z_2^3 \end{array} \Rightarrow \begin{array}{c} |z_1| = |z_2| \\ \varphi_1 = \varphi_2 \end{array} \Rightarrow z_1 = z_2,$$

therefore the function f is injective on Ω.

Exercise 19.

Decide if the given limit exists, and if it does compute it

a) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{z}$;

b) $\lim\limits_{z\to 0} \frac{\operatorname{Im}(z^2)}{z\bar{z}}$;

c) $\lim\limits_{z\to 0} \frac{z \operatorname{Im} z}{|z|}$;

d) $\lim\limits_{z\to 0} \frac{z^2}{|z|^2}$;

e) $\lim\limits_{z\to 0} \frac{z^3}{|z|^2}$;

f) $\lim\limits_{z\to i} \frac{z^2+z(2-i)-2i}{z^2+1}$;

g) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{1+|z|}$.

Solution:

a) $\lim\limits_{z\to 0} \frac{\operatorname{Re} z}{z}$ does not exist, because

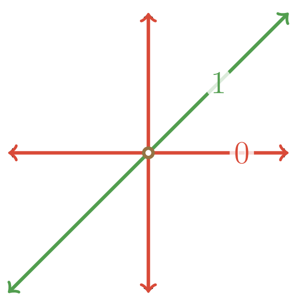
$$0 \neq \frac{1}{n} \to 0 \ \wedge\ \frac{\operatorname{Re}\left(\frac{1}{n} + 0i\right)}{\frac{1}{n}} = 1 \to 1$$

and at the same time

$$0 \neq i\frac{1}{n} \to 0 \ \wedge\ \frac{\operatorname{Re}\left(i\frac{1}{n}\right)}{\frac{1}{n}} = 0 \to 0.$$

b) $\lim\limits_{z\to 0} \frac{\operatorname{Im} z^2}{z\cdot\bar{z}}$ does not exist, because for $0 \neq z = x + iy$ we have that

$$\frac{\operatorname{Im} z^2}{z\cdot\bar{z}} = \frac{2xy}{x^2+y^2} = \begin{cases} 1, & x = y \neq 0, \\ 0, & x\cdot y = 0,\ x^2+y^2 \neq 0. \end{cases}$$

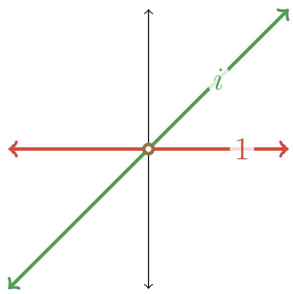


c) $\underline{\lim_{z\to 0} \frac{z \operatorname{Im} z}{|z|} = 0}$, because

$$0 \neq z_n \to 0 \;\Rightarrow\; \left|\frac{z_n \operatorname{Im} z_n}{|z_n|}\right| = |\operatorname{Im} z_n| \to 0 \;\Rightarrow\; \frac{z_n \operatorname{Im} z_n}{|z_n|} \to 0.$$

d) $\underline{\lim_{z\to 0} \frac{z^2}{|z|^2} \text{ does not exist}}$, because for $0 \neq z = x + iy$ we have that

$$\frac{z^2}{|z|^2} = \frac{x^2 - y^2 + 2ixy}{x^2 + y^2} = \begin{cases} i, & x = y \neq 0, \\ 1, & y = 0 \neq x. \end{cases}$$



e) $\underline{\lim_{z\to 0} \frac{z^3}{|z|^2} = 0}$, because

$$\lim_{z\to 0} \left|\frac{z^3}{|z|^2}\right| = \lim_{z\to 0} |z| = 0.$$

f)

$$\underline{\lim_{z\to i} \frac{z^2 + z(2-i) - 2i}{z^2 + 1}} = \lim_{z\to i} \frac{(z-i)(z+2)}{(z-i)(z+i)} = \underline{\lim_{z\to i} \frac{z+2}{z+i}} =$$

$$= \lim_{x+iy\to i} \frac{x + 2 + iy}{x + i(y+1)} =$$

$$= \lim_{(x,y)\to(0,1)} \frac{x(x+2) + y(y+1)}{x^2 + (y+1)^2} +$$

$$+ i \lim_{(x,y)\to(0,1)} \frac{xy - (x+2)(y+1)}{x^2 + (y+1)^2} =$$

$$= \frac{1\cdot 2}{2^2} + i\frac{-2\cdot 2}{4} = \underline{\frac{1}{2} - i}.$$

Alternatively we can use the continuity of the function $f(z) := \frac{z+2}{z+i}$ at the point i:

$$\underline{\lim_{z\to i} \frac{z+2}{z+i}} = \frac{2+i}{2i} = \underline{\frac{1}{2} - i}.$$

g)

$$\underline{\lim_{z\to 0} \frac{\operatorname{Re} z}{1 + |z|}} = \lim_{(x,y)\to(0,0)} \frac{x}{1 + \sqrt{x^2 + y^2}} = \frac{0}{1} = \underline{0}.$$

Exercise 20.

Draw the set $\langle\varphi\rangle := \{\varphi(t)\colon\ t \in D\varphi\}$, if

a) $\varphi(t) := 1 - it,\ D\varphi = \langle 0, 2\rangle;$

b) $\varphi(t) := t - it^2,\ D\varphi = \langle -1, 2\rangle;$

c) $\varphi(t) := 1 + e^{-it},\ D\varphi = \langle 0, 2\pi\rangle;$

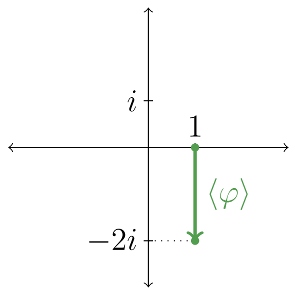
d) $\varphi(t) := e^{2it} - 1,\ D\varphi = \langle 0, 2\pi\rangle;$

e) $\varphi(t) := \begin{cases} e^{i\pi t}, & t \in \langle 0, 1), \\ t - 2, & t \in \langle 1, 3\rangle; \end{cases}$

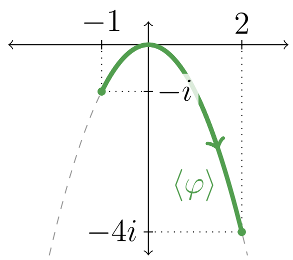
f) $\varphi(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi), \\ \frac{3t}{\pi} - 4, & t \in \langle \pi, 2\pi\rangle. \end{cases}$

Solution:

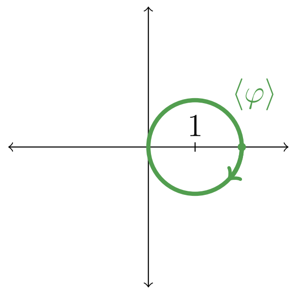
a) $\varphi(t) := 1 - it,\ D\varphi = \langle 0, 2\rangle.$



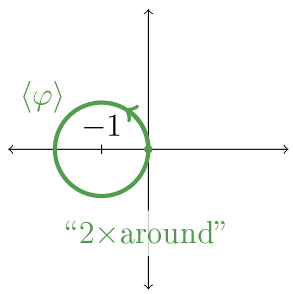
b) $\varphi(t) := t - it^2,\ D\varphi = \langle -1, 2\rangle.$



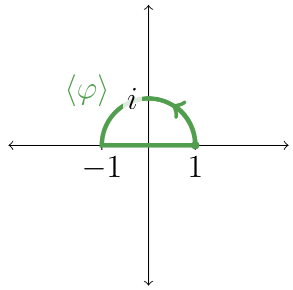
c) $\varphi(t) := 1 + e^{-it},\ D\varphi = \langle 0, 2\pi\rangle.$



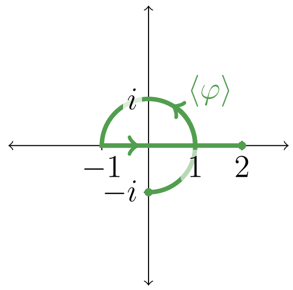
d) $\varphi(t) := e^{2it} - 1,\ D\varphi = \langle 0, 2\pi\rangle.$



e) $\varphi(t) := \begin{cases} e^{i\pi t}, & t \in \langle 0, 1), \\ t - 2, & t \in \langle 1, 3 \rangle. \end{cases}$



f) $\varphi(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi), \\ \frac{3t}{\pi} - 4, & t \in \langle \pi, 2\pi \rangle. \end{cases}$



<u>Exercise 21</u>.
Find a parametrization of the set Ω (i.e. find a curve φ such that $\langle \varphi \rangle = \Omega$), if

a) $\Omega = \{z \in \mathbb{C} \colon \; |z - 2 + 3i| = 2\}$;

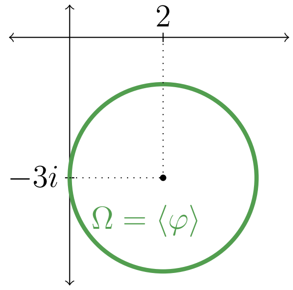
b) Ω is a line segment with the endpoints $a, b \in \mathbb{C}$, $a \neq b$;

c) $\Omega = \{z \in \mathbb{C} \colon \; \operatorname{Re} z = 2 \operatorname{Im} z\}$;

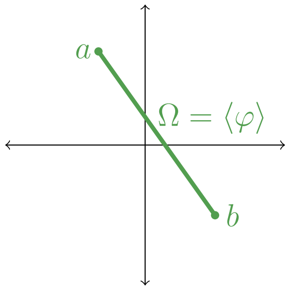
d) $\Omega = \{z \in \mathbb{C} \colon \; \operatorname{Re}\left(\frac{1}{z}\right) = 2\}$.

<u>Solution:</u>

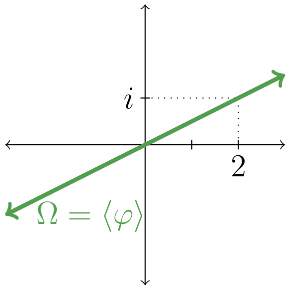
a) $\Omega = \{z \in \mathbb{C} \colon \; |z - 2 + 3i| = 2\}$; $\underline{\varphi(t) := 2 - 3i + 2e^{it}, \; t \in \langle 0, 2\pi \rangle}$.



b) Ω is a line segment with the endpoints $a, b \in \mathbb{C}$, $a \neq b$; $\underline{\varphi(t) := a + (b-a)t,\ t \in \langle 0, 1 \rangle}$.



c) $\Omega = \{z \in \mathbb{C} \colon\ \operatorname{Re} z = 2\ \operatorname{Im} z\}$; $\underline{\varphi(t) := t + \frac{t}{2}i,\ t \in \mathbb{R}}$.



d)

$$\Omega = \left\{ z \in \mathbb{C} \colon\ \operatorname{Re}\left(\frac{1}{z}\right) = 2 \right\} =$$

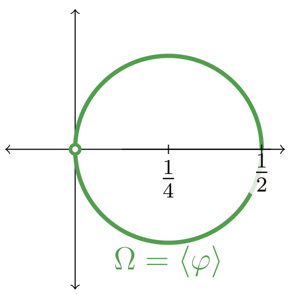
$$= \left\{ x + iy \colon\ \operatorname{Re}\left(\frac{1}{x+iy}\right) = \frac{x}{x^2+y^2} = 2 \right\} =$$

$$= \left\{ x + iy \in \mathbb{C} \setminus \{0\} \colon\ 2\left(x^2 - \frac{x}{2} + y^2\right) = 0 \right\} =$$

$$= \left\{ x + iy \in \mathbb{C} \setminus \{0\} \colon\ 2\left((x - \frac{1}{4})^2 + y^2 - \frac{1}{16}\right) = 0 \right\} =$$

$$= \left\{ x + iy \in \mathbb{C} \setminus \{0\} \colon\ \left(x - \frac{1}{4}\right)^2 + y^2 = \frac{1}{16} \right\};$$

$$\underline{\varphi(t) := \frac{1}{4} + \frac{1}{4}e^{it},\ t \in (-\pi, \pi).}$$



Exercise 22.
Draw the set Ω, and decide if Ω is a domain and if it is an open set, where

a) $\Omega = \{z \in \mathbb{C}\colon\ |z-i| < 1 \ \vee\ |z+i| < 1\}$;

b) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < 1 \ \wedge\ |z-2| < 2\}$;

c) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < |z+1|\}$;

d) $\Omega = \{z \in \mathbb{C}\colon\ |z+1| > 2|z|\}$;

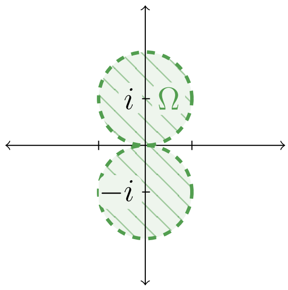
e) $\Omega = \{z \in \mathbb{C}\colon\ 1 < |z| < 2\}$;

f) $\Omega = \big\{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \arg z \in (-\pi, \pi\rangle \setminus \{0\}\big\}$;

g) $\Omega = \{z \in \mathbb{C}\colon\ |2z| < |1+z^2|\}$.

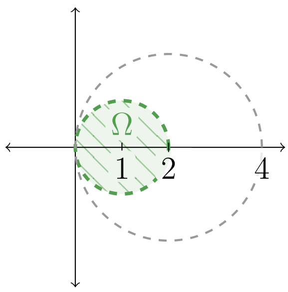
Solution:

a) $\Omega = \{z \in \mathbb{C}\colon\ |z-i| < 1 \ \vee\ |z+i| < 1\}$.



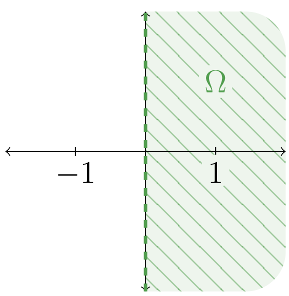
Ω is open, but not connected, and therefore Ω is not a domain.

b) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < 1 \ \wedge\ |z-2| < 2\}$.



Ω is open and connected set, and therefore Ω is a domain.

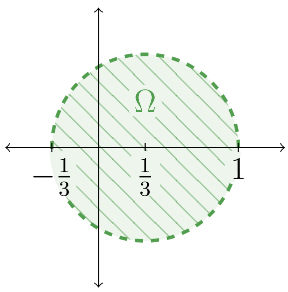
c) $\Omega = \{z \in \mathbb{C}\colon\ |z-1| < |z+1|\}$.



Ω is open and connected set, and therefore Ω is a domain.

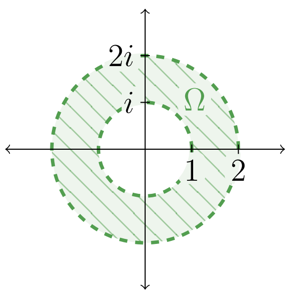
d)

$$\begin{aligned}\Omega &= \{z \in \mathbb{C}\colon\ |z+1| > 2|z|\} = \\ &= \big\{x+iy\colon\ (x+1)^2 + y^2 > 4(x^2+y^2)\big\} = \\ &= \big\{x+iy\colon\ 3x^2 + 3y^2 - 2x - 1 < 0\big\} = \\ &= \left\{x+iy\colon\ x^2 + y^2 - \frac{2}{3}x - \frac{1}{3} < 0\right\} = \\ &= \left\{x+iy\colon\ \left(x - \frac{1}{3}\right)^2 + y^2 < \frac{4}{9}\right\}.\end{aligned}$$



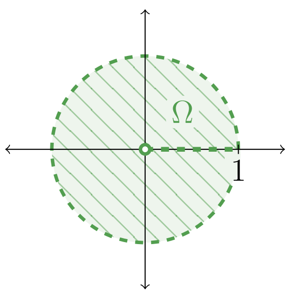
Ω is open and connected set, and therefore Ω is a domain.

e) $\Omega = \{z \in \mathbb{C}\colon\ 1 < |z| < 2\}$.



Ω is open and connected set, and therefore Ω is a domain.

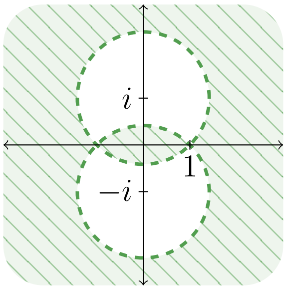
f) $\Omega = \big\{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \arg z \in (-\pi, \pi\rangle \setminus \{0\}\big\}$.



Ω is open and connected set, and therefore Ω is a domain.

g)

$$\begin{aligned}\Omega = \{z \in \mathbb{C}\colon\ |2z| < |1+z^2|\} = \\ = \left\{x+iy\colon\ 4(x^2+y^2) < (1+x^2-y^2)^2 + 4x^2y^2\right\} = \\ = \left\{x+iy\colon\ 4x^2+4y^2 < 1+x^4+y^4+2x^2-2y^2-2x^2y^2+4x^2y^2\right\} = \\ = \left\{x+iy\colon\ 0 < 1+x^4+y^4-2x^2-6y^2+2x^2y^2\right\} = \\ = \left\{x+iy\colon\ (x^2+y^2-1)^2-4y^2 > 0\right\} = \\ = \left\{x+iy\colon\ (x^2+y^2-1+2y)(x^2+y^2-1-2y) > 0\right\} = \\ = \left\{x+iy\colon\ [x^2+(y+1)^2-2][x^2+(y-1)^2-2] > 0\right\}.\end{aligned}$$



<u>Ω is open</u>, but not connected set, and therefore <u>Ω is not a domain</u>.

<u>Exercise 23.</u>
Find all of the points where the function f has a derivative and the points where it is holomorphic, if

a) $f(z) := \operatorname{Re} z$;

b) $f(z) := |z^2|$;

c) $f(z) := ze^z$;

d) $f(z) := \overline{z}|z|$;

e) $f(z) := \frac{\operatorname{Re} z}{z}$;

f) $f(z) := z^2\overline{z}$;

g) $f(z) := z^2 + 2z - 1$.

<u>Solution:</u>

a)

$$f(x+iy) = \underbrace{x}_{=:u(x,y)} + \underbrace{0}_{=:v(x,y)} \cdot i.$$

For every $(x,y) \in \mathbb{R}^2$ we have that

$$\frac{\partial u}{\partial x}(x,y) = 1 \neq 0 = \frac{\partial v}{\partial y}(x,y),$$

and because of that it follows that the function <u>f does not have a derivative anywhere</u> and the function <u>f is not holomorphic at any point</u>.

b) $f(x+iy) = |(x+iy)^2| = (|x+iy|)^2 = x^2+y^2$. So $f = u+iv$, where $u(x,y) := x^2+y^2$ and $v(x,y) := 0$.

$$\left.\begin{array}{rcl} \dfrac{\partial u}{\partial x}(x,y) = 2x & = & \dfrac{\partial v}{\partial y}(x,y) = 0 \\ \dfrac{\partial u}{\partial y}(x,y) = 2y & = & -\dfrac{\partial v}{\partial x}(x,y) = 0 \end{array}\right\} \Leftrightarrow (x,y) = (0,0),$$

and at the same time the functions u and v are differentiable in $\mathbb{R}^2$, and therefore f has a derivative (only) in the point 0 ant it is not holomorphic anywhere.

c)

$$\begin{aligned} f(x+iy) &= (x+iy)e^x(\cos y + i\sin y) = \\ &= \underbrace{xe^x\cos y - ye^x\sin y}_{=:u(x,y)} + i\underbrace{(xe^x\sin y + ye^x\cos y)}_{=:v(x,y)}. \end{aligned}$$

Functions u and v are differentiable in $\mathbb{R}^2$,

$$\frac{\partial u}{\partial x}(x,y) = e^x\cos y + xe^x\cos y - ye^x\sin y,$$

$$\frac{\partial v}{\partial y}(x,y) = xe^x\cos y + e^x\cos y - ye^x\sin y,$$

and

$$\frac{\partial u}{\partial y}(x,y) = -xe^x\sin y - e^x\sin y - ye^x\cos y,$$

$$-\frac{\partial v}{\partial x}(x,y) = -\left(e^x\sin y + xe^x\sin y + ye^x\cos y\right).$$

So $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in $\mathbb{R}^2$, and therefore f is holomorphic everywhere in $\mathbb{C}$ and $f'(z)$ exists at every $z \in \mathbb{C}$.

$$\left(f'(z) = f'(x+iy) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(x,y) = \cdots = e^z + ze^z.\right)$$

d)

$$f(x+iy) = (x-iy)\sqrt{x^2+y^2} = \underbrace{x\sqrt{x^2+y^2}}_{=:u(x,y)} + i\underbrace{\left(-y\sqrt{x^2+y^2}\right)}_{=:v(x,y)}.$$

From this it follows that for every $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we have that

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} > 0, \\ \frac{\partial v}{\partial y}(x,y) &= -\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}} < 0, \end{aligned}$$

and therefore: if $z \neq 0$, then $f'(z)$ does not exist.

It remains to prove or disprove the existence of the derivative at the point 0:

$$\underline{f'(0)} = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\overline{z}\cdot|z|}{z} =$$

$$= \lim_{z\to 0} \frac{|z|\left(\cos(\arg z) - i\sin(\arg z)\right)\cdot|z|}{|z|\left(\cos\left(\arg z\right) + i\sin\left(\arg z\right)\right)} =$$

$$= \lim_{z\to 0} \left[|z|\cdot\left(\cos\left(-2\arg z\right) + i\sin\left(-2\arg z\right)\right)\right] = \underline{0},$$

$$\text{because } \forall z\neq 0:\ \left|\cos\left(-2\arg z\right) + i\sin\left(-2\arg z\right)\right| = 1.$$

Summary: the function f has a derivative only at the point 0, and therefore f is not holomorphic at any point.

e)

$$f(x+iy) = \frac{x}{x+iy} = \frac{x(x-iy)}{x^2+y^2} = \underbrace{\frac{x^2}{x^2+y^2}}_{=:u(x,y)} + i\underbrace{\left(-\frac{xy}{x^2+y^2}\right)}_{=:v(x,y)}.$$

For every $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ we have that

$$\frac{\partial u}{\partial x}(x,y) = \frac{2x(x^2+y^2)-x^2 2x}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial y}(x,y) = \frac{-x(x^2+y^2)+xy2y}{(x^2+y^2)^2} = \frac{-x^3+xy^2}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y}(x,y) = \frac{-x^2 2y}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x}(x,y) = \frac{-y(x^2+y^2)+xy2x}{(x^2+y^2)^2} = \frac{x^2y-y^3}{(x^2+y^2)^2},$$

and therefore the derivative can exist only in the points $x+iy$ where

$$\left(\frac{2xy^2}{(x^2+y^2)^2} = \frac{x(-x^2+y^2)}{(x^2+y^2)^2}\right)\wedge\left(\frac{2x^2y}{(x^2+y^2)^2} = \frac{y(x^2-y^2)}{(x^2+y^2)^2}\right),$$

that is

$$\left(\frac{xy^2}{(x^2+y^2)^2} = \frac{-x^3}{(x^2+y^2)^2}\right)\wedge\left(\frac{x^2y}{(x^2+y^2)^2} = \frac{-y^3}{(x^2+y^2)^2}\right).$$

It is easy to observe that this system of equations has no solution.

Summary: the function f does not have a derivative at any point, and therefore it is not holomorphic at any point.

f)

$$f(x+iy) = (x^2-y^2+2ixy)(x-iy) =$$

$$= x^3 - xy^2 + 2xy^2 + i(-x^2y + y^3 + 2x^2y) =$$

$$= \underbrace{x^3+xy^2}_{=:u(x,y)} + i\underbrace{(y^3+x^2y)}_{=:v(x,y)}.$$

The functions u and v are differentiable in $\mathbb{R}^2$, and for every $(x, y) \in \mathbb{R}^2$ we have that

$$\frac{\partial u}{\partial x}(x, y) = 3x^2 + y^2, \qquad \frac{\partial u}{\partial y}(x, y) = 2xy,$$
$$\frac{\partial v}{\partial y}(x, y) = 3y^2 + x^2, \qquad -\frac{\partial v}{\partial x}(x, y) = -2xy.$$

From this it follows that the derivative exists in all such points $x+iy$ for which $2x^2 = 2y^2$ and $4xy = 0$. There is only one such point, which is $z = 0 + i0 = 0$.

Summary: the function f has a derivative only in the point 0 $(f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0)$, it is not holomorphic anywhere.

g)

$$f(x+iy) = x^2 - y^2 + 2ixy + 2x + 2iy - 1 = \underbrace{x^2 - y^2 + 2x - 1}_{=:u(x,y)} + i\,\underbrace{(2xy + 2y)}_{=:v(x,y)}.$$

For every $(x, y) \in \mathbb{R}^2$ we have that

$$\frac{\partial u}{\partial x}(x, y) = 2x + 2 = \frac{\partial v}{\partial y}(x, y),$$
$$\frac{\partial u}{\partial y}(x, y) = -2y = -\frac{\partial v}{\partial x}(x, y),$$

and because the functions u a v are moreover differentiable in $\mathbb{R}^2$, we have

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \cdots = 2z + 2$$

for every $z = x + iy$. The function f has the derivative at every point $z \in \mathbb{C}$ and it is holomorphic at every point $z \in \mathbb{C}$.

EXERCISE 24.

Determine, if the function Φ is harmonic on the domain Ω, where

a) $\Phi(x, y) := x^2 - y^2 + 2022$, $\Omega = \mathbb{C}$;

b) $\Phi(x, y) := \frac{x}{x^2+y^2} + x^2 - y^2 + x - y$, $\Omega = \mathbb{C} \setminus \{0\}$.

Solution:

a) Obviously $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2)$ and for every $(x, y) \in \mathbb{R}^2$ we have that

$$\Delta\Phi(x, y) = \frac{\partial^2 \Phi}{\partial x^2}(x, y) + \frac{\partial^2 \Phi}{\partial y^2}(x, y) = 2 - 2 = 0,$$

therefore Φ is harmonic in $\mathbb{C}$.

b) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ and for every $(x,y) \neq (0,0)$ we have that

$$\frac{\partial \Phi}{\partial x}(x,y) = \frac{x^2+y^2-x2x}{(x^2+y^2)^2} + 2x + 1,$$

$$\frac{\partial^2 \Phi}{\partial x^2}(x,y) = \frac{-2x(x^2+y^2)^2 - (-x^2+y^2)2(x^2+y^2)2x}{(x^2+y^2)^4} + 2,$$

$$\frac{\partial \Phi}{\partial y}(x,y) = \frac{-x2y}{(x^2+y^2)^2} - 2y - 1,$$

$$\frac{\partial^2 \Phi}{\partial y^2}(x,y) = \frac{-2x(x^2+y^2)^2 + 2xy2(x^2+y^2)2y}{(x^2+y^2)^4} - 2.$$

From this it follows that

$$\Delta\Phi(x,y) = \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3} + \frac{-2x(x^2+y^2)+8xy^2}{(x^2+y^2)^3} = 0,$$

and therefore $\underline{\Phi \text{ is harmonic in } \mathbb{C} \setminus \{0\}}$.

Exercise 25.
Find (if it exists) a holomorphic function $f = u + iv$, $f\colon\ \Omega \to \mathbb{C}$, where

a) $u(x,y) := x^3 - 3xy^2 - 2y$, $\Omega = \mathbb{C}$;

b) $u(x,y) := \frac{x}{x^2+y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$;

c) $u(x,y) := 3x^2 - y^2 + 3x + y$, $\Omega = \mathbb{C}$;

d) $u(x,y) := x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$.

Solution:

a)

$$\frac{\partial u}{\partial x}(x,y) = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}(x,y) \ \Rightarrow\ v(x,y) = 3x^2y - y^3 + \varphi(x),$$
$$\frac{\partial u}{\partial y}(x,y) = -6xy - 2 = -\frac{\partial v}{\partial x}(x,y) = -6xy - \varphi'(x) \ \Rightarrow\ \varphi(x) = 2x + c,\ c \in \mathbb{R},$$

and therefore

$$\underline{f(x+iy) = (x^3 - 3xy^2 - 2y) + i(3x^2y - y^3 + 2x + c),\ c \in \mathbb{R}.}$$

(You can simplify this to $\underline{f(z) = z^3 + 2zi + ci,\ c \in \mathbb{R}}$.)

b)

$$-\frac{\partial u}{\partial y}(x,y) = \frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial x}(x,y) \;\Rightarrow\; v(x,y) = \int \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}x.$$

After the change of variables $x^2+y^2 = t$ ($2x\,\mathrm{d}x = \mathrm{d}t$) we get

$$\int \frac{2xy}{(x^2+y^2)^2}\,\mathrm{d}x = y\int \frac{\mathrm{d}t}{t^2} = y\frac{-1}{t} = \frac{-y}{x^2+y^2},$$

and therefore

$$v(x,y) = \frac{-y}{x^2+y^2} + \varphi(y).$$

Plugging this to the second Cauchy-Riemann condition we get

$$\frac{\partial u}{\partial x}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} + \varphi'(y) \;\Rightarrow\; \varphi(y) = c,\; c\in\mathbb{R},$$

and therefore

$$\underline{f(z) = f(x+iy) =} \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2} + c\right) = \underline{\frac{1}{z} + ci,\; c\in\mathbb{R}.}$$

c)

$$\frac{\partial u}{\partial x}(x,y) = 6x+3, \qquad \frac{\partial u}{\partial y}(x,y) = -2y+1,$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) = 6, \qquad \frac{\partial^2 u}{\partial y^2}(x,y) = -2,$$

and therefore

$$\Delta u(x,y) \neq 0 \text{ for every } (x,y) \text{ v } \mathbb{R}^2.$$

Function u is not harmonic on Ω, and therefore the required function f does not exist.

d)

$$\frac{\partial u}{\partial x}(x,y) = 2x+5+\frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}(x,y)$$

$$\Downarrow$$

$$v(x,y) = 2xy+5y-\frac{x}{x^2+y^2}+\varphi(x).$$

Furthermore

$$\frac{\partial u}{\partial y}(x,y) = -2y+1+\frac{y^2-x^2}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}(x,y) = -2y+\frac{y^2-x^2}{(x^2+y^2)^2} - \varphi'(x)$$

$$\Downarrow$$

$$\varphi'(x) = -1.$$

From this we can easily obtain that $v(x,y) = 2xy+5y-\frac{x}{x^2+y^2}-x+c,\; c\in\mathbb{R}$.

The sought-after function is

$$\underline{f(x+iy) = \left(x^2-y^2+5x+y-\frac{y}{x^2+y^2}\right) + i\left(2xy+5y-\frac{x}{x^2+y^2}-x+c\right),\; c\in\mathbb{R},}$$

which is

$$\underline{f(z) = z^2+(5-i)z-\frac{i}{z}+ci,\; c\in\mathbb{R}.}$$

Exercise 26.
Let $u(x,y) := x^3 - 3xy^2 - 2y + 2$. Find (if it exists) a holomorphic function $f = u + iv$, $f\colon\ \mathbb{C} \to \mathbb{C}$, where

a) $f(0) = i$;

b) $f(1) = 3 - i$.

Solution:
Similarly to the solution to Exercise 25 a) we can find out that

$$f(x+iy) = (x^3 - 3xy^2 - 2y + 2) + i(3x^2y - y^3 + 2x + c), \text{ where } c \in \mathbb{R}.$$

a) The requirement
$$f(0) = f(0 + 0i) = 2 + ic = i$$
obviously cannot be satisfied by any choice of $c \in \mathbb{R}$. The function f with the required properties does not exist.

b) We want to satisfy the condition
$$f(1) = f(1 + 0i) = 3 + i(2 + c) = 3 - i,$$
and therefore $2 + c = -1$, which is $c = -3$. So the required function exists, it is
$$\underline{f(x+iy) = (x^3 - 3xy^2 - 2y + 2) + i(3x^2y - y^3 + 2x - 3)}.$$

Exercise 27.
Find (if it exists) a holomorphic function $f = u + iv$, $f\colon\ \Omega \to \mathbb{C}$, where

a) $v(x,y) := -3xy^2 + x^3 + 5$, $\Omega = \mathbb{C}$;

b) $v(x,y) := \arctan \frac{y}{x}$, $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}$.

Solution:

a)
$$\frac{\partial v}{\partial y}(x,y) = -6xy = \frac{\partial u}{\partial x}(x,y) \ \Rightarrow\ u(x,y) = -3x^2y + \varphi(y),$$
$$-\frac{\partial v}{\partial x}(x,y) = 3y^2 - 3x^2 = \frac{\partial u}{\partial y}(x,y) = -3x^2 + \varphi'(y) \ \Rightarrow\ \varphi(y) = y^3 + c,\ c \in \mathbb{R},$$
and therefore
$$\underline{f(x+iy) = (-3x^2y + y^3 + c) + i(-3xy^2 + x^3 + 5),\ c \in \mathbb{R}},$$
which is
$$\underline{f(z) = c + iz^3 + 5i,\ c \in \mathbb{R}}.$$

b)

$$\frac{\partial v}{\partial y}(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\partial u}{\partial x}(x,y),$$

and therefore

$$u(x,y) = \frac{1}{2}\ln(x^2+y^2) + \varphi(y).$$

Plugging this into the second C-R condition

$$-\frac{\partial v}{\partial x}(x,y) = \frac{-1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{y}{x^2+y^2} = \frac{\partial u}{\partial y}(x,y) = \frac{y}{x^2+y^2} + \varphi'(y)$$

we get

$$\varphi(y) = c,\ c \in \mathbb{R}.$$

The sought function on the set Ω is

$$\underline{f(x+iy) = \ln\sqrt{x^2+y^2} + c + i\arg(x+iy),\ c \in \mathbb{R}},$$

which is

$$\underline{f(z) = c + \ln z,\ c \in \mathbb{R}}.$$

EXERCISE 28.

Let $\Omega := \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\}$. Let $v(x,y) := 1 + \arctan\frac{y}{x}$. Find (if it exists) a holomorphic function $f = u + iv$, $f\colon\ \Omega \to \mathbb{C}$, where

a) $f(3) = \ln 3 + 6 + i$;

b) $f(e) = 1 - i$.

Solution:

Similarly to the Excercise 27 b) we have

$$f(x+iy) = \ln\sqrt{x^2+y^2} + c + i\left(\arctan\frac{y}{x} + 1\right),\ c \in \mathbb{R}.$$

a)

$$f(3+0i) = \ln 3 + c + i = \ln 3 + 6 + i \ \Rightarrow\ c = 6,$$

and therefore

$$\underline{f(x+iy) = \ln\sqrt{x^2+y^2} + 6 + i\left(\arctan\frac{y}{x} + 1\right) \text{ in } \Omega.}$$

b)

$$\forall c \in \mathbb{R}\colon\ f(e+0i) = 1 + c + i \neq 1 - i.$$

<u>The sought function f does not exist.</u>

Exercise 29.
Prove that even though the function

$$v(x,y) := \ln(x^2+y^2)$$

is harmonic on (*doubly connected*) domain $\mathbb{C}\setminus\{0\}$, there does not exist a function $u:\ \mathbb{R}^2 \to \mathbb{R}$, such that $f := u + iv$ is holomorphic in $\mathbb{C}\setminus\{0\}$.

Solution:
Clearly $v \in \mathcal{C}^\infty(\mathbb{R}^2\setminus\{(0,0)\})$, and for every $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$:

$$\frac{\partial v}{\partial x}(x,y) = \frac{2x}{x^2+y^2},$$

$$\frac{\partial^2 v}{\partial x^2}(x,y) = \frac{2(x^2+y^2)-2x2x}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$

and therefore (the partial derivatives with respects to y are analogous)

$$\Delta v(x,y) = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

We've proven that the function <u>v is on $\mathbb{C}\setminus\{0\}$ harmonic</u>.

Let us assume, for contradiction, that there is a function u with the properties stated above. Then for $(x,y)\in\mathbb{R}^2$, such that $x+iy\in\Omega_1 := \{z\in\mathbb{C}:\ \operatorname{Re} z > 0\}$, we have

$$-\frac{\partial v}{\partial x}(x,y) = -\frac{2x}{x^2+y^2} = \frac{\partial u}{\partial y}(x,y),$$

and therefore (if we use the change of variables $\frac{y}{x} = t,\ \frac{1}{x}\,\mathrm{d}y = \mathrm{d}t$)

$$\begin{aligned} u(x,y) &= \int -\frac{2x}{x^2+y^2}\,\mathrm{d}y = \\ &= \int -\frac{2}{x}\frac{\mathrm{d}y}{1+\left(\frac{y}{x}\right)^2} = -2\arctan\frac{y}{x} + \varphi(x). \end{aligned}$$

Let us plug that into the second C-R condition

$$\frac{\partial v}{\partial y}(x,y) = \frac{2y}{x^2+y^2} = \frac{\partial u}{\partial x}(x,y) = \frac{2y}{x^2+y^2} + \varphi'(x)$$

to figure out that

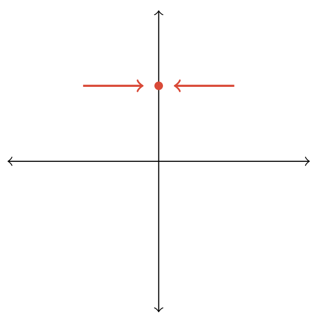
$$u(x,y) = -2\arctan\frac{y}{x} + c_1 \text{ for some } c_1\in\mathbb{R}.$$

Similarly there must exist a $c_2\in\mathbb{R}$, such that for every $x+iy\in\Omega_2 := \{z\in\mathbb{C}:\ \operatorname{Re} z<0\}$ we have

$$u(x,y) = -2\arctan\frac{y}{x} + c_2.$$

At the same time the function u is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ (at every point $\mathbb{R}^2 \setminus \{(0,0)\}$ it must be differentiable), and therefore

$$\begin{array}{ccccc} \lim\limits_{x\to 0-} u(x,1) & = & u(0,1) & = & \lim\limits_{x\to 0+} u(x,1). \\ \parallel & & & & \parallel \\ \pi + c_2 & & & & -\pi + c_1 \end{array}$$

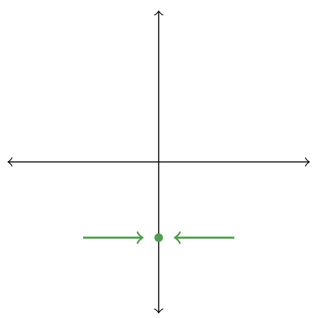


From this it follows that

$$2\pi = c_1 - c_2.$$

Analogously

$$\begin{array}{ccccc} \lim\limits_{x\to 0-} u(x,-1) & = & u(0,-1) & = & \lim\limits_{x\to 0+} u(x,-1), \\ \parallel & & & & \parallel \\ -\pi + c_2 & & & & \pi + c_1 \end{array}$$



and therefore

$$2\pi = c_2 - c_1.$$

This leads us to the fact that

$$2\pi = c_1 - c_2 = -(c_2 - c_1) = -2\pi,$$

which is an contradiction. <u>The sought function u does not exist.</u>

Exercise 30.
Find the rotational angle and extensibility coefficient of the function f at the point z_0, where

a) $f(z) := e^z$, $z_0 = -1 - \frac{\pi}{2}i$;

b) $f(z) := z^3$, $z_0 = -3 + 4i$;

c) $f(z) := \frac{z+i}{z-i}$, $z_0 = 2i$.

Solution:

a)

$$|f'(z_0)| = |e^{z_0}| = |e^{-1-i\frac{\pi}{2}}| = \underline{\frac{1}{e}},$$

which is the extensibility coefficient of the function f at the point z_0 (and $\frac{1}{e} < 1$ implies that it is a *contraction*).

$$\arg f'(z_0) = \arg\left(\frac{1}{e}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)\right) = \arg\left(-\frac{i}{e}\right) = \underline{-\frac{\pi}{2}},$$

which is the rotational angle of the function f at the point z_0.

b) $z_0 = 5\left(\cos\left(\frac{\pi}{2} + \arctan\frac{3}{4}\right) + i\sin\left(\frac{\pi}{2} + \arctan\frac{3}{4}\right)\right)$, and therefore

$$f'(z_0) = 3z_0^2 = 3\cdot 25\left(\cos\left(\pi + 2\arctan\frac{3}{4}\right) + i\sin\left(\pi + 2\arctan\frac{3}{4}\right)\right).$$

From this we get

$|f'(z_0)| = \underline{75}$ ... extensibility coefficient of the function f at z_0 ($75 > 1$, therefore it is a *dilatation*),

$\arg f'(z_0) = \underline{-\pi + 2\arctan\frac{3}{4}}$ ... rotational angle of the function f at the point z_0.

c)

$$f'(z) = \frac{z - i - (z+i)}{(z-i)^2} = \frac{-2i}{(z-i)^2},$$

$$f'(z_0) = \frac{-2i}{i^2} = 2i,$$

and therefore

$1 < |f'(z_0)| = \underline{2}$ ... extensibility coefficient of f at z_0 (*dilatation*),

$\arg f'(z_0) = \underline{\frac{\pi}{2}}$ ... rotational angle f at z_0.

Exercise 31.
Determine in which points of the complex plane is the given mapping a contraction:

a) $f(z) := \frac{2}{z}$;

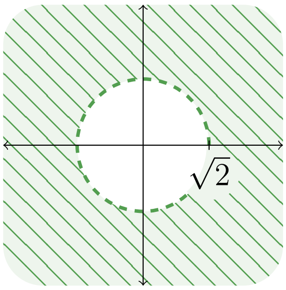
b) $f(z) := \ln(z+4)$.

Solution:

a) $f'(z) = \frac{-2}{z^2}$. Therefore for $z \in \mathbb{C}$:

$$0 < |f'(z)| < 1 \Leftrightarrow \left|\frac{-2}{z^2}\right| < 1 \Leftrightarrow 2 < |z|^2 \Leftrightarrow \sqrt{2} < |z|.$$

The mapping f is a contraction in every point of the set $\left\{z \in \mathbb{C}\colon\ |z| > \sqrt{2}\right\}$.

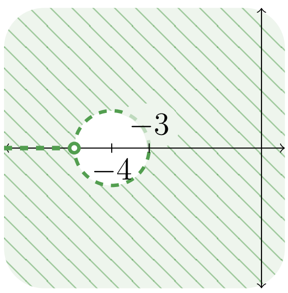


b) $f'(z)$ exists in $\mathbb{C} \setminus \{x+iy\colon\ y=0 \ \wedge\ x \le -4\} =: \Omega$. For every $z \in \Omega$ we have that

$$\begin{aligned} |f'(z)| &= \left|\frac{1}{z+4}\right|, \\ 0 < \frac{1}{|z+4|} < 1 &\Leftrightarrow 1 < |z+4|. \end{aligned}$$

The mapping f is a contraction at every point of the set

$$\{z \in \mathbb{C}\colon\ |z+4| > 1\} \setminus \{x+iy\colon\ y=0 \ \wedge\ x \le -4\}.$$



Exercise 32.
Draw the sets Ω and $f(\Omega)=\{f(z)\colon\ z\in\Omega\}$, where[2]

a) $\Omega=U(1,2),\ f(z):=1-2iz;$

b) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z<1\},\ f(z):=(1+i)z+1;$

c) $\Omega=U(1,2),\ f(z):=\frac{1}{z};$

d) $\Omega=U(1,2),\ f(z):=\frac{2iz}{z+3};$

e) $\Omega=U(1,2),\ f(z):=\frac{z-1}{2z-6};$

f) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z<1\},\ f(z):=\frac{1}{z};$

g) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z<1\},\ f(z):=\frac{z}{z-1+i};$

h) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z<1\},\ f(z):=\frac{z}{z-2};$

i) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z<0\ \wedge\ \operatorname{Im}z<0\},\ f(z):=\frac{1}{z};$

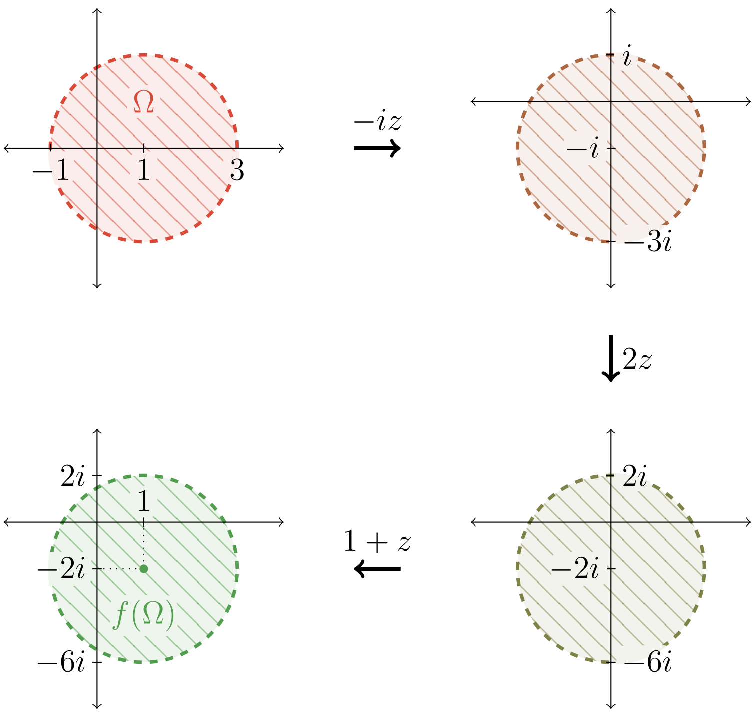
j) $\Omega=\{z\in\mathbb{C}\colon\ \operatorname{Re}z>0\ \wedge\ \operatorname{Im}z>0\},\ f(z):=\frac{z-1}{z+1};$

k) $\Omega=\{z\in\mathbb{C}\colon\ -1<\operatorname{Re}z<0\ \wedge\ \operatorname{Im}z<0\},\ f(z):=\frac{z-i}{z+i};$

l) $\Omega=\{z\in\mathbb{C}\colon\ |z|<1\ \wedge\ \operatorname{Re}z<0\ \wedge\ \operatorname{Im}z>0\},\ f(z):=\frac{z}{z-i}.$

Solution:

a) $\Omega=U(1,2),\ f(z):=1-2iz,\ \underline{f(\Omega)=U(1-2i,4)}.$



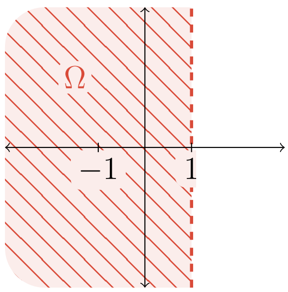
[2] *A hint for some of the following exercises.* Realize (and prove) that:

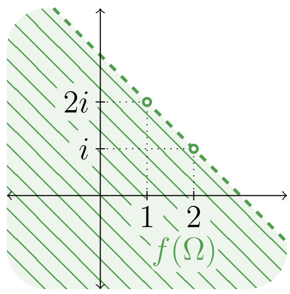
$$\left.\begin{array}{c} f \text{ is conformal in the set } \Omega\subset\mathbb{C}_\infty,\\ A,\,B\subset\Omega\end{array}\right\}\ \Rightarrow\ f(A\cap B)=f(A)\cap f(B).$$

b) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\}$, $f(z) := (1+i)z + 1$,

$$\begin{aligned} f(1) &= 1 + i + 1 = 2 + i, \\ f(1+i) &= 2i + 1, \\ f(0) &= 1, \end{aligned}$$

and therefore (think it through!) $\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Re} z + \operatorname{Im} z < 3\}}$.

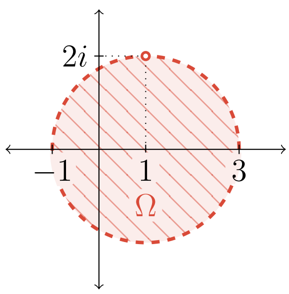


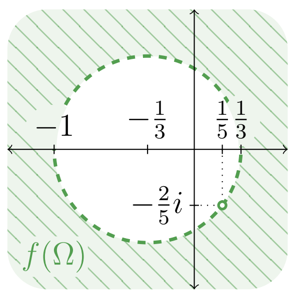


c) $\Omega = U(1,2)$, $f(z) := \frac{1}{z}$,

$$\begin{aligned} f(0) &= \infty, \\ f(-1) &= -1, \\ f(3) &= \frac{1}{3}, \\ f(1+2i) &= \frac{1}{1+2i} = \frac{1-2i}{5}, \end{aligned}$$

and therefore $\underline{f(\Omega) = \mathbb{C}_\infty \setminus \overline{U\left(-\frac{1}{3}, \frac{2}{3}\right)}}$.

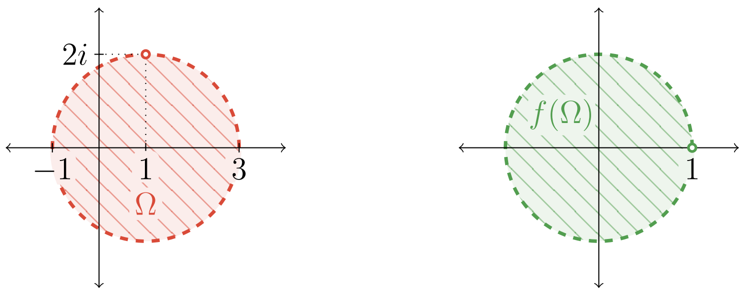




d) $\Omega = U(1,2)$, $f(z) := \frac{2iz}{z+3}$,

$$\begin{aligned} f(-3) &= \infty, \\ f(-1) &= \frac{-2i}{2} = -i, \\ f(3) &= i, \\ f(1+2i) &= \frac{2i(1+2i)}{4+2i} = -\frac{4-2i}{4+2i} = -\frac{3}{5} + \frac{4}{5}i, \end{aligned}$$

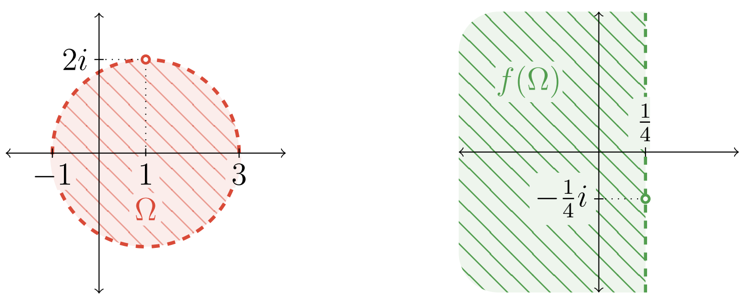
and therefore $\underline{f(\Omega) = U(0,1)}$.



e) $\Omega = U(1,2)$, $f(z) := \frac{z-1}{2z-6}$,

$$\begin{aligned} f(0) &= \frac{1}{6}, \\ f(3) &= \infty, \\ f(-1) &= \frac{-2}{-8} = \frac{1}{4}, \\ f(1+2i) &= \frac{2i}{2+4i-6} = \frac{1}{4} - \frac{1}{4}i, \end{aligned}$$

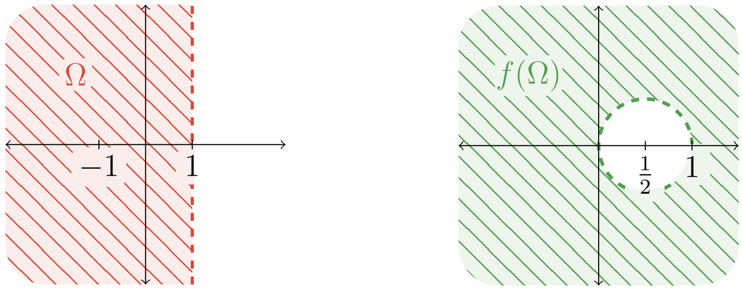
$$\text{and therefore } \underline{f(\Omega) = \left\{ z \in \mathbb{C} \colon \operatorname{Re} z < \frac{1}{4} \right\}}.$$



f) $\Omega = \{z \in \mathbb{C} \colon \operatorname{Re} z < 1\}$, $f(z) := \frac{1}{z}$,

$$\begin{aligned} f(1) &= 1, \\ f(0) &= \infty, \\ f(1+i) &= \frac{1}{1+i} = \frac{1-i}{2}, \\ f(1-i) &= \frac{1}{1-i} = \frac{1+i}{2}, \\ f(\infty) &= 0, \end{aligned}$$

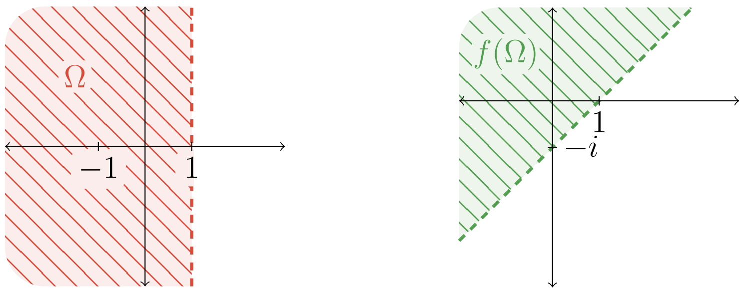
$$\text{and therefore } \underline{f(\Omega) = \mathbb{C}_\infty \setminus \overline{U\left(\frac{1}{2}, \frac{1}{2}\right)}}.$$



g) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{z}{z-1+i}$,

$$\begin{aligned} f(0) &= 0, \\ f(1) &= \frac{1}{i} = -i, \\ f(1-i) &= \infty, \\ f(1+i) &= \frac{1+i}{2i} = \frac{1}{2} - \frac{1}{2}i, \end{aligned}$$

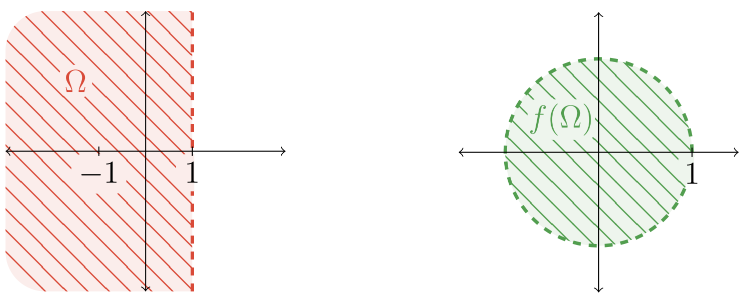
and therefore $\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > \operatorname{Re} z - 1\}}$.



h) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 1\},\ f(z) := \frac{z}{z-2}$,

$$\begin{aligned} f(2) &= \infty, \\ f(1) &= -1, \\ f(1+i) &= \frac{-2i}{2} = -i, \\ f(\infty) &= 1, \end{aligned}$$

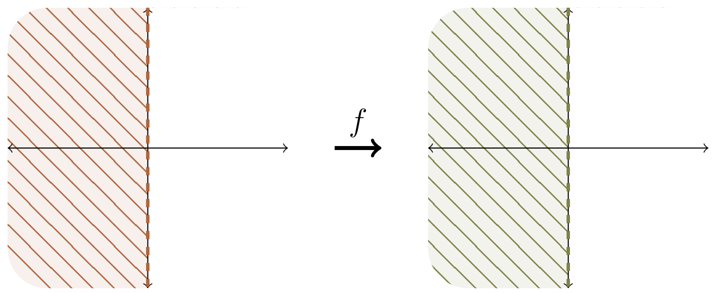
and therefore $\underline{f(\Omega) = U(0,1)}$.

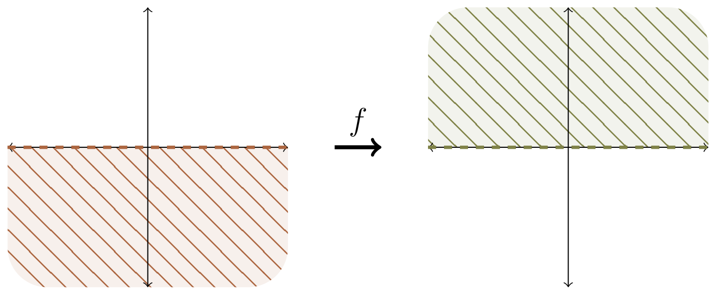


i) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z < 0\}$, $f(z) := \frac{1}{z}$,

$$\begin{aligned} f(0) &= \infty, \\ f(-1) &= -1, \\ f(1) &= 1, \\ f(i) &= -i, \\ f(-i) &= i, \end{aligned}$$

and thereforeže $\Omega = \Omega_1 \cap \Omega_2$, where $\Omega_1 := \{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\}$, $\Omega_2 := \{z \in \mathbb{C}\colon\ \operatorname{Im} z < 0\}$,





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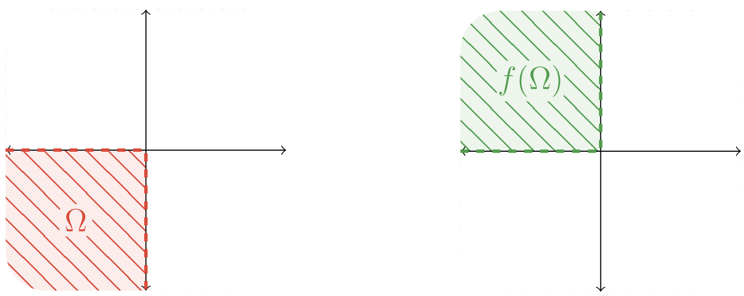
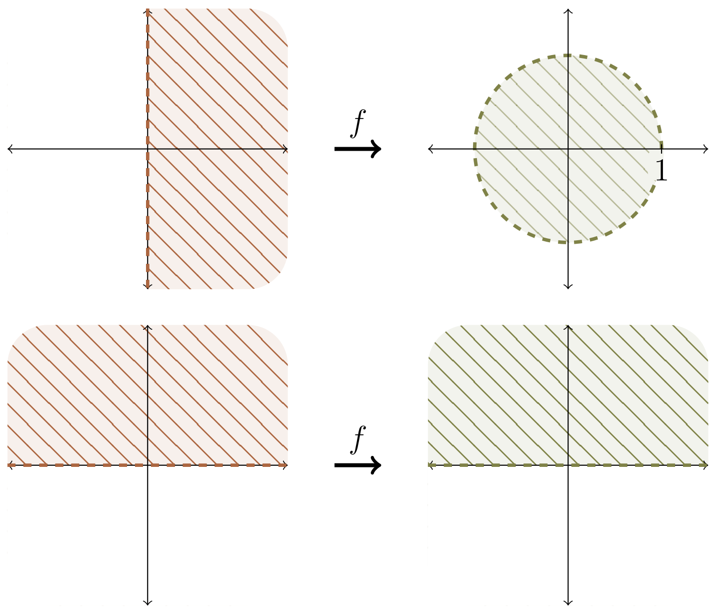
$$\underline{f(\Omega) =} f(\Omega_1) \cap f(\Omega_2) = \underline{\{z \in \mathbb{C}\colon\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z > 0\}}.$$



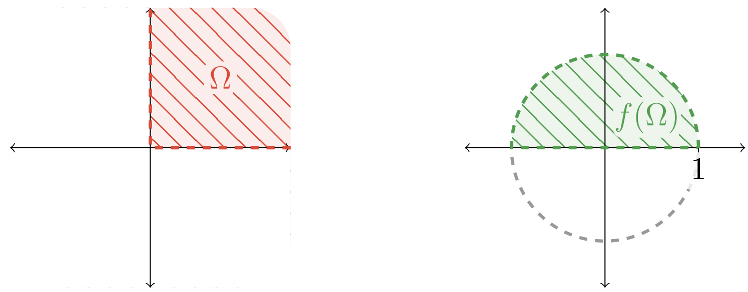
j) $\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > 0\ \wedge\ \operatorname{Im} z > 0\}$, $f(z) := \frac{z-1}{z+1}$,

$$\begin{aligned} f(0) &= -1, \\ f(1) &= 0, \\ f(i) &= \frac{i-1}{i+1} = i, \\ f(-i) &= -i, \\ f(-1) &= \infty, \end{aligned}$$



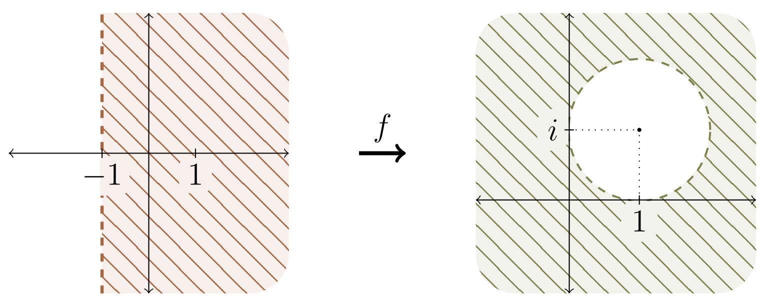
and therefore

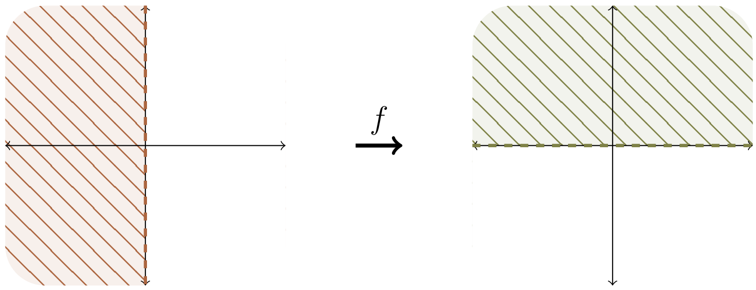
$$\underline{f(\Omega) = \{z \in \mathbb{C}\colon \ |z| < 1 \ \wedge \ \operatorname{Im} z > 0\}.}$$

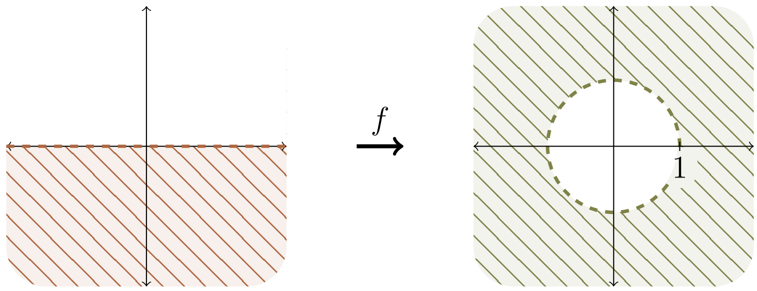


k) $\Omega = \{z \in \mathbb{C}\colon \ -1 < \operatorname{Re} z < 0 \ \wedge \ \operatorname{Im} z < 0\}, \ f(z) := \frac{z-i}{z+i},$

$$\begin{aligned}
f(0) &= -1,\\
f(i) &= 0,\\
f(-i) &= \infty,\\
f(1) &= \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = -i,\\
f(-1) &= \frac{-1-i}{-1+i} = \frac{(-1-i)^2}{2} = i,\\
f(-1+i) &= \frac{-1}{-1+2i} = \frac{1+2i}{5},\\
f(-1-i) &= 1+2i,
\end{aligned}$$

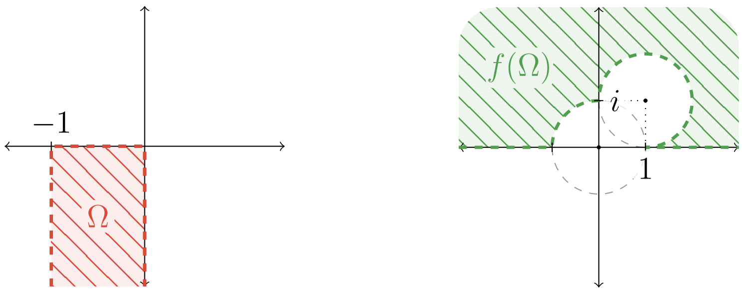






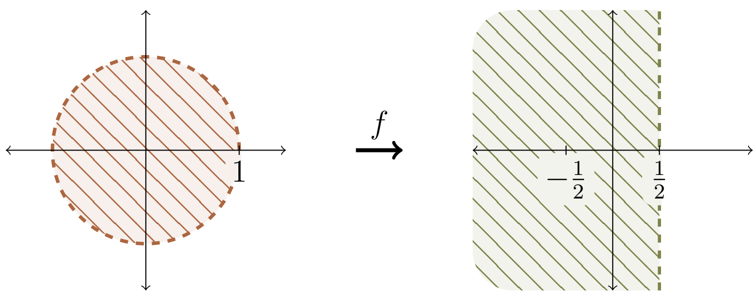
and therefore

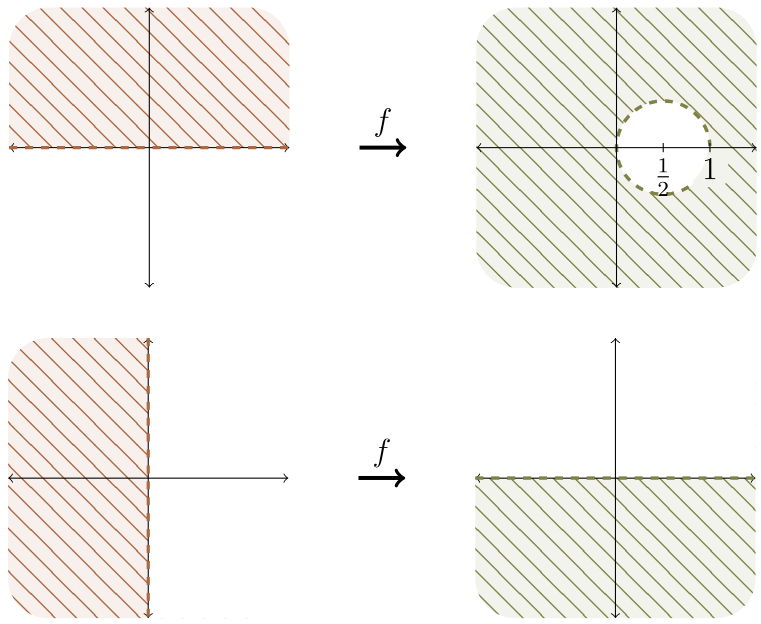
$$\underline{f(\Omega) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\ \wedge\ |z-(1+i)| > 1\ \wedge\ |z| > 1\}}.$$



l) $\Omega = \{z \in \mathbb{C}\colon\ |z| < 1\ \wedge\ \operatorname{Re} z < 0\ \wedge\ \operatorname{Im} z > 0\}$, $f(z) := \frac{z}{z-i}$,

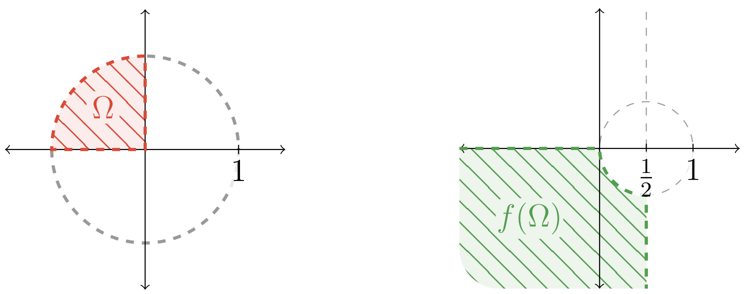
$$\begin{aligned} f(0) &= 0, \\ f(i) &= \infty, \\ f(-1) &= \frac{1-i}{2}, \\ f(1) &= \frac{1+i}{2}, \\ f(-i) &= \frac{1}{2}, \end{aligned}$$





and therefore

$$f(\Omega) = \left\{ z \in \mathbb{C} \colon \ \mathrm{Re}\, z < \frac{1}{2} \ \wedge \ \mathrm{Im}\, z < 0 \ \wedge \ \left| z - \frac{1}{2} \right| > \frac{1}{2} \right\}.$$



<u>Exercise 33.</u>
Find a linear fractional function f such that

a) $f(-1) = 0, \ f(i) = 2i, \ f(1+i) = 1-i;$

b) $f(i) = \infty, \ f(6) = 0, \ f(\infty) = 3;$

c) $f(0) = i, \ f(i) = 0, \ f(-1) = -i.$

<u>Solution:</u>

a) Let us search for the function f of the form

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{for } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{for } z = \infty, \end{cases}$$

where $a, b, c, d \in \mathbb{C}$, $ad \neq bc$.

From the given conditions we get the system of equations

$$\begin{aligned}\frac{-a+b}{-c+d} &= 0,\\ \frac{ai+b}{ci+d} &= 2i,\\ \frac{a(1+i)+b}{c(1+i)+d} &= 1-i.\end{aligned}$$

From the first equation follows that $a = b$. We can choose (think about why!) $a = b = 1$. The remaining two equations then become

$$\begin{aligned}\frac{i+1}{ci+d} &= 2i,\\ \frac{2+i}{c(1+i)+d} &= 1-i,\end{aligned}$$

and therefore

$$\begin{aligned}i+1 &= -2c+2id,\\ 2+i &= 2c+d-di.\end{aligned}$$

Adding these two equations we get $3+2i = d+id$, and therefore

$$d = \frac{3+2i}{1+i} = \frac{5-i}{2}.$$

It remains to compute c:

$$2c = 2+i-d(1-i) = 2+i-\frac{1}{2}(4-6i) = 4i,$$

and therefore $c = 2i$.

Conclusion:

$$f(z) = \begin{cases} \dfrac{z+1}{2iz+\frac{5-i}{2}} = \dfrac{2z+2}{4iz+5-i}, & z \in \mathbb{C}, \\ \dfrac{1}{2i} = -\dfrac{i}{2}, & z = \infty. \end{cases}$$

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b) Let

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{for } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{for } z = \infty. \end{cases}.$$

From the condition

$$f(\infty) = \frac{a}{c} = 3$$

it follows that we can choose $a = 3$ and $c = 1$. And the rest is easy:

$$f(6) = \frac{6a+b}{6z+d} = 0 \;\Rightarrow\; b = -6a = -18,$$
$$f(i) = \infty \;\Rightarrow\; ci + d = 0 \;\Rightarrow\; d = -ci = -i.$$

Summary:

$$f(z) = \begin{cases} \dfrac{3z-18}{z-i}, & z \in \mathbb{C}, \\ 3, & z = \infty. \end{cases}$$

c) Let

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{for } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{for } z = \infty. \end{cases}.$$

Analyzing the conditions we get

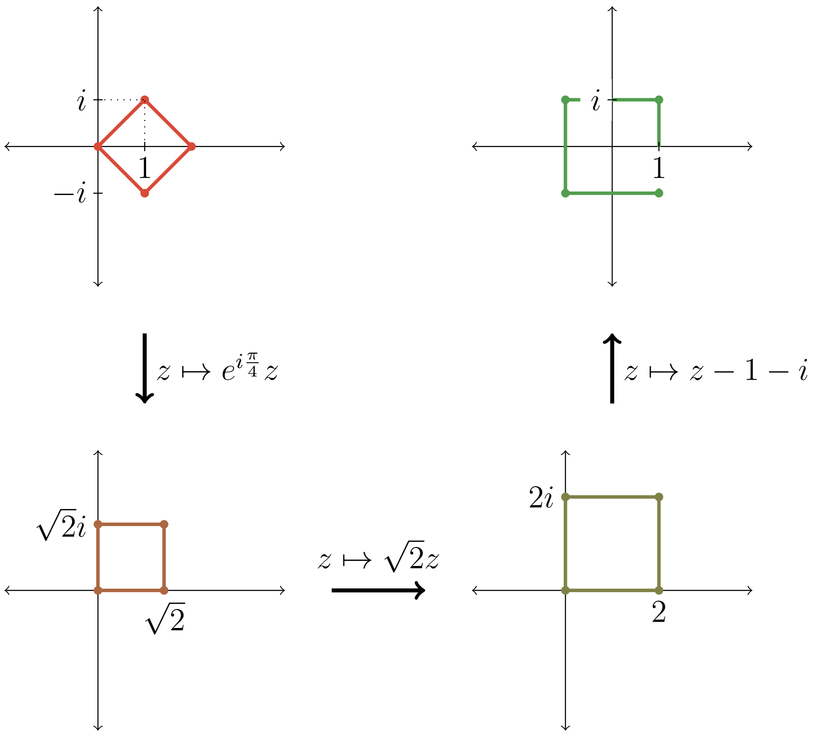
$$f(0) = \frac{b}{d} = i \;\Rightarrow\; \text{we can choose } b = 1,\; d = -i,$$
$$f(i) = \frac{ai+b}{ci+d} = 0 \;\Rightarrow\; ai + b = 0 \;\Rightarrow\; ai + 1 = 0 \;\Rightarrow\; a = i,$$
$$f(-1) = \frac{-a+b}{-c+d} = -i \;\Rightarrow\; -a + b = -i(-c+d) \;\Rightarrow\; c = -1-2i,$$

and therefore

$$f(z) = \begin{cases} \dfrac{iz+1}{(-1-2i)z-i}, & z \in \mathbb{C}, \\ \dfrac{i}{-1-2i} = -\dfrac{2}{5} - \dfrac{i}{5}, & z = \infty. \end{cases}$$

EXERCISE 34.
Find the linear function, which maps the square with the vertices $0,\ 1-i,\ 2,\ 1+i$ onto the square with the vertices $1+i,\ -1+i,\ -1-i,\ 1-i$.

Solution:



Composing the functions $z \mapsto e^{i\frac{\pi}{4}}$, $z \mapsto \sqrt{2}z$ and $z \mapsto z-1-i$ we get

$$\begin{aligned} f(z) &= \left(\sqrt{2}\left(e^{i\frac{\pi}{4}}z\right)\right) - 1 - i = \\ &= (1+i)z - 1 - i, \end{aligned}$$

that is

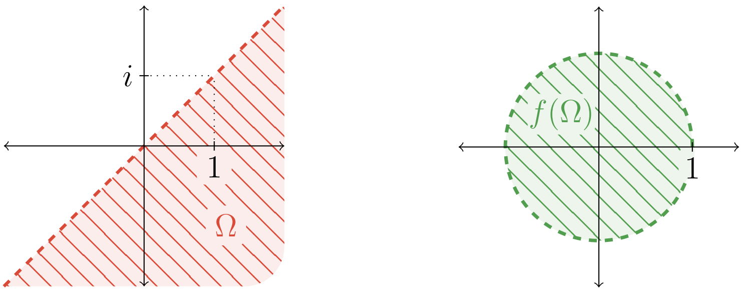
$$\underline{f(z) = (1+i)(z-1)}.$$

EXERCISE 35.
Let

$$\Omega = \{z \in \mathbb{C}\colon\ \operatorname{Re} z > \operatorname{Im} z\}.$$

Find the linear fractional function f, such that $f(\Omega) = U(0,1)$.

Solution:

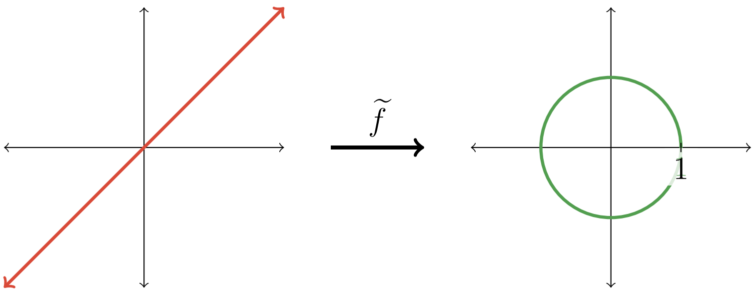


We start with finding the linear fractional function

$$\widetilde{f}(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{for } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{for } z = \infty, \end{cases}$$

such that $\widetilde{f}(0) = -1$, $\widetilde{f}(1+i) = i$ and $\widetilde{f}(\infty) = 1$.

Then



and $\widetilde{f}(\Omega)$ is either $U(0,1)$ (then we would set $f := \widetilde{f}$), or $\widetilde{f}(\Omega) = \mathbb{C}_\infty \setminus \overline{U(0,1)}$ (then we would choose $f := \frac{1}{\widetilde{f}}$).

Solving the system of equations

$$\begin{aligned} \frac{b}{d} &= -1, \\ \frac{a(1+i)+b}{c(1+i)+d} &= i, \\ \frac{a}{c} &= 1 \end{aligned}$$

we get

$$\widetilde{f}(z) = \begin{cases} \dfrac{z-1+i}{z+1-i}, & z \in \mathbb{C}, \\ 1, & z = \infty, \end{cases}$$

and because

$$\left|\widetilde{f}(1)\right| = \left|\frac{i}{2-i}\right| = \left|\frac{-1+2i}{5}\right| = \frac{1}{5}\sqrt{5} < 1,$$

we pick $f := \widetilde{f}$, that is

$$f(z) = \begin{cases} \dfrac{z-1+i}{z+1-i} & \text{for } z \in \mathbb{C}, \\ 1 & \text{for } z = \infty. \end{cases}$$

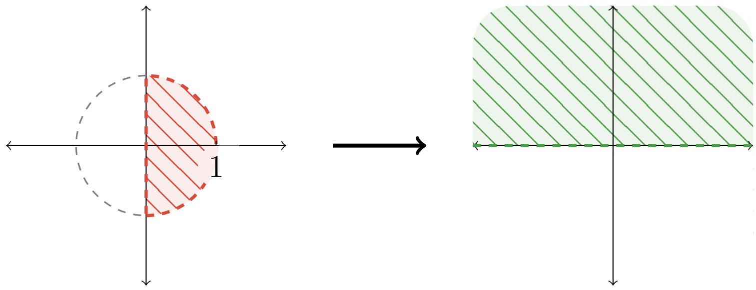
Exercise 36.
Find a conformal mapping, which maps the domain

$$\Omega = \{z \in \mathbb{C}\colon\ |z| < 1 \ \wedge\ \operatorname{Re} z > 0\}$$

onto the domain

$$\{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}.$$

Solution:

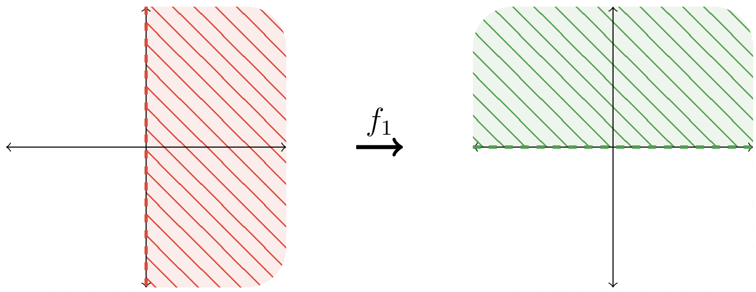


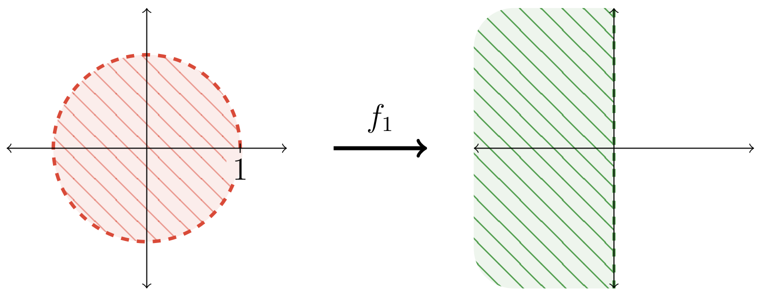
Let us consider the linear fractional function f_1 such that $f_1(i) = \infty$ and $f_1(-i) = 0$. Then the images of the circle $\{z \in \mathbb{C}\colon\ |z| = 1\}$ and the line $\{z \in \mathbb{C}\colon\ \operatorname{Re} z = 0\}$ (by f_1) are clearly lines with the intersection at 0 with the "angle" $\frac{\pi}{2}$. We can choose for example

$$f_1(z) := \begin{cases} \dfrac{z+i}{z-i}, & z \in \mathbb{C}, \\ 1, & z = \infty. \end{cases}$$

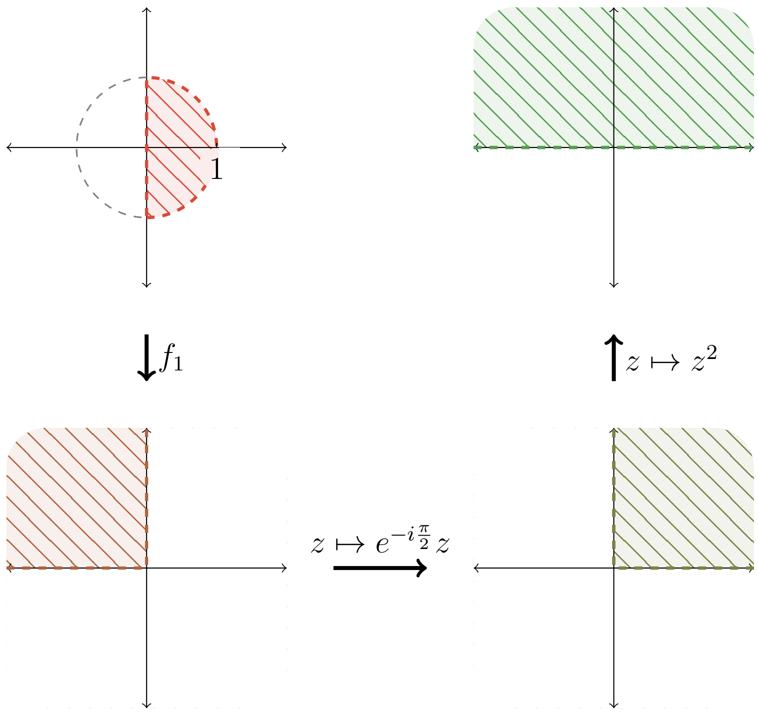
Then

$$f_1(0) = -1,\ f_1(i) = \infty,\ f_1(-i) = 0 \text{ a } f_1(1) = i,$$





and therefore



Summary: one of the functions with the required properties is the function defined on Ω

$$\underline{f(z) :=} \left(e^{-i\frac{\pi}{2}} \cdot \frac{z+i}{z-i}\right)^2 = \left(-i\frac{z+i}{z-i}\right)^2 = \underline{-\left(\frac{z+i}{z-i}\right)^2}.$$

EXERCISE 37.

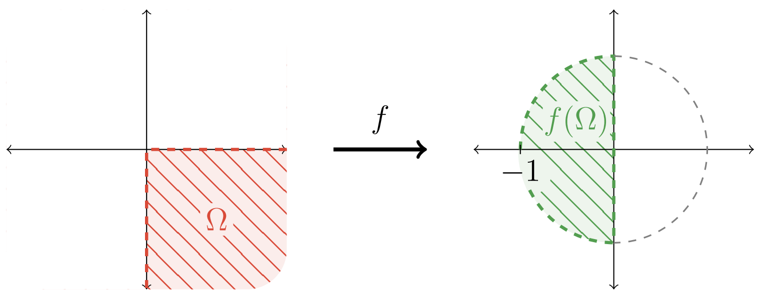
Let

$$\Omega = \{z \in \mathbb{C} \colon \ \operatorname{Re} z > 0 \ \wedge \ \operatorname{Im} z < 0\}.$$

Find the linear fractional function f such that

$$f(\Omega) = \{z \in \mathbb{C} \colon \ |z| < 1 \ \wedge \ \operatorname{Re} z < 0\}.$$

Solution:



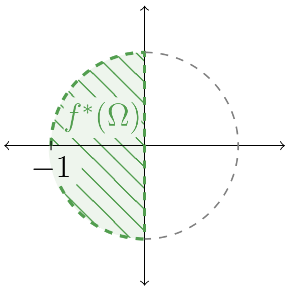
We firstly find the linear fractional function f^* such that

$$f^*(0) = i,$$
$$f^*(\infty) = -i,$$
$$f^*(-1) = \infty,$$

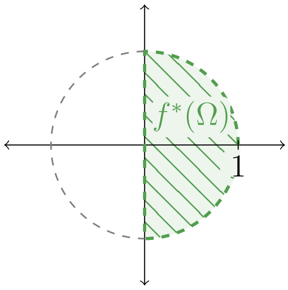
that is the function

$$f^*(z) = \begin{cases} \dfrac{-iz+i}{z+1}, & z \in \mathbb{C}, \\ -i, & z = \infty. \end{cases}$$

Then clearly either



(then we would define $f := f^*$), or



(which would lead us to the definition $f := -f^*$).

Because $f^*(i) = \frac{1+i}{i+1} = 1$ (the first possibility is realized), we choose

$$f(z) := f^*(z) = \begin{cases} \dfrac{-iz+i}{z+1}, & z \in \mathbb{C}, \\ -i, & z = \infty. \end{cases}.$$

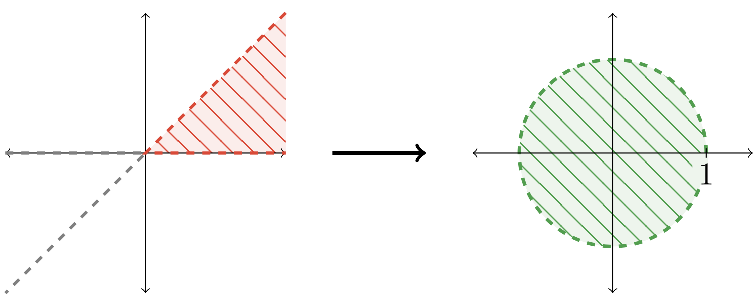
EXERCISE 38.

Find the conformal mapping which maps the domain

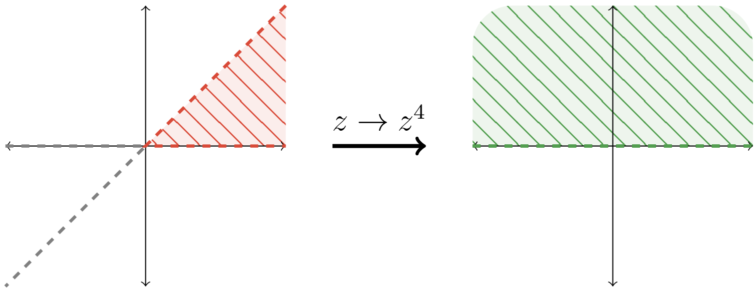
$$\Omega = \{z \in \mathbb{C} \colon \ \operatorname{Re} z > \operatorname{Im} z > 0\}$$

onto $U(0,1)$.

Solution:



Let us firstly consider the mapping $z \mapsto z^4$.



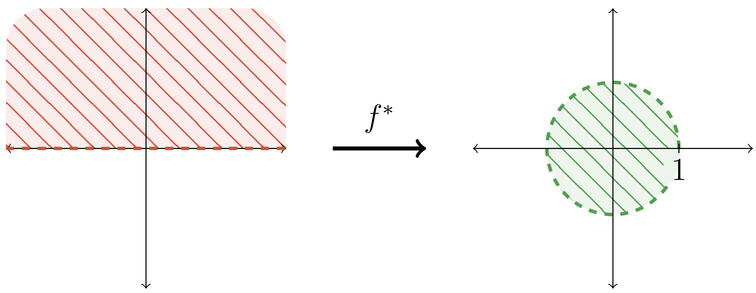
Then we find the linear fractional function f^* such that

$$\begin{aligned} f^*(-1) &= -1, \\ f^*(0) &= i, \\ f^*(1) &= 1, \end{aligned}$$

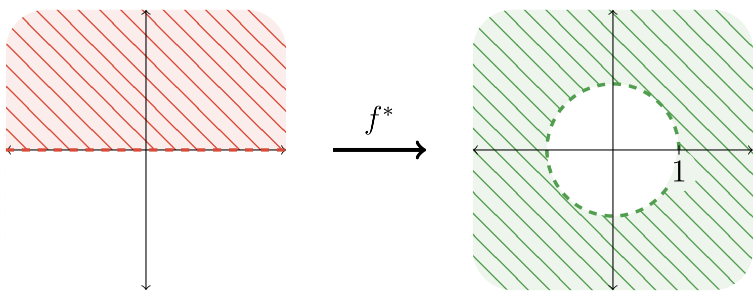
that is

$$f^*(z) = \begin{cases} \dfrac{z+i}{iz+1}, & z \in \mathbb{C}, \\ \dfrac{1}{i} = -i, & z = \infty. \end{cases}$$

Clearly either



(in which case we would (for $z \in \Omega$) define $f(z) := f^*(z^4)$), or[3]



(then we would define $f(z) := \frac{1}{f^*(z^4)}$ in Ω).

Because $f^*(i) = \infty$, the second case arose. We choose (for $z \in \Omega$)

$$\underline{f(z) := \frac{1}{f^*(z^4)} = \frac{iz^4+1}{z^4+i}}.$$

[3] For the right-hand-side image we need to imagine that $\infty = f^*(i)$.

EXERCISE 39.
Find the images of the lines parallel to the real and imaginary axes by the mapping $f(z) := \frac{1}{z}$ (consider the lines together with the point ∞).

Solution:
For $0 < c \in \mathbb{R}$ we have that

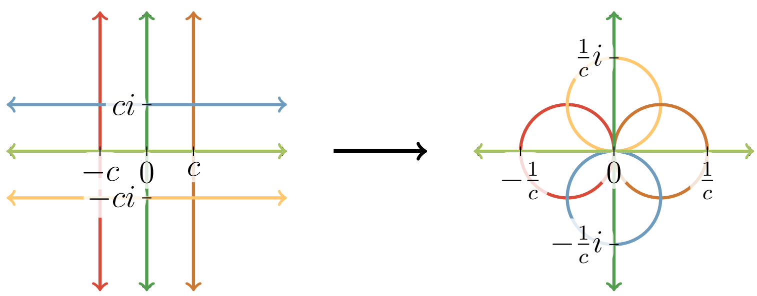
$$\begin{aligned} f(0) &= \infty, & f(\infty) &= 0, \\ f(i) &= -i, & f(c) &= \frac{1}{c}, \\ f(-i) &= \ \ i, & f(-c) &= -\frac{1}{c}, \\ f(1) &= \ \ 1, & f(ci) &= -\frac{1}{c}i, \\ f(-1) &= -1, & f(-ci) &= \ \ \frac{1}{c}i, \end{aligned}$$

and therefore also

$$\{z \in \mathbb{C}\colon\ \operatorname{Re} z = 0\} \cup \{\infty\} \to \{z \in \mathbb{C}\colon\ \operatorname{Re} z = 0\} \cup \{\infty\},$$
$$\{z \in \mathbb{C}\colon\ \operatorname{Im} z = 0\} \cup \{\infty\} \to \{z \in \mathbb{C}\colon\ \operatorname{Im} z = 0\} \cup \{\infty\},$$

$$\{z \in \mathbb{C}\colon\ \operatorname{Re} z = c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon\ \left|z - \frac{1}{2c}\right| = \frac{1}{2c}\right\},$$
$$\{z \in \mathbb{C}\colon\ \operatorname{Im} z = c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon\ \left|z + \frac{1}{2c}i\right| = \frac{1}{2c}\right\},$$

$$\{z \in \mathbb{C}\colon\ \operatorname{Re} z = -c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon\ \left|z + \frac{1}{2c}\right| = \frac{1}{2c}\right\},$$
$$\{z \in \mathbb{C}\colon\ \operatorname{Im} z = -c\} \cup \{\infty\} \to \left\{z \in \mathbb{C}\colon\ \left|z - \frac{1}{2c}i\right| = \frac{1}{2c}\right\}.$$



Exercise 40.
Find the images of the sets

$$M_\alpha = \{z \in \mathbb{C}\colon\ \arg z = \alpha\} \text{ and } N_r = \{z \in \mathbb{C}\colon\ |z| = r\},$$

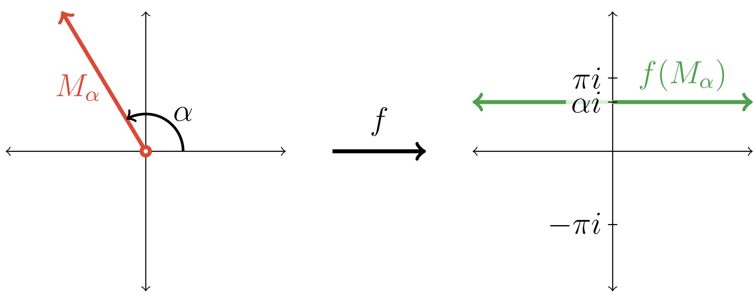
where $\alpha \in (-\pi, \pi\rangle$ and $r \in \mathbb{R}^+$, by the mapping $f(z) := \ln z$.

Solution:

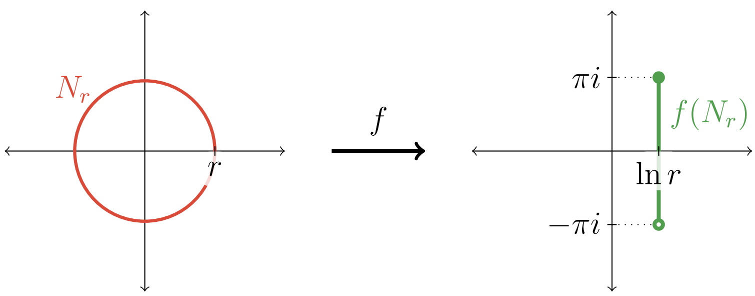
$$\ln z = \ln |z| + i \arg z,$$

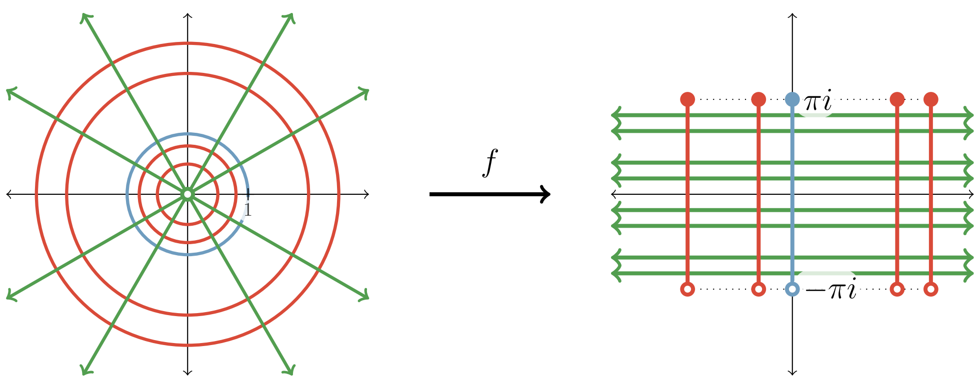
and therefore

$$f(M_\alpha) = \{z \in \mathbb{C}\colon\ \operatorname{Im} z = \alpha\},$$



$$f(N_r) = \{\ln r + ik\colon\ k \in (-\pi, \pi\rangle\}.$$





Exercise 41.
Compute

$$\int_\gamma |z| \, \mathrm{d}z,$$

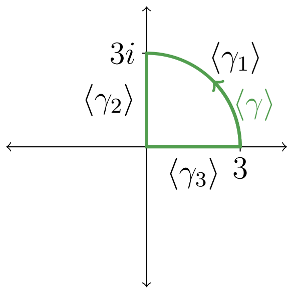
where

$$\gamma(t) := \begin{cases} 3e^{it}, & t \in \langle 0, \frac{\pi}{2} \rangle, \\ i\big(3 + \frac{\pi}{2} - t\big), & t \in \langle \frac{\pi}{2}, \frac{\pi}{2} + 3 \rangle, \\ t - \frac{\pi}{2} - 3, & t \in \langle \frac{\pi}{2} + 3, \frac{\pi}{2} + 6 \rangle. \end{cases}$$

Solution:
Let us choose

$$\gamma_1(t) := 3e^{it}, \ t \in \langle 0, \frac{\pi}{2} \rangle,$$
$$\gamma_2(t) := ti, \ i \in \langle 0, 3 \rangle,$$
$$\gamma_3(t) := t, \ i \in \langle 0, 3 \rangle.$$

Then



and

$$\gamma_1'(t) = 3ie^{it},$$
$$\gamma_2'(t) = i,$$
$$\gamma_3'(t) = 1,$$

and therefore

$$\underline{\int_\gamma |z| \, \mathrm{d}z} = \int_{\gamma_1} |z| \, \mathrm{d}z - \int_{\gamma_2} |z| \, \mathrm{d}z + \int_{\gamma_3} |z| \, \mathrm{d}z =$$
$$= \int_0^{\frac{\pi}{2}} 3 \cdot 3ie^{it} \, \mathrm{d}t - \int_0^3 ti \, \mathrm{d}t + \int_0^3 t \, \mathrm{d}t =$$
$$= 9i \int_0^{\frac{\pi}{2}} (\cos t + i \sin t) \, \mathrm{d}t + (1 - i) \int_0^3 t \, \mathrm{d}t =$$
$$= 9i[\sin t]_0^{\frac{\pi}{2}} + 9[\cos t]_0^{\frac{\pi}{2}} + (1 - i) \left[\frac{t^2}{2}\right]_0^3 =$$
$$= 9i - 9 + \frac{9}{2}(1 - i) =$$
$$= \underline{-\frac{9}{2} + \frac{9}{2} i}.$$

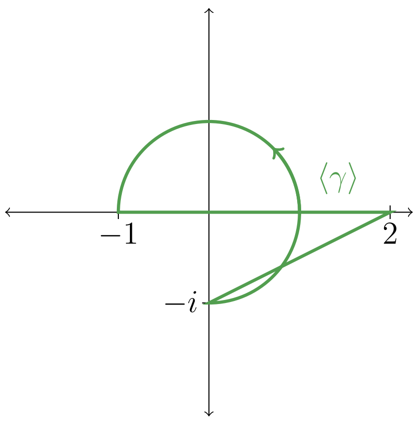
Exercise 42.
Compute

$$\int_\gamma z^3 \,\mathrm{d}z,$$

where

$$\gamma(t) := \begin{cases} e^{it}, & t \in \langle -\frac{\pi}{2}, \pi\rangle, \\ \frac{3}{\pi}t - 4, & t \in \langle \pi, 2\pi\rangle, \\ -\frac{2+i}{\pi}t + 6 + 2i, & t \in \langle 2\pi, 3\pi\rangle. \end{cases}$$

Solution:



It is enough to apply Cauchy's theorem.

$$\underline{\int_\gamma z^3 \,\mathrm{d}z = 0,}$$

because $f(z) := z^3$ is a holomorphic function on the simply connected domain $\mathbb{C}$ and γ is piecewise smooth closed curve in $\mathbb{C}$.

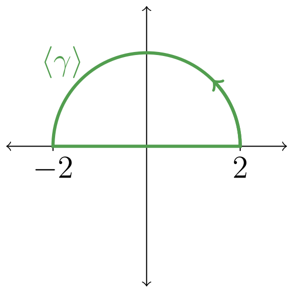
Exercise 43.
Compute

$$\int_\gamma |z|\,\overline{z} \,\mathrm{d}z,$$

where γ is a simple, closed, piecewise smooth and positively oriented curve such that $\langle\gamma\rangle$ is the boundary of the set

$$\{z \in \mathbb{C}\colon\ |z| < 2 \ \wedge\ \operatorname{Im} z > 0\}.$$

Solution:



Let us define the curves

$$\gamma_1(t) := 2e^{it},\ t \in \langle 0, \pi\rangle,$$
$$\gamma_2(t) := t,\ t \in \langle -2, 2\rangle.$$

Then

$$\gamma_1'(t) = 2ie^{it},$$
$$\gamma_2'(t) = 1,$$

and therefore

$$\underline{\int_\gamma |z|\,\overline{z}\,\mathrm{d}z} = \int_{\gamma_1} |z|\,\overline{z}\,\mathrm{d}z + \int_{\gamma_2} |z|\,\overline{z}\,\mathrm{d}z =$$
$$= \int_0^\pi 2\cdot 2e^{-it}\cdot 2ie^{it}\,\mathrm{d}t + \underbrace{\int_{-2}^{2} |t|\,t\,\mathrm{d}t}_{=0} =$$
$$= 8i\int_0^\pi 1\,\mathrm{d}t = \underline{8\pi i}.$$

<u>Exercise 44</u>.
Using the Cauchy's integral formulas calculate the given integrals[4]

a)

$$\int_k \frac{z^2+i}{z}\,\mathrm{d}z, \quad \text{where} \quad k = \{z\in\mathbb{C}\colon\ |z-2i| = 1\};$$

b)

$$\int_k \frac{\sin z}{z+i}\,\mathrm{d}z, \quad \text{where} \quad k = \{z\in\mathbb{C}\colon\ |z+i| = 1\};$$

[4] *Convention.* By the symbol $\int_k f(z)\,\mathrm{d}z$, where $k\subset\mathbb{C}$, we mean $\int_\gamma f(z)\,\mathrm{d}z$, where γ is a simple, closed, piecewise smooth and positively oriented curve such that $\langle\gamma\rangle = k$.

c)

$$\int_k \frac{\sin z}{z^2 - 7z + 10}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon\ |z| = 3\};$$

d)

$$\int_k \frac{\sin z}{(z - 2i)^3}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon\ |z| = 3\};$$

e)

$$\int_k \frac{\cos z}{z^2 - \pi^2}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon\ |z| = 4\};$$

f)

$$\int_k \frac{e^{\frac{1}{z}}}{(z^2 - 4)^2}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon\ |z - 2| = 1\};$$

g)

$$\int_\gamma \frac{e^z \cos(\pi z)}{z^2 + 2z}\,\mathrm{d}z, \quad \text{where} \quad \gamma(t) := \frac{3}{2}e^{it},\ t \in \langle 0, 2\pi\rangle;$$

h)

$$\int_\gamma \frac{\mathrm{d}z}{(z^2 - 1)^3}, \quad \text{where} \quad \gamma(t) := \frac{-2 + e^{-4\pi i t}}{2},\ t \in \langle 0, 4\rangle;$$

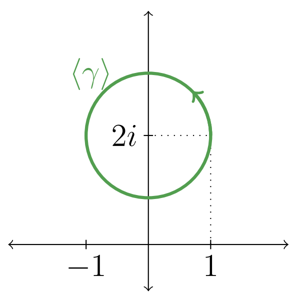
i)

$$\int_\gamma \frac{\mathrm{d}z}{(1 - z)(z + 2)(z - i)^2},$$

where γ is a simple, closed, piecewise smooth and positively oriented curve such that $-2 \in \operatorname{int}\gamma,\ i \in \operatorname{int}\gamma,\ 1 \in \operatorname{ext}\gamma$.

Solution:

a)



The function "$\frac{z^2+i}{z}$" is holomorphic on a simply connected domain

$$\Omega := \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}$$

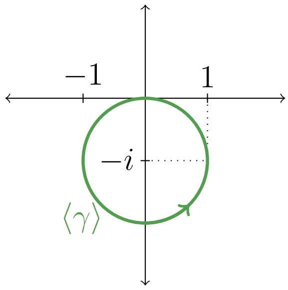
and $k = \langle\gamma\rangle \subset \Omega$, and therefore it follows from the Cauchy's theorem that

$$\int_k \frac{z^2 + i}{z}\,\mathrm{d}z = \int_\gamma \frac{z^2 + i}{z} = 0.$$

But we were supposed to use the Cauchy's integral formulas. Which we can do for example as

$$\underline{\int_k \frac{z^2+i}{z}\,\mathrm{d}z} = \int_\gamma \frac{\frac{z^2+i}{z}(z-2i)}{z-2i}\,\mathrm{d}z = 2\pi i\left[\frac{z^2+i}{z}(z-2i)\right]_{z=2i} = \underline{0}.$$

b)



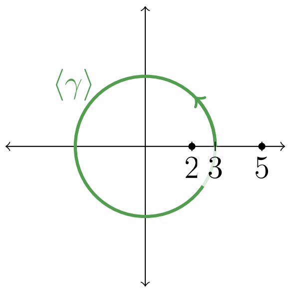
$$\underline{\int_k \frac{\sin z}{z+i}\,\mathrm{d}z} = \int_\gamma \frac{\sin z}{z-(-i)}\,\mathrm{d}z =$$

$$= 2\pi i\,[\sin z]_{z=-i} =$$

$$= 2\pi i\,\frac{e^{i(-i)}-e^{-i(-i)}}{2i} =$$

$$= \pi(e-e^{-1}) = \underline{2\pi\sinh 1}.$$

c)



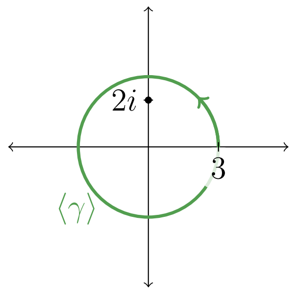
$z^2-7z+10=(z-5)(z-2)$, and therefore

$$\underline{\int_k \frac{\sin z}{z^2-7z+10}\,\mathrm{d}z} = \int_\gamma \frac{\frac{\sin z}{z-5}}{z-2}\,\mathrm{d}z =$$

$$= 2\pi i\left[\frac{\sin z}{z-5}\right]_{z=2} =$$

$$= 2\pi i\,\frac{\sin 2}{-3} = \underline{-\left(\frac{2}{3}\pi\sin 2\right)i}.$$

d)

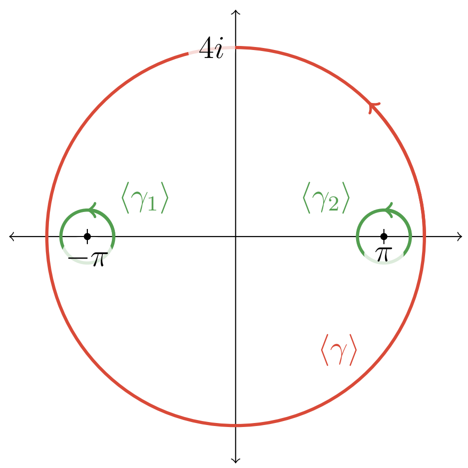


$$\underline{\int_k \frac{\sin z}{(z-2i)^3}\,\mathrm{d}z} = \frac{2\pi i}{2!}\left[(\sin z)''\right]_{z=2i} =$$

$$= \pi i\left[-\sin z\right]_{z=2i} =$$

$$= -\pi i\,\frac{e^{-2}-e^{2}}{2i} = \underline{\pi \sinh 2}.$$

e)

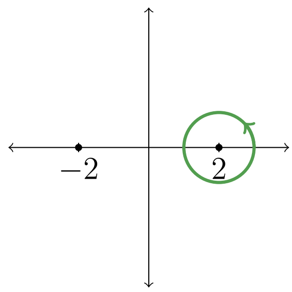


$$\underline{\int_k \frac{\cos z}{z^2-\pi^2}\,\mathrm{d}z} = \int_{\gamma_1} \frac{\frac{\cos z}{z-\pi}}{z-(-\pi)}\,\mathrm{d}z + \int_{\gamma_2} \frac{\frac{\cos z}{z+\pi}}{z-\pi}\,\mathrm{d}z =$$

$$= 2\pi i\left(\left[\frac{\cos z}{z-\pi}\right]_{z=-\pi} + \left[\frac{\cos z}{z+\pi}\right]_{z=\pi}\right) =$$

$$= 2\pi i\left(\frac{-1}{-2\pi} + \frac{-1}{2\pi}\right) = \underline{0}.$$

f)



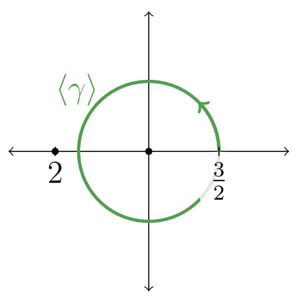
$$\underline{\int_k \frac{e^{\frac{1}{z}}}{(z^2-4)^2}\,\mathrm{d}z} = \int_k \frac{\frac{e^{\frac{1}{z}}}{(z+2)^2}}{(z-2)^2}\,\mathrm{d}z =$$

$$= 2\pi i \left[\left(\frac{e^{\frac{1}{z}}}{(z+2)^2}\right)'\right]_{z=2} =$$

$$= 2\pi i \left[\frac{-\frac{1}{z^2}e^{\frac{1}{z}}(z+2) - e^{\frac{1}{z}}2}{(z+2)^3}\right]_{z=2} =$$

$$= 2\pi i \frac{\sqrt{e}\left(-\frac{1}{4}\cdot 4 - 2\right)}{16\cdot 4} = \underline{-\frac{3\pi\sqrt{e}}{32}i}.$$

g)

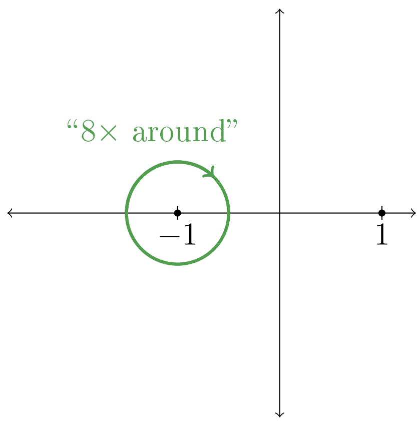


$$\underline{\int_\gamma \frac{e^z\cos(\pi z)}{z^2+2z}\,\mathrm{d}z} = \int_\gamma \frac{\frac{e^z\cos(\pi z)}{z+2}}{z-0}\,\mathrm{d}z =$$

$$= 2\pi i \left[\frac{e^z\cos(\pi z)}{z+2}\right]_{z=0} =$$

$$= 2\pi i \frac{1}{2} = \underline{\pi i}.$$

h)



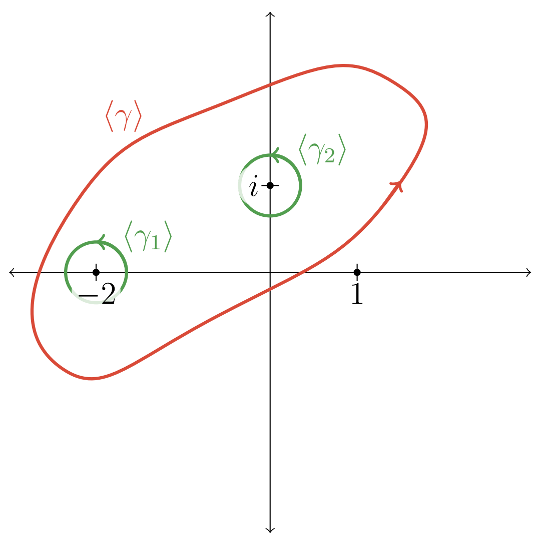
$$\underline{\int_\gamma \frac{\mathrm{d}z}{(z^2-1)^3}} = \int_\gamma \frac{\frac{1}{(z-1)^3}}{(z+1)^3}\,\mathrm{d}z =$$

$$= -8 \cdot \frac{2\pi i}{2!}\left[\left(\frac{1}{(z-1)^3}\right)''\right]_{z=-1} =$$

$$= -8\pi i\left[-3\left(\frac{1}{(z-1)^4}\right)'\right]_{z=-1} =$$

$$= -8 \cdot 12\pi i\left[\frac{1}{(z-1)^5}\right]_{z=-1} = \underline{3\pi i}.$$

i)



$$\underline{\int_\gamma \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2}} = \int_{\gamma_1} \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2} + \int_{\gamma_2} \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2} =$$

$$= \int_{\gamma_1} \frac{\frac{1}{(1-z)(z-i)^2}}{z+2}\,\mathrm{d}z + \int_{\gamma_2} \frac{\frac{1}{(1-z)(z+2)}}{(z-i)^2}\,\mathrm{d}z =$$

$$= 2\pi i \left(\left[\frac{1}{(1-z)(z-i)^2} \right]_{z=-2} + \left[\left(\frac{1}{(1-z)(z+2)} \right)' \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{1}{3(-2-i)^2} + \left[\left(\frac{1}{-z^2-z+2} \right)' \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{1}{3(3+4i)} + \left[\frac{2z+1}{(-z^2-z+2)^2} \right]_{z=i} \right) =$$

$$= 2\pi i \left(\frac{3-4i}{75} + \frac{1+2i}{(3-i)^2} \right) = 2\pi i \left(\frac{3-4i}{75} + \frac{(1+2i)(8+6i)}{100} \right) =$$

$$= 2\pi i \left(\frac{3}{75} - \frac{4}{100} - \frac{4}{75}i + \frac{22}{100}i \right) = 2\pi i \frac{-\frac{8}{3}+11}{50} i = \underline{-\frac{1}{3}\pi}.$$

<u>Exercise 45</u>.
Compute

a) $\int_0^{1+i} e^z \,\mathrm{d}z$;

b) $\int_0^{1+i} z^3 \,\mathrm{d}z$;

c) $\int_0^{i} z^2 \sin z \,\mathrm{d}z$;

d) $\int_0^{i} z \sin z \,\mathrm{d}z$.

<u>Solution:</u>

a)

$$\underline{\int_0^{1+i} e^z \,\mathrm{d}z} = [e^z]_0^{1+i} = e^1(\cos 1 + i \sin 1) - 1 =$$
$$= \underline{e\cos 1 - 1 + i(\sin 1)e}.$$

b)

$$\underline{\int_0^{1+i} z^3 \,\mathrm{d}z} = \left[\frac{z^4}{4} \right]_0^{1+i} = \frac{1}{4}(2i)^2 = \underline{-1}.$$

c)

$$\underline{\int_0^i z^2 \sin z \, \mathrm{d}z} = \int_0^i \underbrace{z^2}_{=:u} \underbrace{\sin z}_{=:v'} \, \mathrm{d}z =$$

$$= \left[-z^2 \cos z\right]_0^i + \int_0^i \underbrace{2z}_{=:u} \underbrace{\cos z}_{=:v'} \, \mathrm{d}z =$$

$$= \cos i + [2z \sin z]_0^i - 2 \int_0^i \sin z \, \mathrm{d}z =$$

$$= \cos i + 2i \sin i + 2[\cos z]_0^i =$$

$$= \frac{e^{-1} + e^1}{2} + 2i \frac{e^{-1} - e^1}{2i} + 2\frac{e^{-1} + e^1}{2} - 2 =$$

$$= \underline{3 \cosh 1 - 2 \sinh 1 - 2}$$

(twice we used integration by parts).

d)

$$\underline{\int_0^i z \sin z \, \mathrm{d}z} = [-z \cos z]_0^i + [\sin z]_0^i =$$

$$= -i \cos i + \sin i =$$

$$= -i \cosh 1 + i \sinh 1 =$$

$$= i(\sinh 1 - \cosh 1) = \underline{-\frac{1}{e} i}$$

(we again integrated by parts).

Exercise 46.
Decide if the given series converges

a) $\sum\limits_{n=1}^{\infty} \frac{i^n}{n2^n}$;

b) $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n$;

c) $\sum\limits_{n=1}^{\infty} \frac{(-i)^n}{3n-17}$.

Solution:

a) $\underline{\sum\limits_{n=1}^{\infty} \frac{i^n}{n2^n} \text{ converges absolutely}}$, because

$$\sqrt[n]{\left|\frac{i^n}{n2^n}\right|} = \frac{1}{\sqrt[n]{n}\,2} \to \frac{1}{2} < 1.$$

b) $\underline{\sum\limits_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n \text{ converges absolutely, because}}$

$$\sqrt[n]{\left|\frac{n}{3^n}(1+i)^n\right|} = \frac{\sqrt[n]{n}}{3}\sqrt{2} \to \frac{\sqrt{2}}{3} < 1.$$

c) $\underline{\sum\limits_{n=1}^{\infty} \frac{(-i)^n}{3n-17} \text{ converges conditionally, because the series}}$

$$\sum_{n=1}^{\infty} \operatorname{Re}\left(\frac{(-i)^n}{3n-17}\right) \quad \text{i} \quad \sum_{n=1}^{\infty} \operatorname{Im}\left(\frac{(-i)^n}{3n-17}\right)$$

converges (it is enough to realize that

$$\sum_{n=1}^{\infty} \frac{(-i)^n}{3n-17} = \frac{-i}{3\cdot 1-17} + \frac{-1}{3\cdot 2-17} + \frac{i}{3\cdot 3-17} + \frac{1}{3\cdot 4-17} + \frac{-i}{3\cdot 5-17} + \cdots,$$

and to use the Leibniz criterion on the series, and the easy observation that from the convergence of the series $a_1 + a_2 + a_3 + \cdots$ follows the convergence of the series $0 + a_1 + 0 + a_2 + 0 + a_3 + \cdots$), and furthermore

$$\sum_{n=1}^{\infty} \left|\frac{(-i)^n}{3n-17}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{3n-17}\right| = \infty$$

(see the integral criterion).

Exercise 47.

Find the domain of convergence of a given series (that is find all $z \in \mathbb{C}$, for which the given series converges).

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n$;

b) $\sum\limits_{n=1}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$.

Solution:

a) For $z = 1$ the series clearly diverges. For $z \in \mathbb{C} \setminus \{1\}$ we have that

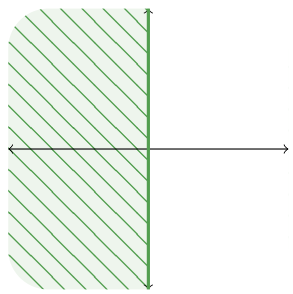
$$\sqrt[n]{\left|\frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n\right|} = \frac{1}{(\sqrt[n]{n})^2} \cdot \left|\frac{z+1}{z-1}\right| \to \left|\frac{z+1}{z-1}\right|,$$

and therefore the given series converges absolutely for every $z \in \mathbb{C}$ such that $\left|\frac{z+1}{z-1}\right| < 1$, and diverges for every $z \in \mathbb{C}$ for which $\left|\frac{z+1}{z-1}\right| > 1$.

If $\left|\frac{z+1}{z-1}\right| = 1$, we have $\left|\frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n\right| = \frac{1}{n^2}$, and therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n$ converges absolutely.

Summary: the given series converges (absolutely) for every

$$z \in \left\{ z \in \mathbb{C}\colon \ \left| \frac{z+1}{z-1} \right| \le 1 \right\} = \{ z \in \mathbb{C}\colon \ \operatorname{Re} z \le 0 \}.$$



b) Because for every $z \in \mathbb{C} \setminus \{0\}$

$$\left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \frac{|z|}{n+1} \to 0 < 1,$$

the series $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ converges absolutely in $\mathbb{C}$. Because for every $z \in \mathbb{C} \setminus \{0\}$

$$\sqrt[n]{\left| \frac{n^2}{z^n} \right|} = \frac{\left(\sqrt[n]{n}\right)^2}{|z|} \to \frac{1}{|z|},$$

the series $\sum_{n=1}^{\infty} \frac{n^2}{z^n}$ converges absolutely for $|z| > 1$ and diverges for $|z| < 1$. If $|z| = 1$ we have

$$\left| \frac{n^2}{z^n} \right| = n^2 \to \infty \neq 0,$$

and therefore the series $\sum_{n=1}^{\infty} \frac{n^2}{z^n}$ diverges.

Let us now define

$$s_n(z) := \sum_{k=1}^{n} \left(\frac{z^k}{k!} + \frac{k^2}{z^k} \right),$$
$$s_n^*(z) := \sum_{k=1}^{n} \frac{z^k}{k!},$$
$$s_n^{**}(z) := \sum_{k=1}^{n} \frac{k^2}{z^k}.$$

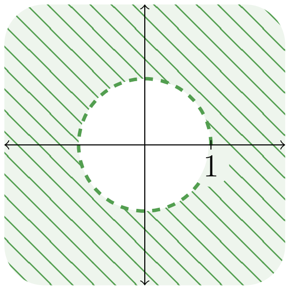
Then for every $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have that

$$s_n(z) = s_n^*(z) + s_n^{**}(z),$$
$$s_n^{**} = s_n(z) - s_n^*(z),$$

and furthermore (we already know that) $\lim s_n^*(z) \in \mathbb{C}$ for every $z \in \mathbb{C}$, and therefore for every $z \in \mathbb{C}$ we have that

$$\lim s_n(z) \in \mathbb{C} \ \Leftrightarrow \ \lim s_n^{**}(z) \in \mathbb{C}.$$

Summary: the given series covnerges (absolutely) on the set $\{z \in \mathbb{C}\colon\ |z| > 1\}$.



Exercise 48.

Find the radius of convergence R of the given power series

a) $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^{2011}}$;

b) $\sum\limits_{n=1}^{\infty} n^n (z-1)^n$;

c) $\sum\limits_{n=1}^{\infty} \frac{3^n (z-1)^n}{\sqrt{(3n-2)2^n}}$;

d) $\sum\limits_{n=0}^{\infty} \frac{(z+1+i)^n}{3^n (n-i)}$;

e) $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!} z^n$;

f) $\sum\limits_{n=0}^{\infty} \big(\cos(in)\big) z^n$;

g) $\sum\limits_{n=0}^{\infty} (n^2-n-2) z^n$;

h) $\sum\limits_{n=0}^{\infty} \frac{z^n}{(n+8)!}$.

Solution:

a)

$$\sqrt[n]{\frac{1}{n^{2011}}} = \frac{1}{(\sqrt[n]{n})^{2011}} \to 1,$$

and therefore

$$\underline{R = 1}.$$

b)

$$\sqrt[n]{n^n} = n \to \infty,$$

and therefore

$$\underline{R = \frac{1}{\infty} = 0}.$$

c) Because

$$\sqrt[n]{\frac{3^n}{\sqrt{(3n-2)2^n}}} = \frac{3}{\sqrt{2}} \frac{1}{\sqrt{\sqrt[n]{3n-2}}} \to \frac{3}{\sqrt{2}}$$

(it is enough to realize that for $n \geq 3$ we have that $1 \leq \sqrt[n]{3n-2} \leq \sqrt[n]{n} \cdot \sqrt[n]{n} \to 1$), and therefore

$$\underline{R = \frac{\sqrt{2}}{3}}.$$

d)

$$\left|\frac{\frac{1}{3^{n+1}(n+1-i)}}{\frac{1}{3^n(n-i)}}\right| = \frac{1}{3}\left|\frac{n-i}{n+1-i}\right| = \frac{1}{3}\left|\frac{1-\frac{i}{n}}{1+\frac{1-i}{n}}\right| \to \frac{1}{3},$$

and therefore

$$\underline{R = 3}.$$

e)

$$\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{1}{n+1}\frac{(n+1)^n(n+1)}{n^n} = \left(1+\frac{1}{n}\right)^n \to e,$$

and therefore

$$\underline{R = \frac{1}{e}}.$$

f)

$$\left|\frac{\cos(i(n+1))}{\cos(in)}\right| = \left|\frac{e^{i(i(n+1))}+e^{-i(i(n+1))}}{e^{iin}+e^{-iin}}\right| =$$

$$= \frac{e^{-(n+1)}+e^{n+1}}{e^{-n}+e^{n}} \cdot \frac{\frac{1}{e^n}}{\frac{1}{e^n}} =$$

$$= \frac{\frac{1}{e^n e^{n+1}}+e}{\frac{1}{e^n e^n}+1} \to e,$$

and therefore

$$\underline{R = \frac{1}{e}}.$$

g)

$$\left|\frac{(n+1)^2-(n+1)-2}{n^2-n-2}\right| \to 1,$$

and therefore

$$\underline{R = 1}.$$

h)

$$\frac{\frac{1}{(n+9)!}}{\frac{1}{(n+8)!}} = \frac{1}{n+9} \to 0,$$

and therefore

$$\underline{R = \infty}.$$

<u>Exercise 49</u>.
Find the sum of the power series in the disk of convergence

a) $\sum_{n=1}^{\infty} nz^n$; b) $\sum_{n=1}^{\infty} \frac{z^n}{n}$;

c) $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$;

e) $\sum_{n=0}^{\infty} (n^2 - n - 2) z^n$.

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n+1}$;

Solution:

a) $\sqrt[n]{n} \to 1$, and therefore the radius of convergence of a given series is 1.

For every $z \in \mathbb{C}$, $|z| < 1$ we have that

$$\underline{\sum_{n=1}^{\infty} n z^n} = z \sum_{n=1}^{\infty} n z^{n-1} = z \left(\sum_{n=1}^{\infty} n \frac{z^n}{n} \right)' =$$

$$= z \left(\sum_{n=1}^{\infty} z^n \right)' = z \left(\frac{z}{1-z} \right)' =$$

$$= z \frac{1 - z + z}{(1-z)^2} = \underline{\frac{z}{(1-z)^2}}.$$

b) $\sqrt[n]{\frac{1}{n}} \to 1$, therefore the radius of convergence is 1.

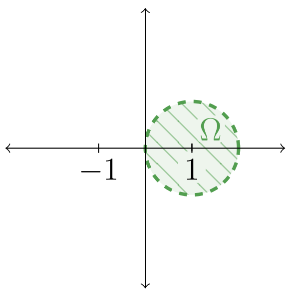
Let us define the function $f(z) := \sum_{n=1}^{\infty} \frac{z^n}{n}$. Then for every $z \in \mathbb{C}$, $|z| < 1$ we have that

$$f'(z) = \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z}.$$

From this, because

$$|z| < 1 \;\Rightarrow\; 1 - z \in \Omega := \{w \in \mathbb{C} \colon\; |w - 1| < 1\},$$

$$\ln' w = \frac{1}{w} \text{ v } \Omega,$$



there is a $c \in \mathbb{C}$ such that for each $z \in \mathbb{C}$, $|z| < 1$ we have that

$$f(z) = -\ln(1-z) + c.$$

Furthermore $f(0) = -\ln 1 + c = 0$, and therefore $c = 0$.

Summary: for each $z \in \mathbb{C}$, $|z| < 1$ we have that

$$\underline{\sum_{n=1}^{\infty} \frac{z^n}{n}} = f(z) = \underline{-\ln(1-z)}.$$

c)

$$\left|\frac{\frac{1}{2n+3}}{\frac{1}{2n+1}}\right| \to 1,$$

and therefore the radius of convergence is 1.

Let us define $f(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$. Then for every $z \in \mathbb{C},\ |z| < 1$ we have that

$$f'(z) = \sum_{n=0}^{\infty} z^{2n} = \frac{1}{1-z^2} = \\ = \frac{1}{2}\frac{-1}{z-1} + \frac{1}{2}\frac{1}{z+1}.$$

From that it follows that there is $c \in \mathbb{C}$, such that for each $z \in \mathbb{C}, |z| < 1$ we have that

$$f(z) = -\frac{1}{2}\ln(1-z) + \frac{1}{2}\ln(1+z) + c.$$

And because $0 = f(0) = c$ for each $z \in \mathbb{C},\ |z| < 1$, we have that

$$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} = -\frac{1}{2}\ln(1-z) + \frac{1}{2}\ln(1+z).$$

d) $\sqrt[n]{\frac{1}{n+1}} \to 1$, and therefore the radius of convergence is 1.

Let $f(z) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{n+1}}{n+1}$. Then for $z \in \mathbb{C},\ |z| < 1$, we have that

$$f'(z) = \sum_{n=1}^{\infty} (-1)^{n+1} z^n = -\sum_{n=1}^{\infty} (-z)^n = \\ = \frac{z+1-1}{1+z}.$$

From that it follows that there is a $c \in \mathbb{C}$ such that

$$f(z) = z - \ln(1+z) + c,$$

and because $0 = f(0) = c$ we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n+1} = \begin{cases} \frac{1}{z} f(z) = 1 - \frac{\ln(1+z)}{z}, & 0 < |z| < 1, \\ 0, & z = 0. \end{cases}$$

e)

$$\left|\frac{(n+1)^2 - (n+1) - 2}{n^2 - n - 2}\right| \to 1,$$

and therefore the radius of convergence is 1.

For every $z \in \mathbb{C},\ |z| < 1$, we have that

$$\sum_{n=0}^{\infty} (n^2 - n - 2) z^n = \sum_{n=0}^{\infty} n^2 z^n - \sum_{n=0}^{\infty} n z^n - 2\sum_{n=0}^{\infty} z^n$$

(it is enough to realize that each of the series is absolutely convergent).

Furthermore ($|z| < 1$):

- $$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

- $$\begin{aligned}\sum_{n=0}^{\infty} nz^n &= \sum_{n=1}^{\infty} nz^n = z\sum_{n=1}^{\infty} nz^{n-1} = \\ &= z\left(\sum_{n=1}^{\infty} n\frac{z^n}{n}\right)' = z\left(\frac{z}{1-z}\right)' = \\ &= z\,\frac{1-z+z}{(1-z)^2} = \frac{z}{(1-z)^2},\end{aligned}$$

- $$\begin{aligned}\sum_{n=0}^{\infty} n^2z^n &= \sum_{n=1}^{\infty} n^2z^n = z\sum_{n=1}^{\infty} n^2z^{n-1} = \\ &= z\left(\sum_{n=1}^{\infty} n^2\frac{z^n}{n}\right)' = z\left(\sum_{n=1}^{\infty} nz^n\right)' = \\ &= z\left(\frac{z}{(1-z)^2}\right)' = z\frac{(1-z)^2 + z\,2\,(1-z)}{(1-z)^4} = \\ &= z\,\frac{z+1}{(1-z)^3},\end{aligned}$$

and therefore <u>for every $z \in \mathbb{C},\ |z| < 1$, we have that</u>

$$\underline{\sum_{n=0}^{\infty}(n^2-n-2)z^n = \frac{z^2+z-z(1-z)-2(1-z)^2}{(1-z)^3} = \frac{2-4z}{(z-1)^3}.}$$

Exercise 50.
Find the sum of the given series

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n2^n}$; b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$.

Solution:
Let us consider the function

$$f(z) := \sum_{n=1}^{\infty} \frac{z^n}{n2^n}.$$

Because $\frac{1}{\sqrt[n]{n2^n}} \to \frac{1}{2}$, the power series in the definition of the function f has the radius of convergence 2. Therefore for every $z \in \mathbb{C},\ 0 < |z| < 2$, we have that

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{2^n} = \frac{1}{z} \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n = $$
$$= \frac{1}{z} \frac{\frac{z}{2}}{1 - \frac{z}{2}} = \frac{1}{2 - z}.$$

Therefore there is a $c \in \mathbb{C}$ for which $f(z) = -\ln(2-z)+c$. And because $f(0) = 0 = -\ln 2+c$, for every $z \in \mathbb{C},\ |z| < 2$ we have that

$$f(z) = -\ln(2 - z) + \ln 2.$$

a)

$$\underline{\sum_{n=1}^{\infty} \frac{1}{n2^n}} = f(1) = \underline{\ln 2},$$

b)

$$\underline{\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}} = f(-1) = \underline{-\ln 3 + \ln 2 = \ln \frac{2}{3}}.$$

Exercise 51.
Find the Taylor series of the function f centered at z_0 and find its radius of convergence, where

a) $f(z) := \frac{z+1}{z^2+4z-5},\ z_0 = -1$;

b) $f(z) := \frac{z}{z^2+i},\ z_0 = 0$;

c) $f(z) := \ln \frac{1+z}{1-z},\ z_0 = 0$;

d) $f(z) := e^{3z-2},\ z_0 = 1$;

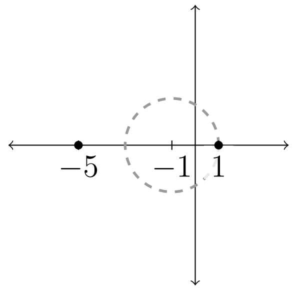
e) $f(z) := \sin(3z^2 + 2),\ z_0 = 0$;

f) $f(z) := \frac{1}{(z-1)^3},\ z_0 = 3$;

g) $f(z) := \sin^2 z,\ z_0 = 0$.

Solution:

a) $f(z) = \frac{2}{3}\frac{1}{z+5} + \frac{1}{3}\frac{1}{z-1}$,



and therefore the radius of convergence is 2 and for every $z \in \mathbb{C},\ |z+1| < 2$, it holds, that

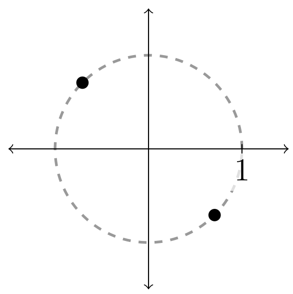
$$\underline{f(z)} = \frac{2}{3}\cdot\frac{1}{4+z+1} + \frac{1}{3}\cdot\frac{1}{-2+z+1} = \frac{2}{12}\cdot\frac{1}{1+\frac{z+1}{4}} - \frac{1}{6}\cdot\frac{1}{1-\frac{z+1}{2}} =$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\frac{(z+1)^n}{4^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^n} =$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{6\cdot 4^n} - \frac{1}{6\cdot 2^n}\right)(z+1)^n =$$

$$= \underline{\sum_{n=1}^{\infty}\frac{(-1)^n - 2^n}{6\cdot 4^n}(z+1)^n}.$$

b) $z^2 + i = 0$ if and only if $z = \pm\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$,



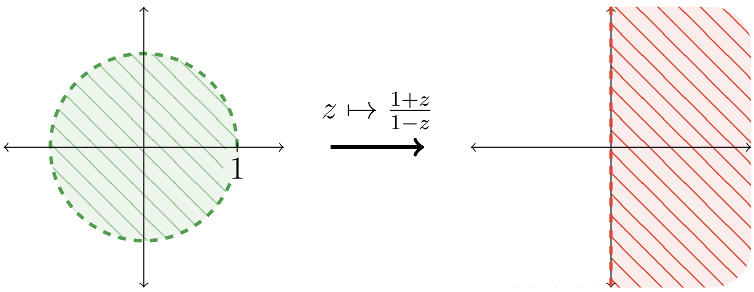
and therefore the sought Taylor series has the radius of convergence 1. For every $z \in \mathbb{C},\ |z| < 1$, we have that

$$\underline{f(z)} = \frac{z}{i}\cdot\frac{1}{1+\frac{z^2}{i}} = \frac{z}{i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^2}{i}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}z^{2n+1} =$$

$$= \underline{\sum_{n=0}^{\infty} i^{n-1}z^{2n+1}}.$$

c) Because clearly



($0 \mapsto 0$, $1 \mapsto \infty$, $-1 \mapsto 0$), the radius of convergence is 1. For every $z \in \mathbb{C}$, $|z| < 1$ we have that

$$f'(z) = \frac{1-z}{1+z} \cdot \frac{1-z+(1+z)}{(1-z)^2} = \frac{2}{(1+z)(1-z)} =$$
$$= \frac{2}{1-z^2} = \sum_{n=0}^{\infty} 2z^{2n},$$

and therefore there is a $c \in \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} 2\,\frac{z^{2n+1}}{2n+1} + c.$$

And because $f(0) = 0 = c$, for every $z \in \mathbb{C}$, $|z| < 1$ we have that

$$f(z) = \sum_{n=0}^{\infty} 2\,\frac{z^{2n+1}}{2n+1}.$$

d) Clearly the radius of convergence is ∞. We know that for every $z \in \mathbb{C}$ we have $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, and therefore

$$f(z) = e^{3z-2} = e^{3(z-1)+1} = e\,e^{3(z-1)} =$$
$$= \sum_{n=0}^{\infty} \frac{e \cdot 3^n}{n!}(z-1)^n.$$

e) The radius of convergence is ∞ and for any $z \in \mathbb{C}$ we have that

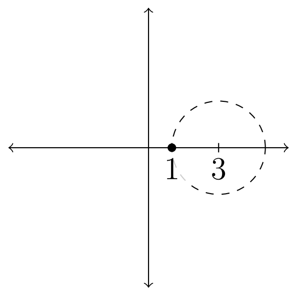
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

From this it follows that for every $z \in \mathbb{C}$ we have that

$$\begin{aligned}
f(z) &= \sin(3z^2)\cos 2 + \cos(3z^2)\sin 2 = \\
&= \sum_{n=0}^{\infty} \underbrace{\cos 2 \cdot (-1)^n \frac{3^{2n+1}}{(2n+1)!}}_{=:\alpha_n} z^{4n+2} + \sum_{n=0}^{\infty} \underbrace{\sin 2 \cdot (-1)^n \frac{3^{2n}}{(2n)!}}_{=:\beta_n} z^{4n} = \\
&= \sum_{n=0}^{\infty} a_n z^{2n},
\end{aligned}$$

where $a_{2k} := \beta_k$ and $a_{2k+1} := \alpha_k$ for every $k \in \mathbb{N} \cup \{0\}$.

f)



Clearly the radius of convergence is 2. For every $z \in U(3,2)$ we have that

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{2+z-3} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-3}{2}} = \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-3}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-3)^n
\end{aligned}$$

and

$$\left(\frac{1}{z-1}\right)'' = \left(-\frac{1}{(z-1)^2}\right)' = 2\,\frac{1}{(z-1)^3}.$$

From this it easily follows, that for each $z \in U(3,2)$ we have that

$$\begin{aligned}
f(z) &= \frac{1}{2}\left(\frac{1}{z-1}\right)'' = \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} n(z-3)^{n-1}\right)' = \\
&= \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^n}{2^{n+1}} n(n-1)(z-3)^{n-2} = \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n}{2^{n+2}} n(n-1)(z-3)^{n-2} = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+4}} (n+2)(n+1)(z-3)^n.
\end{aligned}$$

g) <u>The radius of convergence is ∞</u> and <u>for any $z \in \mathbb{C}$ we have that</u>

$$\underline{f(z)} = \sin^2 z = \frac{1-\cos(2z)}{2} = $$
$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n}}{(2n)!} = $$
$$= \underline{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}z^{2n}}.$$

<u>**Exercise 52**.</u>

Find the domain of convergence of the given Laurent series (that is find all $z \in \mathbb{C}$, for which the given series converges).

a) $\sum\limits_{n=-\infty}^{\infty} 2^{-|n|}z^n$;

b) $\sum\limits_{n=-\infty}^{\infty} \frac{(z-i)^n}{n^2+1}$.

<u>**Solution:**</u>

a)

$$\sum_{n=-\infty}^{\infty} 2^{-|n|}z^n = \sum_{n=0}^{\infty} 2^{-n}z^n + \sum_{n=1}^{\infty} 2^{-n}z^{-n} = $$
$$= \sum_{n=0}^{\infty}\frac{1}{2^n}z^n + \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{z^n}.$$

Because the power series $\sum\limits_{n=0}^{\infty}\frac{1}{2^n}z^n$ has the radius of convergence 2 (clearly $\sqrt[n]{\frac{1}{2^n}} \to \frac{1}{2}$), the following implications hold:

$$|z| < 2 \;\Rightarrow\; \sum_{n=0}^{\infty}\frac{1}{2^n}z^n \text{ converges absolutely,}$$

$$|z| > 2 \;\Rightarrow\; \sum_{n=0}^{\infty}\frac{1}{2^n}z^n \text{ diverges.}$$

If $|z| = 2$, then $\left|\frac{1}{2^n}z^n\right| = 1 \to 1 \neq 0$, and therefore the series $\sum\limits_{n=0}^{\infty}\frac{1}{2^n}z^n$ diverges.

Let us now consider the regular part of the given series, that is the series $\sum\limits_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{z^n}$. We have found out that

$$\left|\frac{1}{z}\right| < 2 \left(\text{tj. } |z| > \frac{1}{2}\right) \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{z^n} \text{ converges absolutely,}$$

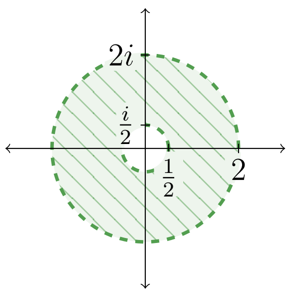
$$\left|\frac{1}{z}\right| > 2 \left(\text{tj. } |z| < \frac{1}{2}\right) \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{z^n} \text{ diverges,}$$

$$\left|\frac{1}{z}\right| = 2 \left(\text{tj. } |z| = \frac{1}{2}\right) \;\Rightarrow\; \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{1}{z^n} \text{ diverges.}$$

Conclusion: <u>the given series converges (absolutely) for every</u>

$$\underline{z \in P\left(0, \frac{1}{2}, 2\right) = \left\{z \in \mathbb{C}\colon\; \frac{1}{2} < |z| < 2\right\},}$$

<u>elsewhere it diverges.</u>



b)

$$\sum_{n=-\infty}^{\infty} \frac{(z-i)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n + \sum_{n=1}^{\infty} \frac{1}{n^2+1}\,\frac{1}{(z-i)^n}.$$

Because

$$\frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n^2+1}} \to 1,$$

we know that

$$|z-i| < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n \text{ converges absolutely,}$$

$$|z-i| > 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}(z-i)^n \text{ diverges.}$$

If $|z-i| = 1$, we have that

$$\sum_{n=0}^{\infty} \left|\frac{(z-i)^n}{n^2+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n^2+1},$$

and therefore (see the integral criterion) the series $\sum\limits_{n=0}^{\infty} \frac{(z-i)^n}{n^2+1}$ converges absolutely.

We have (also) found that

$$\left|\frac{1}{z-i}\right| < 1 \text{ (tj. } |z-i| > 1) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{1}{(z-i)^n} \text{ converges absolutely,}$$

$$\left|\frac{1}{z-i}\right| > 1 \text{ (tj. } |z-i| < 1) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{1}{(z-i)^n} \text{ diverges,}$$

$$\left|\frac{1}{z-i}\right| = 1 \text{ (tj. } |z-i| = 1) \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{1}{(z-i)^n} \text{ converges absolutely.}$$

Summary: the given series converges (absolutely) for every $z \in \{z \in \mathbb{C}\colon\ |z-i| = 1\}$. Elsewhere it diverges.

Exercise 53.

Find the Laurent series of the function f on the given annulus

a) $f(z) := \frac{\cos z}{z^2},\ 0 < |z| < 1$;

b) $f(z) := \frac{1}{z^2+1},\ |z| > 1$;

c) $f(z) := \frac{z^2+1}{z(z-i)},\ \frac{1}{2} < |z-i| < 1$;

d) $f(z) := \frac{1}{2z-5},\ |z| > \frac{5}{2}$;

e) $f(z) := \frac{1}{z(z-2)},\ 1 < |z-2| < 2$;

f) $f(z) := \frac{z}{(z^2+1)^2},\ 0 < |z-i| < 2$;

g) $f(z) := \frac{z-\sin z}{z^4},\ 0 < |z| < \infty$;

h) $f(z) := \frac{z+2}{z^2-4z+3},\ 2 < |z-1| < \infty$;

i) $f(z) := \frac{1}{z(z-3)^2},\ 1 < |z-1| < 2$.

Solution:

a) For every $z \in \mathbb{C},\ 0 < |z| < 1$ we have that

$$f(z) = \frac{\cos z}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-2}}{(2n)!}.$$

(This holds even for every $z \in \mathbb{C} \setminus \{0\}$.)

b) For every $z \in \mathbb{C},\ |z| > 1$ we have that

$$f(z) = \frac{1}{z^2+1} = \frac{1}{z^2} \frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}}.$$

c) For every $z \in \mathbb{C},\ \frac{1}{2} < |z-i| < 1$ we have that

$$\begin{aligned} f(z) = \frac{z^2+1}{z(z-i)} &= \frac{z+i}{z} = 1 + \frac{i}{z} = \\ &= 1 + \frac{i}{i+z-i} = 1 + \frac{1}{1+\frac{z-i}{i}} = \\ &= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{i^n} (z-i)^n = \\ &= 1 + \sum_{n=0}^{\infty} i^n (z-i)^n. \end{aligned}$$

(This holds even for every $z \in \mathbb{C}$ such that $0 < |z-i| < 1$.)

d) For every $z \in \mathbb{C},\ |z| > \frac{5}{2}$ we have that

$$\underline{f(z) =} \frac{1}{2z-5} = \frac{1}{2z} \cdot \frac{1}{1-\frac{5}{2z}} = \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n \frac{1}{z^n} =$$
$$= \sum_{n=0}^{\infty} \frac{5^n}{2^{n+1}} \frac{1}{z^{n+1}} =$$
$$= \underline{\sum_{n=1}^{\infty} \frac{5^{n-1}}{2^n} \frac{1}{z^n}}.$$

e) For every $z \in \mathbb{C},\ 1 < |z-2| < 2$ we have that

$$\underline{f(z) =} \frac{1}{z(z-2)} = \frac{1}{z-2} \cdot \frac{1}{2+z-2} =$$
$$= \frac{1}{z-2} \cdot \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}} =$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-1} =$$
$$= \underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-1}}.$$

(This holds even for every $z \in \mathbb{C}$ such that $0 < |z-2| < 2$.)

f) Because for every $z \in \mathbb{C},\ 0 < |z-i| < 2$ we have that

$$f(z) = \frac{z}{(z^2+1)^2} = \frac{1}{(z-i)^2} \frac{z+i-i}{(z+i)^2} = \frac{1}{(z-i)^2} \left(\frac{1}{z+i} - \frac{i}{(z+i)^2} \right)$$

and furthermore

$$\frac{1}{z+i} = \frac{1}{2i+z-i} = \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2i)^{n+1}} (z-i)^n,$$

$$-\left(\frac{1}{z+i}\right)^2 = \left(\frac{1}{z+i}\right)' = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2i)^{n+1}}\, n\, (z-i)^{n-1} =$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2i)^{n+2}} (n+1)(z-i)^n.$$

<u>For every $z \in \mathbb{C},\ 0 < |z-i| < 2$ we have</u>

$$\underline{f(z) =} \sum_{n=0}^{\infty} \left((-1)^n \frac{1}{(2i)^{n+1}} + \frac{(-1)^{n+1}}{(2i)^{n+2}}\, i\, (n+1) \right) (z-i)^{n-2} =$$
$$= \underline{\sum_{n=0}^{\infty} \left(\frac{-i^{n+1}}{2^{n+1}} + \frac{i^{n+1}}{2^{n+2}} (n+1) \right) (z-i)^{n-2}}.$$

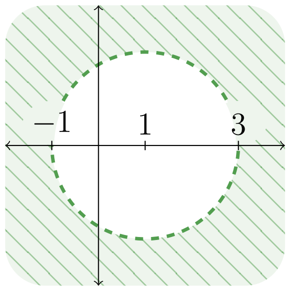
g) For every $z \in \mathbb{C}$ we have that

$$z - \sin z = z - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!},$$

and therefore for every $z \in \mathbb{C},\ z \neq 0$ we have that

$$f(z) = \frac{z - \sin z}{z^4} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-3}}{(2n+1)!}.$$

h) For every $z \in \mathbb{C},\ 2 < |z-1| < \infty$ we have



$$f(z) = \frac{z+2}{z^2 - 4z + 3} = \frac{3}{2} \cdot \frac{1}{1-z} + \frac{5}{2} \cdot \frac{1}{z-3}.$$

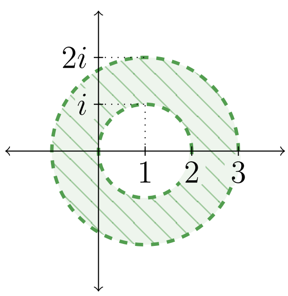
In addition

$$\frac{1}{z-3} = \frac{1}{-2+z-1} = \frac{1}{z-1} \cdot \frac{1}{1 - \frac{2}{z-1}} = \sum_{n=0}^{\infty} \frac{2^n}{(z-1)^{n+1}},$$

and therefore for every $z \in \mathbb{C},\ 2 < |z-1|$, we have

$$f(z) = -\frac{3}{2} \frac{1}{z-1} + \sum_{n=1}^{\infty} \frac{5 \cdot 2^{n-2}}{(z-1)^n} =$$
$$= \frac{1}{z-1} + \sum_{n=2}^{\infty} \frac{5 \cdot 2^{n-2}}{(z-1)^n}.$$

i) Because for every $z \in \mathbb{C},\ 1 < |z-1| < 2$, we have that



$$f(z) = \frac{1}{z(z-3)^2} = \frac{1}{9} \cdot \frac{1}{z} + \frac{1}{9} \cdot \frac{1}{3-z} + \frac{1}{3} \cdot \frac{1}{(z-3)^2},$$

and in addition

$$\frac{1}{z} = \frac{1}{1+z-1} = \frac{1}{z-1}\cdot\frac{1}{1+\frac{1}{z-1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(z-1)^n},$$

$$\frac{1}{z-3} = \frac{1}{-2+z-1} = -\frac{1}{2}\cdot\frac{1}{1-\frac{z-1}{2}} = -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}},$$

$$\left(\frac{1}{z-3}\right)^2 = -\left(\frac{1}{z-3}\right)' = \sum_{n=1}^{\infty}\frac{n(z-1)^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}}(z-1)^n,$$

for every $z \in \mathbb{C}$, $1 < |z-1| < 2$ we have that

$$\underline{f(z)} = \sum_{n=1}^{\infty}\frac{1}{9}\frac{(-1)^{n-1}}{(z-1)^n} + \sum_{n=0}^{\infty}\left(\frac{1}{9}\cdot\frac{1}{2^{n+1}} + \frac{1}{3}\cdot\frac{n+1}{2^{n+2}}\right)(z-1)^n =$$

$$= \underline{\sum_{n=1}^{\infty}\frac{1}{9}\frac{(-1)^{n-1}}{(z-1)^n} + \sum_{n=0}^{\infty}\frac{3n+5}{9\cdot 2^{n+2}}(z-1)^n.}$$

Exercise 54.

Find the Laurent series of the function f on all "maximal anuli" centered at z_0, on which the function f is holomorphic, where

a) $f(z) := \frac{z^2-z+3}{z^3-3z+2}$, $z_0 = 0$;

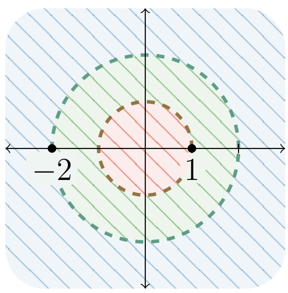
b) $f(z) := \frac{z+1}{z^2}$, $z_0 = 1+i$.

Solution:

a)

$$f(z) = \frac{z^2-z+3}{z^3-3z+2} = \frac{z^2-z+3}{(z-1)^2(z+2)} = \frac{1}{z+2} + \frac{1}{(z-1)^2},$$

and because f is clearly holomorphic on $\mathbb{C}\setminus\{-2,\ 1\}$, we have precisely three "maximal anuli":



α) $P(0,0,1)$,

β) $P(0,1,2)$,

γ) $P(0,2,\infty)$.

α) If $z \in \mathbb{C},\ |z| < 1$, we have

$$
\begin{aligned}
f(z) &= \frac{1}{z+1} + \frac{1}{(z-1)^2} = \frac{1}{2} \cdot \frac{1}{1+\frac{z}{2}} + \left(\frac{-1}{z-1}\right)' = \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n + \left(\frac{1}{1-z}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \left(\sum_{n=0}^{\infty} z^n\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=1}^{\infty} n(z^{n-1}) = \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2^{n+1}} + n + 1\right) z^n.
\end{aligned}
$$

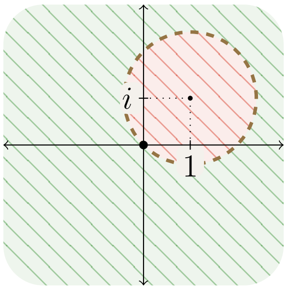
β) For every $z \in \mathbb{C},\ 1 < |z| < 2$, we have that

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \left(\frac{-1}{z} \frac{1}{1-\frac{1}{z}}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n - \left(\sum_{n=0}^{\infty} z^{-n-1}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=0}^{\infty} (n+1) \frac{1}{z^{n+2}} = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=2}^{\infty} \frac{n-1}{z^n}.
\end{aligned}
$$

γ) For every $z \in \mathbb{C}$ such that $|z| > 2$ we have that

$$
\begin{aligned}
f(z) &= \frac{1}{z} \frac{1}{1+\frac{2}{z}} + \sum_{n=2}^{\infty} \frac{n-1}{z^n} = \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{z^{n+1}} + \sum_{n=2}^{\infty} \frac{n-1}{z^n} = \\
&= \frac{1}{z} + \sum_{n=2}^{\infty} \frac{(-2)^{n-1} + n - 1}{z^n}.
\end{aligned}
$$

b) Because f is clearly holomorphic on $\mathbb{C} \setminus \{0\}$ and $|z_0 - 0| = \sqrt{2}$, we have precisely two "maximal anluli":



α) $P(1+i, 0, \sqrt{2})$,

β) $P(1+i, \sqrt{2}, \infty)$.

α) For $z \in \mathbb{C},\ |z-1-i| < \sqrt{2}$, we have that

$$f(z) = \frac{z+1}{z^2} = \frac{1}{z} + \frac{1}{z^2}$$

and furthermore

$$\frac{1}{z} = \frac{1}{1+i+z-1-i} = \frac{1}{1+i} \cdot \frac{1}{1+\frac{z-1-i}{1+i}} = \\ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} (z-1-i)^n,$$

$$\frac{1}{z^2} = -\left(\frac{1}{z}\right)' = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(1+i)^{n+1}}\, n\, (z-1-i)^{n-1},$$

and therefore

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(1+i)^{n+1}} + \frac{(-1)^{n+2}}{(1+i)^{n+2}} (n+1) \right) (z-1-i)^n.$$

β) For every $z \in \mathbb{C},\ |z-1-i| > \sqrt{2}$ we have that

$$\frac{1}{z} = \frac{1}{1+i+z-1-i} = \frac{1}{z-1-i} \cdot \frac{1}{1+\frac{1+i}{z-1-i}} = \sum_{n=0}^{\infty} \frac{(-1)^n (1+i)^n}{(z-1-i)^{n+1}},$$

$$\frac{1}{z^2} = -\left(\frac{1}{z}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^n (1+i)^n (n+1)}{(z-1-i)^{n+2}},$$

and therefore

$$f(z) = \frac{1}{z-1-i} + \sum_{n=1}^{\infty} \frac{(-1)^n (1+i)^n + (-1)^{n-1} (1+i)^{n-1}\, n}{(z-1-i)^{n+1}}.$$

Exercise 55.
Classify each of the isolated singularities of the function f, where

a) $f(z) := z^5 + 4z^3 - 2 + \frac{2}{z} + \frac{3}{z^2}$;

b) $f(z) := \frac{z^2-4}{z-2}$;

c) $f(z) := \frac{1}{z-z^3}$;

d) $f(z) := \frac{z^4}{z^4+1}$;

e) $f(z) := \frac{e^z}{z^2+4}$;

f) $f(z) := \frac{z^2+4}{e^z}$;

g) $f(z) := \frac{1-e^z}{2+e^z}$;

h) $f(z) := e^{\frac{1}{z^2}}$;

i) $f(z) := \frac{1}{(z-3)^2(2-\cos z)}$;

j) $f(z) := \frac{z}{\sin z}$;

k) $f(z) := z^2 \sin \frac{z}{z+1}$;

l) $f(z) := \frac{1-\cos z}{\sin^2 z}$.

Solution:

a) The function $f(z) = z^5 + 4z^3 - 2 + \frac{2}{z} + \frac{3}{z^2}$ has two isolated singularities: 0 and ∞. Clearly we have that

- 0 is a pole of the order 3 of f,
- ∞ is a pole of the order 5 of f.

b) The function $f(z) = \frac{z^2-4}{z-2}$ has two isolated singularities: 2 and ∞.

- Because
$$\lim_{z\to 2} \frac{z^2-4}{z-2} = \lim_{z\to 2} (z+2) = 4,$$
2 is a removable singularity of f.
- $$\lim_{z\to\infty} \frac{f(z)}{z} = 1 \neq 0,$$
and therefore ∞ is a simple pole of f.

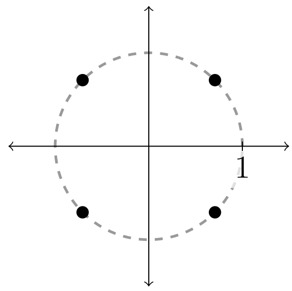
c) The function $f(z) = \dfrac{1}{z-z^3} = \dfrac{1}{z(1-z)(1+z)}$ has four isolated singularities: 0, 1, -1 and ∞.

- $0, 1$ and -1 are simple poles of f.
- Because
$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{1}{z^3\left(\frac{1}{z^2}-1\right)} = \frac{1}{-\infty} = 0,$$
∞ is a removable singularity of f.

d) $f(z) = \frac{z^4}{z^4+1}$ and because

$$z^4 + 1 = 0 \;\Leftrightarrow\; z \in \left\{ \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}} \right\},$$

the function f has five isolated singularities:



- $\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, $\frac{-1-i}{\sqrt{2}}$ and $\frac{-1+i}{\sqrt{2}}$ are simple poles of f
- and, because $\lim_{z\to\infty} f(z) = 1$, ∞ is a removable singularity of f.

e) The function $f(z) = \frac{e^z}{z^2+4}$ has three isolated singularities: $2i$, $-2i$ and ∞.

- $2i$ and $-2i$ are simple poles of f.
- Because

$$\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{x^2+4} \overset{\text{l'H.}}{=} \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{2x} \overset{\text{l'H.}}{=} \lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{e^x}{2} = \infty,$$

$$f(2n\pi i) = \frac{e^{2n\pi i}}{(2n\pi i)^2 + 4} = \frac{1}{-4n^2\pi^2 + 4} \to 0,$$

$\lim_{z\to\infty} f(z)$ does not exist, and therefore ∞ is an essential singularity of f.

f) The function $f(z) = \frac{z^2+4}{e^z}$ has only one isolated singularity, which is ∞.

- Because

$$\lim_{\substack{x\to\infty \\ x\in\mathbb{R}}} f(x) = 0,$$

$$\lim_{\substack{x\to-\infty \\ x\in\mathbb{R}}} f(x) = \infty\cdot\infty = \infty,$$

the limit $\lim_{z\to\infty} f(z)$ does not exist. From this fact it follows that ∞ is an essential singularity of f.

g) Because $f(z) = \frac{1-e^z}{2+e^z}$, and at the same time

$$2 + e^z = 0 \;\Leftrightarrow\; z = \mathrm{Ln}(-2) = \ln 2 + (2k+1)\pi i =: z_k,\; k \in \mathbb{Z},$$

f has isolated singularities precisely in the points z_k.

- Furthermore

$$\begin{aligned} [(2+e^z)']_{z=z_k} = [e^z]_{z=z_k} &= -2 \neq 0, \\ [1-e^z]_{z=z_k} &= \quad 3 \neq 0, \end{aligned}$$

  and therefore $z_k = \ln 2 + (2k+1)\pi i$, $k \in \mathbb{Z}$, are simple poles of f.

Be careful: ∞ is not an isolated singularity of f.

h) $f(z) = e^{\frac{1}{z^2}}$ and for every $z \in \mathbb{C} \setminus \{0\}$ we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{1}{z^{2n}}.$$

From this it follows that

- 0 is an essential singularity of f,
- ∞ is a removable singularity of f.

i) $f(z) = \frac{1}{(z-3)^2(2-\cos z)}$ and because

$$2 = \cos z = \frac{e^{iz} + e^{-iz}}{2} \;\Leftrightarrow\; 4 = e^{iz} + e^{-iz} \;\Leftrightarrow$$

$$\Leftrightarrow\; e^{2iz} - 4e^{iz} + 1 = 0 \;\Leftrightarrow\; e^{iz} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} > 0 \;\Leftrightarrow$$

$$\Leftrightarrow\; iz = \mathrm{Ln}(2 \pm \sqrt{3}) = \ln(2 \pm \sqrt{3}) + 2k\pi i,\; k \in \mathbb{Z} \;\Leftrightarrow$$

$$\Leftrightarrow\; z = z_k := 2k\pi - i\ln(2 \pm \sqrt{3}),\; k \in \mathbb{Z},$$

the function f has isolated singularities in the points 3 and z_k, $k \in \mathbb{Z}$.

- Easily we can compute that

$$[(2-\cos z)']_{z=z_k} = [\sin z]_{z=z_k} \neq 0,$$

  and therefore f has in the points $z_k = 2k\pi - i\ln(2 \pm \sqrt{3})$, where $k \in \mathbb{Z}$, simple poles.
- It is clear that 3 is a pole of the order 2 of the function f.

(∞ is not an isolated singularity of f.)

j) The function $f(z) = \frac{z}{\sin z}$ has clearly isolated singularities in the roots of the function sinus.

Think through the fact that

- 0 is a removable singularity of f,
- $k\pi$, where $k \in \mathbb{Z} \setminus \{0\}$, are simple poles of f.

(∞ is not an isolated singularity of f.)

k) The function $f(z) = z^2 \sin \frac{z}{z+1}$ has precisely two isolated singularities: -1 and ∞.

- -1 is an essential singularity of f (because $\lim\limits_{z\to -1} f(z)$ does not exist),
- ∞ is a pole of the order two of f (because $\lim\limits_{z\to\infty} \frac{f(z)}{z^2} = \lim\limits_{z\to\infty} \sin\left(\frac{z}{z+1}\right) = \sin 1 \neq 0$).

l) The function $f(z) = \frac{1-\cos z}{\sin^2 z}$ clearly has isolated singularities in the roots of the function sinus.

- Because[5]
$$\lim_{z\to 2k\pi} \frac{1-\cos z}{\sin^2 z} \overset{\text{l'H.}}{=} \lim_{z\to 2k\pi} \frac{\sin z}{2\sin z\cos z} = \frac{1}{2},$$
we have that the points $2k\pi$, where $k \in \mathbb{Z}$, are removable singularities of f.
- Because
$$\lim_{z\to(2k+1)\pi} \big(z-(2k+1)\pi\big)^2 \cdot \frac{1-\cos z}{\sin^2 z} \overset{\text{l'H.}}{=} 2 \lim_{z\to(2k+1)\pi} \frac{2\big(z-(2k+1)\pi\big)}{2\sin z\cos z} \overset{\text{l'H.}}{=}$$
$$\overset{\text{l'H.}}{=} -2 \lim_{z\to(2k+1)\pi} \frac{1}{\cos z} = 2 \neq 0,$$
the points $(2k+1)\pi$, where $k \in \mathbb{Z}$, are poles of the order 2 of the function f.

(∞ is not an isolated singularity of f.)

Exercise 56.

Prove the L'Hôpital's rule:

Let f and g be a holomorphic, non-constant functions on some ring neighbourhood of a point $z_0 \in \mathbb{C}$ and let $\lim\limits_{z\to z_0} f(z) = \lim\limits_{z\to z_0} g(z) = 0$.
Then we have that
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

Solution:

From the assumptions it follows that there are $p, q \in \mathbb{N}$, neighbourhood $U(z_0)$ of z_0 and functions f_1 and g_1, which are holomorphic and `non-zero` on $U(z_0)$ such that for every $z \in U(z_0) \setminus \{z_0\}$ we have that
$$f(z) = (z-z_0)^p f_1(z),$$
$$g(z) = (z-z_0)^q g_1(z).$$

[5] We are using L'Hôpital's rule proven in the following exercise.

Therefore

$$\underline{\lim_{z\to z_0}\frac{f(z)}{g(z)}} = \lim_{z\to z_0}(z-z_0)^{p-q}\frac{f_1(z)}{g_1(z)} = \underline{\begin{cases}\infty, & p<q,\\ 0, & p>q,\\ \frac{f_1(z_0)}{g_1(z_0)}, & p=q,\end{cases}}$$

and

$$\underline{\lim_{z\to z_0}\frac{f'(z)}{g'(z)}} = \lim_{z\to z_0}\frac{p(z-z_0)^{p-1}f_1(z)+(z-z_0)^p f_1'(z)}{q(z-z_0)^{q-1}g_1(z)+(z-z_0)^q g_1'(z)} =$$

$$= \lim_{z\to z_0}(z-z_0)^{p-q}\frac{pf_1(z)+(z-z_0)f_1'(z)}{qg_1(z)+(z-z_0)g_1'(z)} =$$

$$= \underline{\begin{cases}\infty, & p<q,\\ 0, & p>q,\\ \frac{f_1(z_0)}{g_1(z_0)}, & p=q.\end{cases}}$$

The theorem is proven.

<u>Exercise 57</u>.
Compute the residue of the function f in all of its isolated singularities, where

a) $f(z) := \frac{1}{z+z^3}$;

b) $f(z) := \frac{z^2}{(1+z)^3}$;

c) $f(z) := \frac{1}{(z^2+1)^3}$;

d) $f(z) := \frac{z^3+1}{z-2}$;

e) $f(z) := \frac{1}{z^6(z^2+1)^2}$;

f) $f(z) := \tan z$;

g) $f(z) := \frac{1}{\sin z}$;

h) $f(z) := \operatorname{cotg}^3 z$;

i) $f(z) := \sin z \cdot \sin\frac{1}{z}$;

j) $f(z) := \frac{\sin(\pi z)}{(z-1)^3}$.

<u>Solution:</u>

a) The function $f(z) = \dfrac{1}{z+z^3} = \dfrac{1}{z(z-i)(z+i)}$ has clearly four isolated singularities: 0, i, $-i$ a ∞. Now we will use (as in several following exercises) the [1, Theorem 44, part (iii)]:

- $\underline{\operatorname{res} f(0)} = \left[\frac{1}{1+3z^2}\right]_{z=0} = \underline{1}$,
- $\underline{\operatorname{res} f(i)} = \left[\frac{1}{1+3z^2}\right]_{z=i} = \frac{1}{1-3} = \underline{-\frac{1}{2}}$,
- $\underline{\operatorname{res} f(-i)} = \left[\frac{1}{1+3z^2}\right]_{z=-i} = \frac{1}{1-3} = \underline{-\frac{1}{2}}$

and [1, Theorem 44, part (v)]:

- $\underline{\operatorname{res} f(\infty)} = -\left(1-\frac{1}{2}-\frac{1}{2}\right) = \underline{0}$.

b) The function $f(z) = \frac{z^2}{(1+z)^3}$ clearly has two isolated singularities.

- -1 is a pole of the order three of f, and therefore

$$\underline{\operatorname{res} f(-1)} = \frac{1}{2}\left[(z^2)''\right]_{z=-1} = \underline{1}.$$

- $\underline{\operatorname{res} f(\infty) = -1}.$

c) The function $f(z) := \frac{1}{(z^2+1)^3}$ has three isolated singularities: poles of the order three at the points i and $-i$ and a removable singularity at ∞.

-

$$\underline{\operatorname{res} f(\pm i)} = \frac{1}{2}\left[\left(\frac{1}{(z\pm i)^3}\right)''\right]_{z=\pm i} = \frac{1}{2}\left[3\cdot 4\,\frac{1}{(z\pm i)^5}\right]_{z=\pm i} =$$

$$= 6\,\frac{1}{(\pm 2i)^5} = \underline{\mp\frac{3}{16}i},$$

- $\underline{\operatorname{res} f(\infty) = 0}.$

d) The function $f(z) = \frac{z^3+1}{z-2}$ has two isolated singularities: 2 (simple pole) and ∞.

-

$$\underline{\operatorname{res} f(2)} = \left[\frac{z^3+1}{1}\right]_{z=2} = \underline{9},$$

- $\underline{\operatorname{res} f(\infty) = -9}.$

e) Because $f(z) = \dfrac{1}{z^6(z^2+1)^2} = \dfrac{1}{z^6(z+i)^2(z-i)^2}$, the numbers $\pm i$ are poles of the order two of f, 0 is a pole of the order 6 of f and ∞ is a removable singularity of f.

-

$$\underline{\operatorname{res} f(\pm i)} = \left[\left(\frac{1}{z^6(z\pm i)^2}\right)'\right]_{z=\pm i} =$$

$$= -\left[\frac{6z^5(z\pm i)^2 + z^6\,2(z\pm i)}{z^{12}(z\pm i)^4}\right]_{z=\pm i} =$$

$$= \underline{\pm\frac{7}{4}i}.$$

- Because $f(z) = 1 : (z^6 + 2z^8 + z^{10}) = \frac{1}{z^6} + \cdots,$[6] we have that $\underline{\operatorname{res} f(\infty) = 0}$,
- $\underline{\operatorname{res} f(0)} = -\frac{7}{4}i + \frac{7}{4}i - 0 = \underline{0}.$

f) The function $f(z) = \tan z = \frac{\sin z}{\cos z}$ has simple poles in the points $\frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$, and

$$\underline{\operatorname{res} f\left(\frac{\pi}{2} + k\pi\right)} = \left[\frac{\sin z}{-\sin z}\right]_{z=\frac{\pi}{2}+k\pi} = \underline{-1}.$$

(∞ is not an isolated singularity of f.)

[6]In the Laurent series of f the coefficient of $\frac{1}{z}$ is equal to 0.

g) $f(z) = \frac{1}{\sin z}$ has simple poles in the points $k\pi$, where $k \in \mathbb{Z}$, and

$$\underline{\operatorname{res} f(k\pi) =} \left[\frac{1}{\cos z}\right]_{z=k\pi} = \underline{(-1)^k}.$$

h) $f(z) = \operatorname{cotg}^3 z = \frac{\cos^3 z}{\sin^3 z}$ has poles of the order three in the points $k\pi$, where $k \in \mathbb{Z}$, and

$$\frac{\cos z}{\sin z} = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right) : \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \cdots,$$

and therefore

$$\left(\frac{\cos z}{\sin z}\right)^3 = \left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \cdots\right)\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \cdots\right)\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \cdots\right).$$

$\operatorname{res} f(0)$ is a "coefficient of $\frac{1}{z}$", therefore

$$\operatorname{res} f(0) = 3\left(-\frac{1}{3}\right) = -1.$$

Because the function f has a period π, that is $f(z) = f(z - k\pi)$, we have that

$$\underline{\operatorname{res} f(k\pi) =} \operatorname{res} f(0) = \underline{-1 \text{ for every } k \in \mathbb{Z}}.$$

i) $f(z) = \sin z \cdot \sin \frac{1}{z}$ has an isolated singularity at 0 and at ∞.
Because for every $z \in \mathbb{C} \setminus \{0\}$ we have that

$$f(z) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} \cdot \frac{1}{z^{2k+1}}\right) = \cdots,$$

we have that[7]

$$\underline{\operatorname{res} f(0) = \operatorname{res} f(\infty) = 0}.$$

j) The function $f(z) = \frac{\sin(\pi z)}{(z-1)^3}$ has two isolated singularities: 1 and ∞.

- Because 1 is a pole of the order two of f,

$$\underline{\operatorname{res} f(1)} = \lim_{z\to 1}\left(\frac{\sin(\pi z)}{(z-1)^3}(z-1)^2\right)' = \lim_{z\to 1}\left(\frac{\sin(\pi z)}{z-1}\right)' =$$

$$= \lim_{z\to 1} \frac{\pi\cos(\pi z)(z-1) - \sin(\pi z)}{(z-1)^2} \overset{\text{l'H}}{=}$$

$$\overset{\text{l'H}}{=} \lim_{z\to 1} \frac{-\pi^2 \sin(\pi z)(z-1) + \pi\cos(\pi z) - \pi\cos(\pi z)}{2(z-1)} =$$

$$= \lim_{z\to 1}\left(-\frac{\pi^2}{2}\sin(\pi z)\right) = \underline{0}.$$

[7] The Laurent series of f has non-zero coefficients only for the "even powers" of z.

Other possible solution:

$$f(z) = \frac{\sin(\pi z)}{(z-1)^3} = -\frac{\sin(\pi(z-1))}{(z-1)^3} =$$
$$= -\frac{1}{(z-1)^3}\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{(2n+1)!}(z-1)^{2n+1} =$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\pi^{2n+1}}{(2n+1)!}(z-1)^{2n-2}.$$

The just computed Laurent series of the function f has non-zero coefficients only for the "even powers" of $(z-1)$, and therefore $\underline{\operatorname{res} f(1) = 0}$.

- $\underline{\operatorname{res} f(\infty) = 0}$.

EXERCISE 58.
Using the residue theorem compute the integrals

a)
$$\int_\gamma \frac{\cos z}{z^3}\,\mathrm{d}z, \quad \text{where} \quad \gamma(t) := 3e^{it}, \ t \in \langle 0, 2\pi\rangle;$$

b)
$$\int_\gamma \frac{1}{z+2}\cos\frac{1}{z}\,\mathrm{d}z, \quad \text{where} \quad \gamma(t) := 18e^{it}, \ t \in \langle 0, 2\pi\rangle;$$

c)
$$\int_k \frac{z^3}{z^4-1}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon \ |z| = 2\};$$

d)
$$\int_k \frac{z^3}{z+1}e^{\frac{1}{z}}\,\mathrm{d}z, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon \ |z| = 2\};$$

e)
$$\int_\gamma z\sin\frac{z+1}{z-1}\,\mathrm{d}z, \quad \text{where} \quad \gamma(t) := 2e^{-it}, \ t \in \langle 0, 6\pi\rangle;$$

f)
$$\int_\gamma \frac{e^{\pi z}}{2z^2 - i}\,\mathrm{d}z,$$

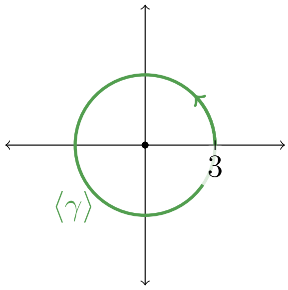
where γ is simple, closed, piecewise smooth positively oriented curve such that

$$\operatorname{int}\gamma = \{z \in \mathbb{C}\colon \ |z| < 1 \ \wedge \ 0 < \arg z < \frac{\pi}{2}\};$$

g)
$$\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)}, \quad \text{where} \quad k = \{z \in \mathbb{C}\colon \ |z| = 2\}.$$

Solution:

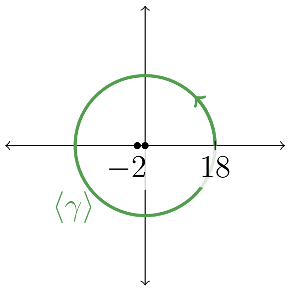
a)



Clearly $z = 0$ is a pole of the order three of the function $\dfrac{\cos z}{z^3}$, and therefore

$$\underline{\int_\gamma \frac{\cos z}{z^3}\,\mathrm{d}z} = 2\pi i \operatorname*{res}_{z=0} \frac{\cos z}{z^3} = $$
$$= 2\pi i \, \frac{1}{2}\left[(\cos z)''\right]_{z=0} =$$
$$= \pi i \left[(-\sin z)'\right]_{z=0} =$$
$$= \pi i \left[-\cos z\right]_{z=0} = \underline{-\pi i}.$$

b)



Clearly

$$\int_\gamma \underbrace{\frac{1}{z+2}\cos\frac{1}{z}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(-2) + \operatorname{res} f(0)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

For every $z \in \mathbb{C},\ |z| > 2$ we have that

$$\frac{1}{z+2} = \frac{1}{z}\,\frac{1}{1+\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}},$$

and therefore

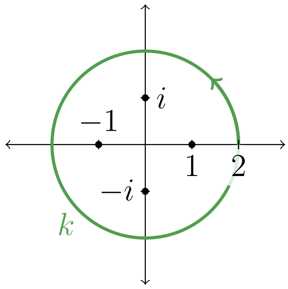
$$f(z) = \left(\sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\,\frac{1}{z^{2n}}\right).$$

From this it follows that

$$\operatorname{res} f(\infty) = -(-2)^0\cdot(-1)^0\cdot\frac{1}{0!} = -1,$$

$$\underline{\int_\gamma \frac{1}{z+2}\cos\frac{1}{z}\,\mathrm{d}z} = -2\pi i \operatorname{res} f(\infty) = \underline{2\pi i}.$$

c)



$$\int_k \underbrace{\frac{z^3}{z^4-1}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(1) + \operatorname{res} f(-1) + \operatorname{res} f(i) + \operatorname{res} f(-i)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

For every $z \in \mathbb{C},\ |z| > 1$ we have that

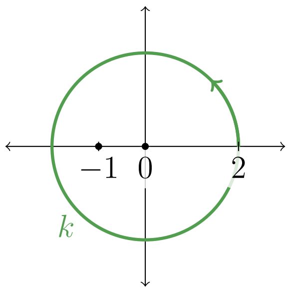
$$f(z) = \frac{z^3}{z^4-1} = z^3\frac{1}{z^4}\frac{1}{1-\frac{1}{z^4}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^4}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{4n+1}},$$

and therefore

$$\operatorname{res} f(\infty) = -1,$$

$$\underline{\int_k \frac{z^5}{z^4-1}\,\mathrm{d}z} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{2\pi i}.$$

d)



Clearly

$$\int_k \underbrace{\frac{z^3}{z+1}e^{\frac{1}{z}}}_{=:f(z)}\,\mathrm{d}z = 2\pi i\Big(\operatorname{res} f(-1) + \operatorname{res} f(0)\Big) = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

Because for every $z \in \mathbb{C},\ |z| > 1$ we have that

$$\frac{z^3}{z+1} = z^2\frac{1}{1+\frac{1}{z}} = z^2\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n-2}},$$

$$f(z) = \left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n-2}}\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}\right),$$

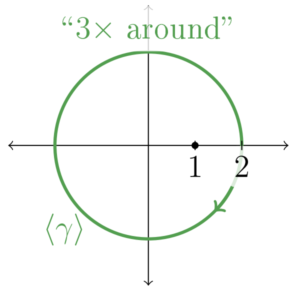
we have

$$\operatorname{res} f(\infty) = -\left((-1)^0\cdot\frac{1}{3!} + (-1)^1\cdot\frac{1}{2!} + (-1)^2\cdot\frac{1}{1!} + (-1)^3\cdot\frac{1}{0!}\right) = \frac{1}{3}.$$

Therefore

$$\underline{\int_k \frac{z^3}{z+1} e^{\frac{1}{z}}\,\mathrm{d}z} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{-\frac{2\pi i}{3}}.$$

e)



Clearly

$$\int_\gamma \underbrace{z \sin\frac{z+1}{z-1}}_{=:f(z)}\,\mathrm{d}z = -3\cdot 2\pi i \operatorname{res} f(1).$$

Because

$$\sin\frac{z+1}{z-1} = \sin\left(\frac{z-1}{z-1} + \frac{2}{z-1}\right) = \sin\left(1+\frac{2}{z-1}\right) =$$
$$= \sin 1\ \cos\frac{2}{z-1} + \cos 1\ \sin\frac{2}{z-1},$$

we have for every $z\in\mathbb{C},\ z\neq 1$,

$$f(z) = (z-1+1)\left(\sin 1 \cos\frac{2}{z-1} + \cos 1 \sin\frac{2}{z-1}\right) =$$
$$= \sin 1\left[\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\frac{1}{(z-1)^{2n-1}} + \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\frac{1}{(z-1)^{2n}}\right] +$$
$$+ \cos 1\left[\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}}{(2n+1)!}\frac{1}{(z-1)^{2n}} + \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}}{(2n+1)!}\frac{1}{(z-1)^{2n+1}}\right].$$

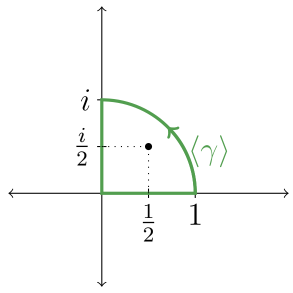
Therefore

$$\operatorname{res} f(1) = \left((-1)\frac{2^2}{2!}\right)\sin 1 + \cos 1\left((-1)^0\,\frac{2^1}{1!}\right) = -2\sin 1 + 2\cos 1,$$

$$\underline{\int_\gamma z\sin\frac{z+1}{z-1}\,\mathrm{d}z} = -3\cdot 2\pi i \operatorname{res} f(1) = \underline{12\pi i\big(\sin 1 - \cos 1\big)}.$$

f) Because

$$2z^2 - i = 0 \;\Leftrightarrow\; z^2 = \frac{i}{2} \;\Leftrightarrow\; z = \pm\frac{1+i}{2},$$

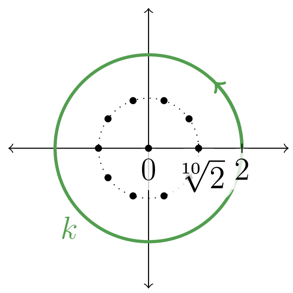


we have that

$$\underline{\int_\gamma \frac{e^{\pi z}}{2z^2 - i}\,\mathrm{d}z} = 2\pi i \operatorname*{res}_{z=\frac{1+i}{2}} \left(\frac{e^{\pi z}}{2z^2 - i}\right) =$$

$$= 2\pi i \frac{e^{\pi\left(\frac{1+i}{2}\right)}}{4\left(\frac{1+i}{2}\right)} = \underline{\frac{\pi}{2}(i-1)e^{\frac{\pi}{2}}}.$$

g) Because the function $f(z) := \frac{1}{z^5(z^{10}-2)}$ clearly has 12 isolated singularities, 11 of which (0 and the roots of $z^{10} = 2$) lie "inside" k and the twelfth is ∞,



we have

$$\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)} = 2\pi i\big(-\operatorname{res} f(\infty)\big).$$

For every $z \in \mathbb{C}$, $\sqrt[10]{2} < |z|$, we have that

$$f(z) = \frac{1}{z^{15}}\frac{1}{1 - \frac{2}{z^{10}}} = \frac{1}{z^{15}}\sum_{n=0}^{\infty}\frac{2^n}{z^{10n}} = \sum_{n=0}^{\infty}\frac{2^n}{z^{10n+15}},$$

and therefore

$$\underline{\int_k \frac{\mathrm{d}z}{z^5(z^{10}-2)}} = 2\pi i\big(-\operatorname{res} f(\infty)\big) = \underline{0}.$$

Exercise 59.
Using the residue theorem compute the integrals[8].

a)
$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x};$$

b)
$$\int_{-\infty}^{\infty} \frac{x^2\ \mathrm{d}x}{x^4+6x^2+25};$$

c)
$$\int_{0}^{\infty} \frac{x^4+1}{x^6+1}\ \mathrm{d}x;$$

d)
$$\int_{0}^{\infty} \frac{x^2}{(x^2+1)^3}\ \mathrm{d}x;$$

e)
$$\int_{-\pi}^{\pi} \frac{\cos x}{3+2\sin x}\ \mathrm{d}x;$$

f)
$$\int_{0}^{2\pi} \frac{\cos^2(2x)}{5-4\cos x}\ \mathrm{d}x;$$

g)
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^6};$$

h)
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+x+1}.$$

Solution:

a) Let us define the curve $\gamma(t) := e^{it}$, $t \in \langle 0, 2\pi\rangle$. Then, using the change of variables[9] $e^{ix} = z$,

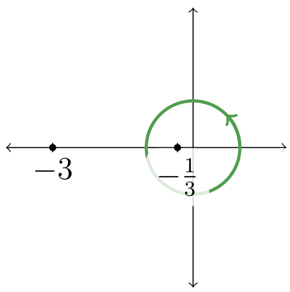
$$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \frac{z+\frac{1}{z}}{2},$$
$$ie^{ix}\,\mathrm{d}x = \mathrm{d}z, \text{ tj. } \mathrm{d}x = \frac{1}{iz}\,\mathrm{d}z,$$

we get

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x} = \int_\gamma \frac{1}{\left(5+3\frac{z+\frac{1}{z}}{2}\right)}\frac{1}{iz}\,\mathrm{d}z = \int_\gamma \frac{2\,\mathrm{d}z}{i(10z+3z^2+3)}.$$

Because

$$3z^2+10z+3 = 0 \iff z = \frac{-10 \pm \sqrt{100-36}}{6} \iff z \in \left\{-\frac{1}{3}, -3\right\},$$



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[8] The integrals should be understood as "*real*" integrals of functions of *real variable*
[9] See [1, Chapter 9.3, part a)].

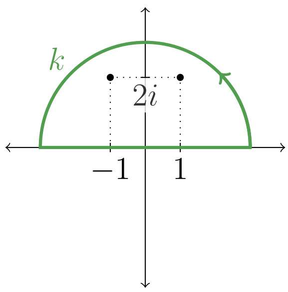
we have that

$$\underline{\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5+3\cos x}} = \int_\gamma \frac{2\,\mathrm{d}z}{i(10z+3z^2+3)} =$$
$$= 2\pi i \operatorname*{res}_{z=-\frac{1}{3}} \left(\frac{2}{i(3z^2+10z+3)} \right) =$$
$$= 4\pi \left[\frac{1}{6z+10} \right]_{z=-\frac{1}{3}} = \frac{4\pi}{8} = \underline{\frac{\pi}{2}}.$$

b) We start by observing that

$$z^4+6z^2+25=0 \;\Leftrightarrow\; z^2=-3\pm 4i \;\Leftrightarrow\; z\in\{1+2i,\ -1-2i,\ -1+2i,\ 1-2i\}\,.$$

Let $k \subset \mathbb{C}$ be the boundary of the set $\{z\in\mathbb{C}\colon\ |z|<3 \ \wedge\ \operatorname{Im} z>0\}$.



Then it holds[10]

$$\underline{\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{x^4+6x^2+25}} = \int_k \frac{z^2\,\mathrm{d}z}{z^4+6z^2+25} =$$
$$= 2\pi i \left(\operatorname*{res}_{z=1+2i} \frac{z^2}{z^4+6z^2+25} + \operatorname*{res}_{z=-1+2i} \frac{z^2}{z^4+6z^2+25} \right) =$$
$$= 2\pi i \left(\left[\frac{z^2}{4z^3+12z} \right]_{z=1+2i} + \left[\frac{z^2}{4z^3+12z} \right]_{z=-1+2i} \right) =$$
$$= 2\pi i \left(\left[\frac{z}{4z^2+12} \right]_{z=1+2i} + \left[\frac{z}{4z^2+12} \right]_{z=-1+2i} \right) =$$
$$= 2\pi i \left(\frac{1+2i}{4(-3+4i)+12} + \frac{-1+2i}{4(-3-4i)+12} \right) =$$
$$= 2\pi i \left(\frac{1+2i}{16i} - \frac{-1+2i}{16i} \right) = \frac{\pi}{8}\Big(1+2i+1-2i\Big) = \underline{\frac{\pi}{4}}.$$

[10] See [1, Chapter 9.3, part b)].

c) Because the problem

$$z^6 + 1 = 0 \;\wedge\; \operatorname{Im} z \geq 0$$

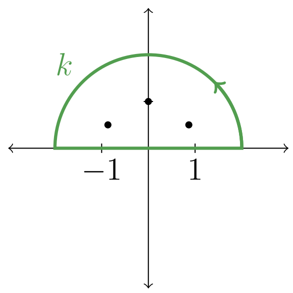
has exactly three solutions:

$$\begin{aligned} z_1 &:= e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \\ z_2 &:= e^{i\frac{\pi}{2}} = i, \\ z_3 &:= e^{\frac{5}{6}\pi i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \end{aligned}$$

and the function $\frac{x^4+1}{x^6+1}$ is even, we have that

$$\begin{aligned} \int_0^{\infty} \frac{x^4+1}{x^6+1}\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^4+1}{x^6+1}\,\mathrm{d}x = \\ &= \frac{1}{2}\int_k \frac{z^4+1}{z^6+1}\,\mathrm{d}z = \\ &= \frac{1}{2}\,2\pi i \sum_{j=1}^{3} \operatorname*{res}_{z=z_j} \frac{z^4+1}{z^6+1}, \end{aligned}$$

where $k \subset \mathbb{C}$ is the boundary of the set $\{z \in \mathbb{C}\colon\; |z| < 2 \;\wedge\; \operatorname{Im} z > 0\}$.



Therefore, because

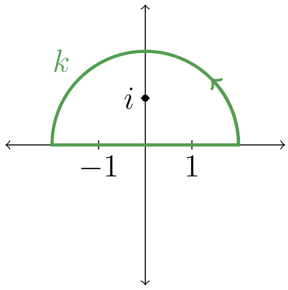
$$\begin{aligned} \operatorname*{res}_{z=z_1} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=\frac{\sqrt{3}}{2}+\frac{1}{2}i} &&= \frac{1}{6}(-i), \\ \operatorname*{res}_{z=z_2} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=i} &&= \frac{2}{6}(-i), \\ \operatorname*{res}_{z=z_3} \frac{z^4+1}{z^6+1} &= \left[\frac{z^4+1}{6z^5}\right]_{z=\frac{-\sqrt{3}}{2}+\frac{1}{2}i} &&= \frac{1}{6}(-i), \end{aligned}$$

we have that

$$\underline{\int_0^{\infty} \frac{x^4+1}{x^6+1}\,\mathrm{d}x = \frac{2}{3}\pi.}$$

d) The function $\frac{x^2}{(x^2+1)^3}$ is even, and therefore for $k \subset \mathbb{C}$, which is the boundary of the set

$$\{z \in \mathbb{C}\colon\ |z| < 2 \ \wedge\ \operatorname{Im} z > 0\}\,,$$



we have that

$$\underline{\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+1)^3}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(x^2+1)^3} = \frac{1}{2}\int_k \frac{z^2\,\mathrm{d}z}{(z^2+1)^3} =$$

$$= \frac{1}{2} 2\pi i \operatorname*{res}_{z=i}\left(\frac{z^2}{(z^2+1)^3}\right) = \pi i \frac{1}{2}\left[\left(\frac{z^2}{(z+i)^3}\right)''\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\left(\frac{2z(z+i)^3 - z^2\,3(z+i)^2}{(z+i)^6}\right)'\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\left(\frac{-z^2+2zi}{(z+i)^4}\right)'\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left[\frac{(-2z+2i)(z+i)^4 - (-z^2+2zi)4(z+i)^3}{(z+i)^8}\right]_{z=i} =$$

$$= \frac{\pi i}{2}\left(\frac{4(2i)^3}{(2i)^8}\right) = \frac{2\pi i}{2^5 i^5} = \underline{\frac{\pi}{16}}.$$

e) Let $\gamma(t) := e^{it}$, where $t \in \langle 0, 2\pi \rangle$. Then

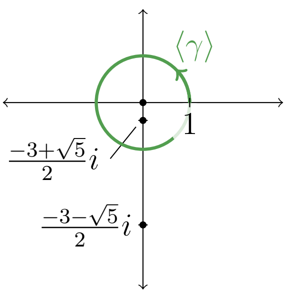
$$\int_{-\pi}^{\pi} \frac{\cos x}{3 + 2\sin x}\,\mathrm{d}x = \int_\gamma \frac{z + \frac{1}{z}}{2}\,\frac{1}{3 + 2\,\frac{z-\frac{1}{z}}{2i}}\,\frac{1}{iz}\,\mathrm{d}z =$$

$$= \frac{1}{2}\int_\gamma \frac{z^2+1}{z}\,\frac{1}{z^2+3iz-1}\,\mathrm{d}z.$$

(We've used the change of variables $e^{ix} = z$, see [1, Chapter 9.3, part a)]).

Because

$$z^2 + 3iz - 1 = 0 \ \Leftrightarrow\ z = \frac{-3 \pm \sqrt{5}}{2} i,$$



we have that

$$\underline{\int_{-\pi}^{\pi} \frac{\cos x}{3+2\sin x}\,\mathrm{d}x} = \frac{1}{2}\int_{\gamma} \underbrace{\frac{z^2+1}{z}\,\frac{1}{z^2+3iz-1}}_{=:f(z)}\,\mathrm{d}z =$$

$$= \frac{1}{2}2\pi i\left(\operatorname{res} f(0) + \operatorname{res} f\left(\frac{-3+\sqrt{5}}{2}i\right)\right) =$$

$$= \pi i\left(\left[\frac{z^2+1}{z^2+3iz-1}\right]_{z=0} + \left[\frac{z^2+1}{z(2z+3i)}\right]_{z=\frac{-3+\sqrt{5}}{2}i}\right) =$$

$$= \pi i\left(-1 + \left[\frac{2-3iz}{2-6iz+3iz}\right]_{z=\frac{-3+\sqrt{5}}{2}i}\right) = \pi i(-1+1) = \underline{0}.$$

f) We will use the change of variables $e^{ix} = z$,

$$\cos x = \frac{z+\frac{1}{z}}{2},$$
$$\cos 2x = \frac{z^2+\frac{1}{z^2}}{2},$$
$$\mathrm{d}x = \frac{1}{iz}\,\mathrm{d}z.$$

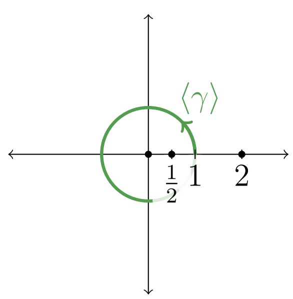
For $\gamma(t) := e^{it}$, where $t \in \langle 0, 2\pi\rangle$, we have that

$$\int_0^{2\pi} \frac{\cos^2 2x}{5-4\cos x}\,\mathrm{d}x = \int_{\gamma} \frac{1}{4}\left(\frac{z^4+1}{z^2}\right)^2 \frac{1}{5-2\frac{z^2+1}{z}}\,\frac{1}{iz}\,\mathrm{d}z =$$

$$= \int_{\gamma} \frac{1}{4i}\,\frac{(z^4+1)^2}{z^4}\,\frac{1}{5z-2z^2-2}\,\mathrm{d}z =$$

$$= \int_{\gamma} \frac{1}{4i}\underbrace{\frac{(z^4+1)^2}{z^4}\,\frac{1}{-2(z-2)\left(z-\frac{1}{2}\right)}}_{=:f(z)}\,\mathrm{d}z =$$

$$= \frac{2\pi i}{4i}\left(\operatorname{res} f(0) + \operatorname{res} f\left(\frac{1}{2}\right)\right).$$



and because

$$\operatorname{res} f(0) = \frac{1}{3!}\left[\left(\frac{(z^4+1)^2}{5z-2z^2-2}\right)'''\right]_{z=0} = \frac{1}{6}\left(-\frac{255}{8}\right) = -\frac{255}{48},$$

$$\operatorname{res} f\left(\frac{1}{2}\right) = \left[\frac{(z^4+1)^2}{z^4}\,\frac{1}{5-4z}\right]_{z=\frac{1}{2}} = \frac{289}{48},$$

we have that

$$\underline{\int_0^{2\pi} \frac{\cos^2 2x}{5-4\cos x}\,\mathrm{d}x = \frac{17}{48}\pi.}$$

g) Because the equation $z^6+1=0$ has, assuming $\operatorname{Im} z \geq 0$, exactly three solutions:

$$z_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i,$$
$$z_2 = e^{i\frac{\pi}{2}} = i,$$
$$z_3 = e^{\frac{5}{6}\pi i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i,$$

for the function $f(z) := \frac{1}{1+z^6}$ we have that

$$\underline{\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^6}} = 2\pi i\Big(\operatorname{res} f(z_1) + \operatorname{res} f(z_2) + \operatorname{res} f(z_3)\Big) =$$

$$= 2\pi i \sum_{k=1}^{3} \frac{1}{6z_k^5} = 2\pi i \sum_{k=1}^{3} \frac{z_k}{6z_k^6} =$$

$$= -\frac{2\pi i}{6}\big(z_1+z_2+z_3\big) =$$

$$= -\frac{\pi}{3}\,i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i + i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$$

$$= -\frac{\pi}{3}\,i\,2i = \underline{\frac{2\pi}{3}}.$$

h) Because

$$z^2 + z + 1 = 0 \iff z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i,$$

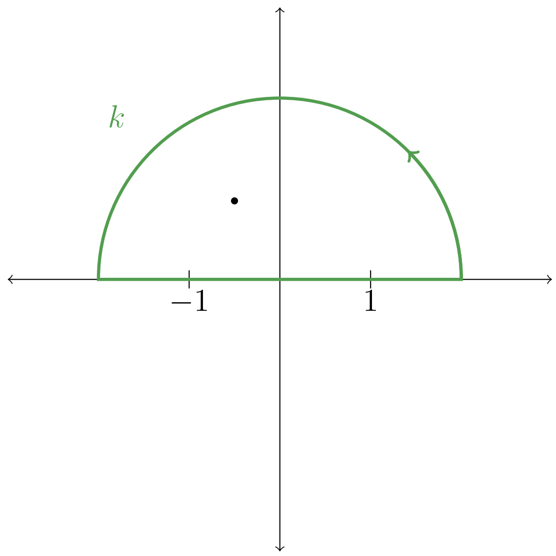
for $k \subset \mathbb{C}$ defined as the boundary of the set $\{z \in \mathbb{C}\colon\ |z| < 2 \ \wedge \operatorname{Im} z > 0\}$ we have that

$$\underline{\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + x + 1}} = \int_k \frac{\mathrm{d}z}{z^2 + z + 1} =$$

$$= 2\pi i \operatorname*{res}_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i} \left(\frac{1}{z^2 + z + 1} \right) =$$

$$= 2\pi i \left[\frac{1}{2z + 1} \right]_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i} =$$

$$= 2\pi i \, \frac{1}{-1 + \sqrt{3}i + 1} = \underline{\frac{2\pi}{\sqrt{3}}}.$$



Bibliography

[1] J. Bouchala, M. Lampart: *An Introduction to Complex Analysis*, 2021, am.vsb.cz/bouchala.

[2] J. Bouchala: *Funkce komplexní proměnné*, 2012, am.vsb.cz/bouchala.

[3] J. Bouchala (and O. Bouchala): *Řešené příklady z komplexní analýzy*, 2020, am.vsb.cz/bouchala.

[4] J. Bouchala, O. Vlach, J. Zapletal: *Line Integrals and Surface Integrals*, 2022, am.vsb.cz/bouchala.

[5] J. Bouchala, P. Vodstrčil, J. Zapletal: *Series*, 2022, am.vsb.cz/bouchala.

[6] I. Černý: *Foundations of Analysis in the Complex Domain*, Academia, Praha, 1992.

[7] Z. Eberhard, E. Zeidler, W. Hackbusch, H. Schwarz, B. Hunt: *Oxford Users' Guide to Mathematics*, OUP, Oxford, 2004.