An Introduction to Complex Analysis

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Preface

Complex analysis is one of the most interesting of the fundamental topics in the undergraduate mathematics course. Its importance to applications means that it can be studied both from a very pure perspective and a very applied perspective. This text book for students takes into account the varying needs and backgrounds for students in mathematics, science, and engineering. It covers all topics likely to feature in this course, including the subjects:

- complex numbers,
- differentiation,
- integration,
- Cauchy's theorem and its consequences,
- Laurent and Taylor series,
- conformal maps and harmonic functions,
- the residue theorem.

Since the topics of complex analysis are not elementary subjects, there are some reasonable assumptions about what readers should know. The reader should be familiar with relevant standard topics taught in the area of real analysis of real functions of one and multiple variables, sequences, and series.

This text is mostly a translation from the Czech original [1].

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Chapter 1

Complex numbers and Gauss plane

1.1 Complex numbers

This chapter is devoted to the fundamental structure of complex numbers and their basic properties.

(i) A **complex number** z is a number

$$z = x + iy \text{ where } x,\ y \in \mathbb{R} \text{ and } i^2 = -1,$$

the numbers x and y are called the **real** and **imaginary** parts of the complex number z respectively, and are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively.[1]

(ii) It is easy to see that **real** and **imaginary** numbers are the special case of complex numbers. Real numbers are those z with $\operatorname{Im} z = 0$, and the imaginary ones are characterized by the condition $\operatorname{Re} z = 0$.

(iii) The two complex numbers z_1 and z_2 are equal if and only if their real and imaginary parts are equal, that is

$$z_1 = z_2 \;\Leftrightarrow\; \left[\operatorname{Re} z_1 = \operatorname{Re} z_2 \;\wedge\; \operatorname{Im} z_1 = \operatorname{Im} z_2\right].$$

(iv) Let $z = x+iy$ be a complex number, then its **absolute value** is a non-negative (real) number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

and the number **complex adjoint** is given by

$$\overline{z} = x - iy = \operatorname{Re} z - i\ \operatorname{Im} z.$$

[1]**Notation:** by $z = x + iy$ it is meant (unless otherwise stated) that $x = \operatorname{Re} z \in \mathbb{R}$ and $y = \operatorname{Im} z \in \mathbb{R}$.

(v) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. Then it is defined

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2), \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned}$$

Moreover, if $z_2 \neq 0 = 0 + 0i$, it is also defined

$$\frac{z_1}{z_2} = \frac{1}{|z_2|^2}\,(z_1 \overline{z_2}).$$

(vi) For every complex number $z = x + iy$ it holds that

$$z\overline{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

Remark 1.1 *The most remarkable difference between real and complex numbers is the fact that* complex numbers are not ordered. *The relation $z_1 < z_2$ between complex numbers z_1 and z_2 is not defined unless they are both real.*

Example 1.1 *Find* Re z_0 *and* Im z_0 *for*

$$z_0 = \frac{2 + 3i}{1 - 2i}.$$

<u>*Solution:*</u>

$$z_0 = \frac{2 + 3i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} = \frac{-4 + 7i}{5} = -\frac{4}{5} + \frac{7}{5}i,$$

hence

$$\operatorname{Re} z_0 = -\frac{4}{5} \quad \textit{and} \quad \operatorname{Im} z_0 = \frac{7}{5}.$$

1.2 Geometric interpretation and argument of complex numbers

Given a complex number $z = x + iy$, its real and imaginary parts uniquely define an element $(x, y) \in \mathbb{R}^2$. The set of all complex numbers is therefore naturally identified with the plane $\mathbb{R}^2$; it is called a **Gauss plane** and is denoted by $\mathbb{C}$.

Points in the plane $\mathbb{R}^2$ can also be represented using polar coordinates, which is a representation of the complex numbers. Let $z \in \mathbb{C},\ z \neq 0$, then there exists $\varphi \in \mathbb{R}$ such that

$$z = |z|(\cos\varphi + i\sin\varphi). \tag{1.1}$$

We call (1.1) a **polar form** of a complex number z.

Since the functions sine and cosine are periodic, the number (angle) φ is not uniquely determined by (1.1).

Definition 1.1 *The set of all real numbers* φ *for which* (1.1) *holds is called an* **argument** *of a complex number* $z \in \mathbb{C} \setminus \{0\}$ *and is denoted by* $\operatorname{Arg} z$*. Accordingly,*

$$\operatorname{Arg} z = \{\varphi \in \mathbb{R} : \ z = |z|(\cos\varphi + i\sin\varphi)\}.$$

Remark 1.2 *If* $z = 0$*, then* $|z| = 0$*, and* (1.1) *holds for arbitrary* $\varphi \in \mathbb{R}$*. This is the reason why an argument of* 0 *is not defined.*

Theorem 1.1 *Let* $z \in \mathbb{C} \setminus \{0\}$ *and* $\varphi \in \operatorname{Arg} z$*. Then*

$$\operatorname{Arg} z = \{\varphi + 2k\pi : \ k \in \mathbb{Z}\}.$$

Proof Let $\varphi \in \operatorname{Arg} z$. Obviously, sine and cosine are periodic functions, and therefore

$$\{\varphi + 2k\pi : \ k \in \mathbb{Z}\} \subset \operatorname{Arg} z.$$

On the other hand, let $\psi \in \operatorname{Arg} z$. We want to find $k \in \mathbb{Z}$ such that $\psi = \varphi + 2k\pi$.

$$\varphi, \psi \in \operatorname{Arg} z \Rightarrow$$

$$\Rightarrow [z = |z|(\cos\varphi + i\sin\varphi) = |z|(\cos\psi + i\sin\psi) \ \wedge \ |z| \neq 0] \Rightarrow$$

$$\Rightarrow \cos\varphi + i\sin\varphi = \cos\psi + i\sin\psi \Rightarrow \begin{bmatrix} \cos\varphi = \cos\psi \\ \wedge \\ \sin\varphi = \sin\psi \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \cos^2\varphi = \cos\psi\cos\varphi \\ \wedge \\ \sin^2\varphi = \sin\psi\sin\varphi \end{bmatrix} \Rightarrow \cos^2\varphi + \sin^2\varphi = \cos\psi\cos\varphi + \sin\psi\sin\varphi \Rightarrow$$

$$\Rightarrow 1 = \cos(\psi - \varphi) \Rightarrow [\exists k \in \mathbb{Z} : \ \psi - \varphi = 2k\pi] \Rightarrow [\exists k \in \mathbb{Z} : \ \psi = \varphi + 2k\pi].$$

□

Definition 1.2 *The argument* $\varphi \in \operatorname{Arg} z$ *such that*

$$-\pi < \varphi \leq \pi$$

is called a **principal value of an argument** *of a complex number* $z \in \mathbb{C} \setminus \{0\}$ *and we denote it* $\arg z$*.*

Example 1.2 *Find* $\operatorname{Arg} z_0$ *and* $\arg z_0$ *for* $z_0 = -\sqrt{3} - i$.
Solution: Obviously[2],

$$\pi + \arcsin\frac{1}{2} = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \in \operatorname{Arg} z_0,$$

therefore[3]

$$\operatorname{Arg} z_0 = \left\{\frac{7\pi}{6} + 2k\pi \;:\; k \in \mathbb{Z}\right\},\ \arg z_0 = -\frac{5\pi}{6}.$$

1.3 Infinity

As we enrich real numbers by the special symbols $+\infty$ and $-\infty$, we will also introduce analogous symbols into the Gauss plane $\mathbb{C}$. It is more efficient to add only *one* such symbol, ∞, **infinity**.

Now, the following is devoted to the another geometrical interpretation of complex numbers, the so called **stereographical projection**, which clearly explains the meaning of ∞.

Let us assume a unit sphere is located in such a way that it touches with its "south pole" the plane of complex numbers at the point 0, and denote the "north pole" by N. Now, join every complex number z with $z^* \neq N$ belonging to the unit sphere in such a way that z^* is just an intersection of this sphere with a straight line joining z with N. In this way, we have generated one-to-one correspondence between the set of complex numbers (zero corresponds to the "south pole") and points of the unit sphere (without N).

We observe that the size of $|z|$ is inversely proportional to the distance between z^* and N. This compels us to add only one special symbol, ∞, to $\mathbb{C}$ and link it with the properties described by the stereographical projection of point N.

The set

$$\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$$

is called an **extended** (sometimes also **closed**) **Gauss plane**.

For each $z \in \mathbb{C}$ we define:

1. $z \pm \infty = \infty \pm z = \infty$,
2. $z \cdot \infty = \infty \cdot z = \infty$ for $z \neq 0$,
3. $\frac{z}{\infty} = 0$,
4. $\frac{z}{0} = \infty$ for $z \neq 0$,
5. $\frac{\infty}{z} = \infty$,
6. $\infty^n = \infty,\ \infty^{-n} = 0,\ 0^{-n} = \infty$ if $n \in \mathbb{N}$,

[2] Try to draw a picture!
[3] See Theorem 1.1 and Definition 1.2.

7. $|\infty| = \infty$, $\overline{\infty} = \infty$. [4]

1.4 Neighbourhood of a point

Definition 1.3 *By the* **neighbourhood of a point** $z_0 \in \mathbb{C}$ *with* **radius** $\varepsilon \in \mathbb{R}^+$ *we mean the set*

$$U(z_0, \varepsilon) = \{z \in \mathbb{C} : \ |z - z_0| < \varepsilon\}.$$

By the **neighbourhood of** ∞ *with* **radius** $\varepsilon \in \mathbb{R}^+$ *we mean the set*

$$U(\infty, \varepsilon) = \left\{z \in \mathbb{C} : \ |z| > \frac{1}{\varepsilon}\right\} \cup \{\infty\}.$$

By the **ring neighbourhood of a point** $z \in \mathbb{C}_\infty$ *with* **radius** $\varepsilon \in \mathbb{R}^+$ *we mean the set*

$$P(z, \varepsilon) = U(z, \varepsilon) \setminus \{z\}.$$

In case we do not care about the value of ε, *the* **neighbourhood** $U(z)$ *or the* **ring neighbourhood** $P(z)$ *of a point* z *is used.*

Definition 1.4 *We call the set* $M \subset \mathbb{C}_\infty$ **open** *if and only if for all* $z \in M$ *there exists a neighbourhood* $U(z)$ *such that*

$$U(z) \subset M.$$

Example 1.3

1. $\emptyset$, $\mathbb{C}$ *a* $\mathbb{C}_\infty$ *are open sets.*
2. $\{z \in \mathbb{C} : \ |z - 3| < |z + 2 - i|\}$ *and* $\{z \in \mathbb{C} : \ \operatorname{Im} z < 1\}$ *are open sets.*
3. $\{2 + \sqrt{3}i\}$, $\{z \in \mathbb{C} : \ \operatorname{Re} z + 2 \operatorname{Im} z = 7\}$, *and* $\{z \in \mathbb{C} : \ \operatorname{Im} z \leq 1\}$ *are not open sets.*

1.5 Sequence of complex numbers

Definition 1.5 *By a* **sequence** *in* $\mathbb{C}_\infty$ *we mean a function* $f : \mathbb{N} \to \mathbb{C}_\infty$.

Definition 1.6 *Let* $z \in \mathbb{C}_\infty$ *and* (z_n) *be a sequence in* $\mathbb{C}_\infty$. *We say that a* **sequence** (z_n) **has a limit** z *if for all* $\varepsilon \in \mathbb{R}^+$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $n \geq n_0$ *it holds that* $z_n \in U(z, \varepsilon)$. *We denote it* $\lim z_n = z$ *or* $z_n \to z$. *We call a sequence* (z_n) *of complex numbers* **convergent** *if there exists* $z \in \mathbb{C}$ *such that*

$$\lim z_n = z.$$

[4] Notice that the following operations are not defined: $\infty \pm \infty$, $0 \cdot \infty$, $\infty \cdot 0$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\operatorname{Arg} \infty$, $\arg \infty$.

Remark 1.3

1. *The definition of a limit in fact means that outside of any neighbourhood of a point z there is at most finite number of terms of sequence (z_n).*

2. *Let (z_n) be a sequence of complex numbers in the Gauss plane and $z \in \mathbb{C}_\infty$. Denote by (z_n^*) an image of this sequence by stereographic projection and z^* a point on a unit sphere in $\mathbb{R}^3$. Then*

$$z_n \to z \ (in\ \mathbb{C}_\infty) \ \Leftrightarrow \ z_n^* \to z^* \ (in\ \mathbb{R}^3).$$

Theorem 1.2 *Let $z_n = x_n + iy_n$ for all sufficiently large $n \in \mathbb{N}$ and $z = x + iy$. Then*

$$\lim z_n = z \ \Leftrightarrow \ \left[\lim x_n = x \ \wedge \ \lim y_n = y\right].$$

Example 1.4 *Find*

$$\lim \frac{(2n-i)i}{n}.$$

<u>*Solution:*</u>

$$\lim \frac{(2n-i)i}{n} = \lim \left(\frac{1}{n} + 2i\right) = \lim \frac{1}{n} + i \lim 2 = 0 + 2i = 2i.$$

Remark 1.4 *The definition of a limit of a complex sequence is equivalent to the definition of a limit of a real sequence. Therefore many theorems are analogous. In what follows, we introduce only several of them.*

Theorem 1.3 *Any sequence of complex numbers has at most one limit.*

Theorem 1.4 *A sequence of complex numbers has a limit $z \in \mathbb{C}_\infty$ if and only if each subsequence also has a limit z.*

Theorem 1.5 *Let (z_n) be a convergent sequence such that for all $n \in \mathbb{N}$ it holds that $z_n \in \mathbb{C}$. Then the sequence (z_n) is bounded, which means that there exists $k \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$ it is $|z_n| \leq k$.*

Chapter 2

Complex functions of a real and a complex variable

2.1 Complex functions

Definition 2.1 *We call a* **complex function of a complex variable** *a function from $\mathbb{C}_\infty$ to the set of all nonempty subsets of $\mathbb{C}_\infty$. In other words, by a complex function f we mean a formula which assigns to each $z \in Df \subset \mathbb{C}_\infty$ one or more complex numbers from $\mathbb{C}_\infty$. The set Df is called a* **domain of a function** *f. A complex number $f(z)$ is called an f –* **image of a point** *z. A function f is called* **single-valued** *if for all $z \in Df$ a set $f(z)$ contains only one point. Otherwise, we call a function f* **multi-valued***, or accordingly* **double-valued**, **triple-valued**, *…,* **infinite-valued**. *A function f is called a* **complex function of a real variable** *if $Df \subset \mathbb{R}$.*

Remark 2.1 *A domain of a function defined only by a formula is a set of all numbers from $\mathbb{C}_\infty$ for which the formula makes sense. For example, the domain of the function f defined as $f(z) = 1/z$ is the set $Df = \mathbb{C}_\infty$.*

Example 2.1

1. *$f(z) = z^2$ is a single-valued function, $Df = \mathbb{C}_\infty$.*
2. *$f(z) = \operatorname{Arg} z$ is an infinite-valued function, $Df = \mathbb{C} \setminus \{0\}$.*

Remark 2.2 *In the following, we use the notation $\operatorname{Arg} z = \arg z + 2k\pi,\ k \in \mathbb{Z}$, instead of the correct notation $\operatorname{Arg} z = \{\arg z + 2k\pi \ : \ k \in \mathbb{Z}\}$.*

Definition 2.2 *Let f be a multi-valued function. A single-valued function φ is called a* **unique-branch of a function** *f if:*

(i) $D\varphi \subset Df$,

(ii) $\forall z \in D\varphi \ : \ \varphi(z) \in f(z)$.

Example 2.2 *Functions*

$$\begin{aligned}\varphi_1(z) &= \arg z,\\ \varphi_2(z) &= \arg z + 2\pi\end{aligned}$$

are a case of two different unique-branches of a function $f(z) = \operatorname{Arg} z$.

2.2 Some important complex functions

2.2.1 Exponential functions

Definition 2.3 *Let* $z = x + iy \in \mathbb{C}$. *We define an* **exponential function** *by* [1]

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

Theorem 2.1 (Properties of exponential functions)

(i) e^z *is a single-valued function.*

(ii) A codomain of e^z *is a set* $\mathbb{C} \setminus \{0\}$.

(iii) Function e^z *is periodic with a period of* $2\pi i$.

Proof Propositions (i) and (ii) follow from the definition and properties of the real functions e^x, $\sin x$, $\cos x$. Periodicity (iii) can be proved as follows:

$$\begin{aligned}e^{z+2\pi i} &= e^{x+iy+2\pi i} = e^x\left(\cos(y+2\pi) + i\sin(y+2\pi)\right) =\\ &= e^x\left(\cos y + i \sin y\right) = e^{x+iy} = e^z.\end{aligned}$$

□

2.2.2 Trigonometric functions

Definition 2.4 *Let* $z = x + iy \in \mathbb{C}$. *We define* **trigonometric functions** *as*

$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i},\\ \cos z &= \frac{e^{iz} + e^{-iz}}{2},\\ \tan z &= \frac{\sin z}{\cos z},\\ \cot z &= \frac{\cos z}{\sin z}.\end{aligned}$$

[1] We use a notation e for two different functions:

$$e^z : \mathbb{C} \to \mathbb{C} \setminus \{0\} \text{ and } e^x : \mathbb{R} \to \mathbb{R}^+.$$

That is alright because for $z = x + 0i = x$ is

$$e^z = e^{x+0i} = e^x(\cos 0 + i \sin 0) = e^x.$$

In other words, complex exponential function is an extension of real exponential function on $\mathbb{C}$. Similarly for sin, cos, sinh, cosh, ln, ...

Theorem 2.2 (Properties of trigonometric functions)

(i) Each of trigonometric functions is single-valued.

(ii) The functions $\sin z$ *and* $\cos z$ *are periodic with a period of* 2π, *the functions* $\tan z$ *and* $\cot z$ *are periodic with a period of* π.

(iii) For all $z \in \mathbb{C}$ *it holds that:*

$$\begin{aligned}
\sin(-z) &= -\sin z, \\
\cos(-z) &= \cos z, \\
\tan(-z) &= -\tan z, \\
\cot(-z) &= -\cot z.
\end{aligned}$$

(iv) For any $z \in \mathbb{C}$ **Euler's formula** *states that*

$$e^{iz} = \cos z + i \sin z.$$

(v)

$$\begin{aligned}
\sin z &= 0 \Leftrightarrow [\exists k \in \mathbb{Z} : \ z = k\pi], \\
\cos z &= 0 \Leftrightarrow \left[\exists k \in \mathbb{Z} : \ z = \frac{\pi}{2} + k\pi\right].
\end{aligned}$$

Example 2.3 *Let* $z_0 = \cos(4+i)$. *Find* $\operatorname{Re} z_0$ *and* $\operatorname{Im} z_0$.
<u>*Solution:*</u>

$$\begin{aligned}
z_0 &= \cos(4+i) = \frac{e^{i(4+i)} + e^{-i(4+i)}}{2} = \\
&= \frac{e^{-1}(\cos 4 + i \sin 4) + e(\cos(-4) + i \sin(-4))}{2} = \\
&= \frac{e^{-1} + e}{2} \cos 4 + i \frac{e^{-1} - e}{2} \sin 4.
\end{aligned}$$

Therefore

$$\operatorname{Re} z_0 = \cosh 1 \ \cos 4, \quad \operatorname{Im} z_0 = -\sinh 1 \ \sin 4.$$

2.2.3 Hyperbolic functions

Definition 2.5 *We define* **hyperbolic functions** *as*

$$\begin{aligned}
\sinh z &= \frac{e^z - e^{-z}}{2}, \\
\cosh z &= \frac{e^z + e^{-z}}{2}, \\
\tanh z &= \frac{\sinh z}{\cosh z}, \\
\coth z &= \frac{\cosh z}{\sinh z}.
\end{aligned}$$

Remark 2.3 *Similarly as for real functions, we can define an* **inverse function** *as a complex function. In contrast with real functions, we define inverse functions also for functions which are not injective. In such cases the inverse function is multi-valued like, e.g., logarithmic function.*

2.2.4 Logarithmic functions

Definition 2.6 *We define a* **logarithmic function** *as*

$$\operatorname{Ln} z = \{w \in \mathbb{C} : \; e^w = z\}.$$

From properties of exponential functions (see Theorem 2.1) it follows that the domain of $\operatorname{Ln} z$ *is a set* $\mathbb{C} \setminus \{0\}$.

Let

$$z = |z|(\cos\varphi + i\sin\varphi),$$

where $|z| > 0$ and $\varphi \in \mathbb{R}$.

$$\operatorname{Ln} z = u + iv.$$

Then

$$e^{u+iv} = z,$$

that is

$$e^u(\cos v + i\sin v) = |z|(\cos\varphi + i\sin\varphi).$$

Therefore [2]

$$u = \ln|z| \;\wedge\; [\exists k \in \mathbb{Z} : v = \varphi + 2k\pi].$$

We proved that for all $z \in \mathbb{C} \setminus \{0\}$ it holds that

$$\operatorname{Ln} z = \ln|z| + i(\varphi + 2k\pi), \; k \in \mathbb{Z},$$

or equivalently

$$\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z.$$

Example 2.4 *Let* $z_0 = \operatorname{Ln}(-1+i)$. *Find* $\operatorname{Re} z_0$ *and* $\operatorname{Im} z_0$.
<u>*Solution:*</u>

$$z_0 = \operatorname{Ln}(-1+i) = \ln\sqrt{2} + \left(\frac{3\pi}{4} + 2k\pi\right)i, \; k \in \mathbb{Z}.$$

Therefore $\operatorname{Re} z_0 = \ln\sqrt{2}$ *and* $\operatorname{Im} z_0 = \left(\frac{3}{4} + 2k\right)\pi i, \; k \in \mathbb{Z}$.

Definition 2.7 *We define a* **principal branch of a logarithm** *as a function on* $\mathbb{C} \setminus \{0\}$ *by*

$$\ln z = \ln|z| + i\arg z.$$

[2] The symbol ln denotes the natural logarithm, which is the function from $\mathbb{R}^+$ to $\mathbb{R}$.

Example 2.5 *Let* $z_0 = \ln(-1-i)$. *Find* $\operatorname{Re} z_0$ *and* $\operatorname{Im} z_0$.
Solution:

$$z_0 = \ln(-1-i) = \ln\sqrt{2} - \frac{3\pi}{4}i.$$

Therefore

$$\operatorname{Re} z_0 = \ln\sqrt{2}, \ \operatorname{Im} z_0 = -\frac{3\pi}{4}.$$

2.2.5 Power functions

Recall that for any $n \in \mathbb{N}$ (resp. $-n \in \mathbb{N}$) is a function $z \mapsto z^n$, defined as

$$z^n = \underbrace{zzz\cdots z}_{n\text{-times}}, \quad (\text{resp.} \quad z^n = \frac{1}{z^{-n}}).$$

Definition 2.8 *Let* $a \in \mathbb{C}$ *such that* $\pm a \notin \mathbb{N}$. *A* **power function** *is defined by*

$$z^a = \{e^{as} : \ s \in \operatorname{Ln} z\} = e^{a \operatorname{Ln} z}.$$

Example 2.6 *Find the real and imaginary parts of* 2^i.
Solution:

$$2^i = e^{i \operatorname{Ln} 2} = e^{i(\ln 2 + 2k\pi i)} = e^{-2k\pi + i\ln 2} = e^{-2k\pi}\left(\cos(\ln 2) + i\sin(\ln 2)\right), \ k \in \mathbb{Z}.$$

Hence,

$$\operatorname{Re}(2^i) = e^{-2k\pi}\cos(\ln 2) \ \textit{and} \ \operatorname{Im}(2^i) = e^{-2k\pi}\sin(\ln 2),$$

where $k \in \mathbb{Z}$.

2.2.6 n-th roots

Definition 2.9 *Let* $n \in \mathbb{N}, \ n \neq 1$. *We define a* **function n-th root** *as*

$$\sqrt[n]{z} = \{w \in \mathbb{C} : \ w^n = z\}.$$

Exercise 2.1

1. *Prove that for all* $z \in \mathbb{C} \setminus \{0\}$ *and* $1 < n \in \mathbb{N}$ *it holds that*

$$\sqrt[n]{z} = z^{\frac{1}{n}}$$

   and that a function $z \mapsto z^{\frac{1}{n}}$ *is n-valued.*

2. *Prove that for* $a = m/n$, *where* $m \in \mathbb{Z} \setminus \{0\}$ *and* $n \in \mathbb{N}$ *are not divisible numbers, the function* $z \mapsto z^a$ *is n-valued.*

3. *Prove that for* $a \in \mathbb{C} \setminus \mathbb{Q}$, *the function* $z \mapsto z^a$ *is infinite-valued.*

Example 2.7 *Find the real and imaginary parts of* $\sqrt[4]{i}$.
Solution:

$$\begin{aligned}\sqrt[4]{i} &= i^{\frac{1}{4}} = e^{\frac{1}{4}\operatorname{Ln} i} = e^{\frac{1}{4}(\frac{\pi}{2}i+2k\pi i)} = e^{\frac{\pi}{8}i+k\frac{\pi}{2}i} = \\ &= \cos\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+k\frac{\pi}{2}\right),\ k\in\{0,1,2,3\}.\end{aligned}$$

Hence,

$$\operatorname{Re}\left(\sqrt[4]{i}\right) = \cos\left(\frac{\pi}{8}+k\frac{\pi}{2}\right) \text{ and } \operatorname{Im}\left(\sqrt[4]{i}\right) = \sin\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)$$

where $k \in \{0,1,2,3\}$.

2.3 Real and imaginary parts of a function

In what follows, any complex function is meant to be a single-valued one.

Definition 2.10 *Let* $f : \mathbb{C} \to \mathbb{C}$. *Function*

$$u : \mathbb{R}^2 \to \mathbb{R} \ \ (resp.\ \ v : \mathbb{R}^2 \to \mathbb{R})$$

defined on the set

$$\{(x,y)\in\mathbb{R}^2 : x+iy \in Df\}$$

by

$$u(x,y) = \operatorname{Re} f(x+iy) \ \ (resp.\ \ v(x,y) = \operatorname{Im} f(x+iy))$$

be called the **real part of a function** f *(resp.* **imaginary part of a function** f*). We denote function* f *as*

$$f = u + iv,$$

where u *is a real and* v *an imaginary part of* f.

Example 2.8 *Find real and imaginary parts of the function*

$$f(z) = \frac{z}{\bar{z}}.$$

Solution: *Firstly,*

$$f(z) = f(x+iy) = \frac{x+iy}{x-iy} = \frac{x^2-y^2}{x^2+y^2} + i\frac{2xy}{x^2+y^2}.$$

Hence

$$\begin{aligned}\operatorname{Re}(f) &= u(x,y) = \frac{x^2-y^2}{x^2+y^2}, \\ \operatorname{Im}(f) &= v(x,y) = \frac{2xy}{x^2+y^2}.\end{aligned}$$

2.4 Limit of a function of a complex variable

Remark 2.4 *By*

$$z_0 \neq z_n \to z_0,$$

we mean that $z_n \to z_0$ and that for all sufficiently large $n \in \mathbb{N}$ it is $z_n \in \mathbb{C}_\infty \setminus \{z_0\}$.

Definition 2.11 *A* **function** $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ **has at the point** $z_0 \in \mathbb{C}_\infty$ **a limit** $a \in \mathbb{C}_\infty$ *if for each sequence (z_n) such that $z_0 \neq z_n \to z_0$ it holds $f(z_n) \to a$.*
It is denoted by

$$\lim_{z \to z_0} f(z) = a.$$

Theorem 2.3 *Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ and $z_0, a \in \mathbb{C}_\infty$. Then $\lim_{z \to z_0} f(z) = a$ if and only if*

$$(\forall U(a)) \left(\exists P(z_0)\right) \left(\forall z \in P(z_0)\right) : f(z) \in U(a).$$

Theorem 2.4 *Let $f = u + iv : \mathbb{C} \to \mathbb{C}$, $z_0 = x_0 + iy_0$, and $a = \alpha + i\beta$. Then $\lim_{z \to z_0} f(z) = a$ if and only if*

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = \alpha \quad \wedge \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = \beta.$$

Example 2.9 *Calculate*

$$\lim_{z \to i} \left(\frac{z - i}{z^2 + 1} \right).$$

<u>Solution:</u>

$$\begin{aligned}
\lim_{z \to i} \left(\frac{z-i}{z^2+1} \right) &= \lim_{z \to i} \left(\frac{1}{z+i} \right) = \lim_{x+iy \to i} \left(\frac{1}{x + i(y+1)} \right) = \\
&= \lim_{x+iy \to i} \left(\frac{x}{x^2 + (y+1)^2} + i \frac{-(y+1)}{x^2 + (y+1)^2} \right) = \\
&= \lim_{(x,y) \to (0,1)} \left(\frac{x}{x^2 + (y+1)^2} \right) + i \lim_{(x,y) \to (0,1)} \left(\frac{-(y+1)}{x^2 + (y+1)^2} \right) = \\
&= 0 - \frac{1}{2} i = -\frac{1}{2} i.
\end{aligned}$$

Example 2.10 *Calculate*

$$\lim_{z \to -1} \arg z.$$

<u>Solution:</u> *The latter limit doesn't exist because*

1. $-1 \neq z_n = \cos\left(\pi + \frac{(-1)^n}{n}\right) + i \sin\left(\pi + \frac{(-1)^n}{n}\right) \to -1,$
2. $\arg\left(z_{2n}\right) \to -\pi,$
3. $\arg\left(z_{2n+1}\right) \to \pi.$

2.5 Continuity of a function of a complex variable

Definition 2.12 *Let* $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$. *The* **function** f *is* **continuous at the point** $z_0 \in \mathbb{C}_\infty$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

We say that a **function** f **is continuous on the set** $M \subset \mathbb{C}_\infty$ *if for all* $z_0 \in M$ *it holds that*

$$\left.\begin{array}{r} z_n \to z_0 \\ \forall n \in \mathbb{N} : \ z_n \in M \end{array}\right\} \Rightarrow f(z_n) \to f(z_0).$$

We say that a **function** f **is continuous** *if it is continuous on its domain* Df.

Theorem 2.5 *Let* $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *and* $z_0 \in \mathbb{C}_\infty$. *Then the following statements are equivalent:*

(i) f *is continuous at the point* z_0,

(ii) $z_n \to z_0 \ \Rightarrow \ f(z_n) \to f(z_0)$,

(iii) $\left(\forall U\left(f(z_0)\right)\right)\left(\exists U(z_0)\right)\left(\forall z \in U(z_0)\right) : \ f(z) \in U\left(f(z_0)\right)$.

Exercise 2.2 *Think over the relation between the continuity of a function* $f = u + iv : \mathbb{C} \to \mathbb{C}$ *and the continuity of the functions* $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Example 2.11

1. *The function* $\arg z$ *is not continuous since it is not continuous, e.g, at the point* -1 *(see Example 2.9).*

2. *A function* $\arg z$ *is continuous on the set* $\mathbb{C} \setminus (-\infty, 0] = \{z \in \mathbb{C} : \ z \notin \mathbb{R}^- \ \wedge \ z \neq 0\}$.

2.6 Complex functions of a real variable and curves

Let $f : \mathbb{R} \to \mathbb{C}_\infty$ be a complex function of a real variable. We can define the limit and continuity of such a function similarly as for complex functions of a complex variable.

Definition 2.13 *Let* $f : \mathbb{R} \to \mathbb{C}_\infty$. *We say that a* **function** f **has a limit** $a \in \mathbb{C}_\infty$ **at a point** $t_0 \in \mathbb{R}$ *if*

$$t_0 \neq t_n \to t_0 \ \ (in\, \mathbb{R}) \ \ \Rightarrow \ \ f(t_n) \to a.$$

It is denoted by

$$\lim_{t \to t_0} f(t) = a.$$

We say that a **function** f **is continuous at a point** $t_0 \in \mathbb{R}$ *if*

$$\lim_{t \to t_0} f(t) = f(t_0).$$

We say that a **function** f **is continuous on the set** $M \subset \mathbb{R}$ *if for all* $t_0 \in M$ *it holds that*

$$\left. \begin{aligned} t_n &\to t_0 \\ \forall n \in \mathbb{N} : \ t_n &\in M \end{aligned} \right\} \Rightarrow \ f(t_n) \to f(t_0).$$

We say that a **function** f **is continuous** *if it is continuous on its domain* Df.

A highly important part of continuous functions are curves.

Definition 2.14 *By a* **curve** *in* $\mathbb{C}_\infty$ *(resp. in* $\mathbb{C}$*) we mean any continuous complex function of a real variable*

$$\gamma : \ I \to \mathbb{C}_\infty \ \ (\textit{resp. } \gamma : \ I \to \mathbb{C}),$$

where $I = D\gamma \subset \mathbb{R}$ *is an interval. The set*

$$\langle \gamma \rangle = \gamma(I) = \{\gamma(t) : \ t \in I\} \subset \mathbb{C}_\infty$$

is called an **image of a curve** γ. *Let* $M = \langle \gamma \rangle$. *Then* γ *is called a* **parametrization of a set** M.

We have already noticed that there is one-to-one correspondence between points from $\mathbb{R}^2$ and points from $\mathbb{C}$:

$$(x, y) \ \leftrightarrow \ x + iy.$$

Similarly, there is one-to-one correspondence between curves in $\mathbb{R}^2$ and curves in $\mathbb{C}$:

$$\gamma = (\gamma_1, \gamma_2) \leftrightarrow \gamma = \gamma_1 + i\gamma_2.$$

That's why all terms for curves in $\mathbb{R}^2$ come up for curves in $\mathbb{C}$, (see [2]). For example:

- a simple curve,
- a closed curve,
- a simple closed curve,
- a oppositely oriented curve,
- a smooth arc,

- a piecewise smooth arc,
- an initial and terminal point of the curve,
- a derivative of a curve at a point,
- a tangent vector to the curve, etc.

Exercise 2.3 *Sketch an image of a curve γ in the Gauss plane if*

1. $\gamma(t) = 2 - 3i + 2e^{-2it}$, $t \in [0, 3\pi/4]$,

2.
$$\gamma(t) = \begin{cases} 4e^{it},\ t \in \left[0, \frac{\pi}{2}\right], \\ i\left(4 + \frac{\pi}{2} - t\right),\ t \in \left[\frac{\pi}{2}, 4 + \frac{\pi}{2}\right], \\ t - 4 - \frac{\pi}{2},\ t \in \left[4 + \frac{\pi}{2}, 8 + \frac{\pi}{2}\right]. \end{cases}$$

Definition 2.15 *Let $M \subset \mathbb{C}_\infty$. A* **closure of the set** *M is defined as a set $\overline{M}$ of all $z \in \mathbb{C}_\infty$ for which there exists a sequence (z_n) in M such that $z_n \to z$.*[3] **Sets** *$A, B \subset \mathbb{C}_\infty$ are called* **disjoint** *if*

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

A **set** *$M \subset \mathbb{C}_\infty$ is called* **connected** *if it cannot be written as a union of two nonempty disjoint sets. A set $\Omega \subset \mathbb{C}_\infty$ is called a* **domain** *if it at the same time holds that:*

(i) Ω is an open set,

(ii) Ω is a connected set.[4]

Definition 2.16 *Let $M \subset \mathbb{C}_\infty$. A set $K \subset M$ is called a* **component of the set** *M if it fulfills the following two conditions:*

(i) K is a connected set,

(ii) for any connected set $K^* \subset M$ such that $K \subset K^*$ it holds $K = K^*$.[5]

[3] If we understand a **closed set** as a complement of an open set, then $\overline{M}$ can be equivalently defined as a smallest closed set containing M.

[4] It means that for any two points $z_1, z_2 \in \Omega$ there exists a curve $\gamma : [a, b] \to \Omega$ such that $\gamma(a) = z_1$, $\gamma(b) = z_2$.

[5] In other words, we call a component of a set any of its maximal connected subsets.

Remark 2.5 *It's easy to show*[6] *that any set* $M \subset \mathbb{C}_\infty$ *is a union of a system of all its components while this system is disjoint.*

Definition 2.17 *Let* $\Omega \subset \mathbb{C}_\infty$ *denote a domain. Then its complement in* $\mathbb{C}_\infty$ *(i.e., the set* $\mathbb{C}_\infty \setminus \Omega$*) which has n different components is called an n* **–times connected domain**. *A one-time connected domain is called* **simply connected**.

Exercise 2.4

1. $\emptyset$, $\mathbb{C}$, $\mathbb{C}_\infty$, $U(z)$, *where* $z \in \mathbb{C}_\infty$, *are simple connected domains.*
2. $P(z)$, $\mathbb{C} \setminus \{z\}$, *where* $z \in \mathbb{C}$, *are two-times connected domains.*
3. $U(1, 2020) \setminus \{2, 4, 5 + i\}$ *is a four-times connected domain.*
4. $U(3, 2) \cup U(4i, 3)$ *is not a domain (it is not connected).*
5. $\mathbb{C}_\infty \setminus \{z \in \mathbb{C} : \arg z \in [0, \pi/4]\}$ *is not a domain (it is not open).*

[6] See, e.g., [5].

Chapter 3

Derivative of a complex function of a complex variable

3.1 Derivative of a function

Definition 3.1 *Let* $f : \mathbb{C} \to \mathbb{C}$. **A derivative of a function** f **at a point** $z_0 \in \mathbb{C}$ *is defined as*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if the limit exists and is finite. A function f *is called* **holomorphic on a set** Ω *if a set* $\Omega \subset \mathbb{C}$ *is open and for all* $z \in \Omega$ *there exists* $f'(z)$. *A function* f *is called* **holomorphic at a point** $z_0 \in \mathbb{C}$ *if* f *is holomorphic on some neighbourhood of a point* z_0 *(i.e., it has a derivative at each point of some neighbourhood* $U(z_0)$*).*

Remark 3.1 *Notice that the above definition is formally identical to the definition of a derivative of a real function of a real variable. Theorems and proofs concerning computing derivatives are also formally identical,*[1] *therefore we do not introduce them.*

Theorem 3.1 *If a function* $f : \mathbb{C} \to \mathbb{C}$ *has a derivative at a point* $z_0 \in \mathbb{C}$*, then it is continuous at* z_0.

Proof We assume

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

Then there exists a ring neighbourhood $P(z_0)$ such that for every $z \in P(z_0)$ it holds that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < |f'(z_0)| + 1.$$

[1]For example, theorems concerning differentiation of a sum, product, quotient, composition of functions, chain rule, etc.

For any $z \in P(z_0)$ it also holds that

$$0 \le |f(z) - f(z_0)| < \big(|f'(z_0)| + 1\big)\,|z - z_0|.$$

Now, let (z_n) be a sequence such that $z_n \to z_0$. From above it follows that $|f(z_n) - f(z_0)| \to 0$ therefore $f(z_n) \to f(z_0)$.

We proved the continuity of the function f at the point z_0 (see Theorem 2.5). □

Theorem 3.2 *A function $f = u + iv$ has a derivative at a point $z_0 = x_0 + iy_0$ if and only if the following two conditions are fulfilled:*

(i) functions u and v are differentiable[2] at the point (x_0, y_0),

(ii) functions u and v fulfill at the point (x_0, y_0) the so called **Cauchy–Riemann equations**

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0), \\ -\frac{\partial u}{\partial y}(x_0, y_0) &= \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Moreover, if $f'(z_0)$ exists then it holds that

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Remark 3.2 *Let us explain why it is possible to formulate f' via partial derivatives of u and v.* [3]
Notice that if $f'(z_0)$ exists, then

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x_0 + h + iy_0) - f(x_0 + iy_0)}{(x_0 + h + iy_0) - (x_0 + iy_0)} = \\ &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{(x_0 + h - x_0) + i(y_0 - y_0)} = \\ &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\lim_{h\to 0} \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} = \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

[2] Recall a sufficient condition of differentiability:

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$. If the functions $\frac{\partial \varphi}{\partial x}$ and $\frac{\partial \varphi}{\partial y}$ are continuous at the point (x_0, y_0), then a function φ has a derivative at the point (x_0, y_0).

[3] Think carefully over the meaning of formulas of the type $\lim\limits_{\substack{h\to 0\\ h\in\mathbb{R}}} \ldots$.

Similarly,

$$
\begin{aligned}
f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(x_0+i(y_0+s))-f(x_0+iy_0)}{(x_0+i(y_0+s))-(x_0+iy_0)} = \\
&= \lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{u(x_0,y_0+s)+iv(x_0,y_0+s)-u(x_0,y_0)-iv(x_0,y_0)}{(x_0-x_0)+i(y_0+s-y_0)} = \\
&= \lim_{s\to 0} \frac{v(x_0,y_0+s)-v(x_0,y_0)}{s} + \frac{1}{i}\lim_{s\to 0}\frac{u(x_0,y_0+s)-u(x_0,y_0)}{s} = \\
&= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).
\end{aligned}
$$

Example 3.1 *Find points at which the function*

$$f(z) = e^z$$

has a derivative. Find $f'(z)$.
Solution: For each $x+iy\in\mathbb{C}$ is:

$$
\begin{aligned}
f(x+iy) &= e^{x+iy} = \underbrace{e^x\cos y}_{u(x,y)} + i\,\underbrace{e^x\sin y}_{v(x,y)}, \\
\frac{\partial u}{\partial x}(x,y) &= e^x\cos y = \frac{\partial v}{\partial y}(x,y), \\
-\frac{\partial u}{\partial y}(x,y) &= e^x\sin y = \frac{\partial v}{\partial x}(x,y).
\end{aligned}
$$

Functions u and v have a derivative at any point $(x,y)\in\mathbb{R}^2$. Therefore for each $z=x+iy\in\mathbb{C}$ it holds that

$$
\begin{aligned}
f'(z) &= f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \\
&= e^x\cos y + ie^x\sin y = e^{x+iy} = f(x+iy) = f(z).
\end{aligned}
$$

3.2 Harmonic functions, harmonic conjugate functions

Definition 3.2 *Let $M\subset\mathbb{R}^2$ be an open set. We call a function $\varphi:\ \mathbb{R}^2\to\mathbb{R}$* **harmonic on the set** *M if for any $(x,y)\in M$ it holds that*

(i) function φ has continuous first and second partial derivatives at the point (x,y) (i.e., φ is of the class C^2 on M),

(ii) $\Delta\varphi(x,y) = \dfrac{\partial^2\varphi}{\partial x^2}(x,y) + \dfrac{\partial^2\varphi}{\partial y^2}(x,y) = 0.$

Example 3.2

1. *Function* $\varphi(x, y) = x + y + e^x \cos y$ *is harmonic on* $\mathbb{R}^2$.
2. *Function* $\varphi(x, y) = \operatorname{Im}(\ln(x + iy))$ *is not harmonic on* $\mathbb{R}^2 \setminus \{(0, 0)\}$.[4]

Remark 3.3 *In the following we use the formulation: "function* φ *is harmonic on the set* $\Omega \subset \mathbb{C}$*" instead of the correct form "function* φ *is harmonic on the set* $\{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$*".*

Now, assume that a function $f = u + iv$ has at any point of a domain $\Omega \subset \mathbb{C}$ a second derivative[5] and that functions u and v are of the class C^2 on the set $\{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$. From Theorem 3.2 it follows that for any point $x + iy \in \Omega$ we have

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y), \\ f''(x+iy) &= \frac{\partial^2 u}{\partial x^2}(x,y) + i\frac{\partial^2 v}{\partial x^2}(x,y) = -\frac{\partial^2 u}{\partial y^2}(x,y) - i\frac{\partial^2 v}{\partial y^2}(x,y). \end{aligned}$$

If we compare real and imaginary parts in the latter equality, we find out that for all $x + iy \in \Omega$:

$$\Delta u(x, y) = 0 = \Delta v(x, y).$$

In other words, functions u and v are harmonic on Ω.

The following theorem generalizes this contemplation.

Theorem 3.3 *Let* $f = u + iv$ *be holomorphic on the domain* $\Omega \subset \mathbb{C}$*. Then functions* u *and* v *are harmonic on* Ω*.*

Definition 3.3 *We call functions* $u, v : \mathbb{R}^2 \to \mathbb{R}$ **harmonic conjugate** *on the domain* $\Omega \subset \mathbb{C}$ *if at the same time it holds that*

(i) functions u *and* v *are harmonic on* Ω*,*

(ii) functions u *and* v *satisfy Cauchy–Riemann equations on* Ω*.*

Remark 3.4 *Notice that harmonic conjugate functions are real and imaginary parts of holomorphic functions.*

---

[4] Why?

[5] Let $n \in \mathbb{N}$. The $(n+1)$**–th derivative of a function** f **at a point** $z_0 \in \mathbb{C}$ is defined by induction:

$$f^{(n+1)}(z_0) = \left(f^{(n)}\right)'(z_0),$$

i.e.,

$$f^{(n+1)}(z_0) = \lim_{z \to z_0} \frac{f^{(n)}(z) - f^{(n)}(z_0)}{z - z_0}$$

if the limit on the right exists and is finite.

Example 3.3 *Find holomorphic function $f = u + iv$ if*

$$u(x, y) = x^2 - y^2 + 2xy.$$

Solution: We search for a function $v : \mathbb{R}^2 \to \mathbb{R}$ joined with a function u by satisfying Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x}(x, y) = 2x + 2y = \frac{\partial v}{\partial y}(x, y) \quad \Rightarrow \quad v(x, y) = 2xy + y^2 + \varphi(x),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$. Now, instituting in the second Cauchy–Riemann equations we get

$$-\frac{\partial u}{\partial y}(x, y) \quad = \quad 2y - 2x = \frac{\partial v}{\partial x}(x, y) = 2y + \varphi'(x).$$

Therefore

$$\begin{aligned} \varphi(x) &= -x^2 + c, \text{ where } c \in \mathbb{R}, \\ v(x, y) &= 2xy + y^2 - x^2 + c. \end{aligned}$$

One can easily make sure that [6] the function

$$f(x + iy) = x^2 - y^2 + 2xy + i(2xy + y^2 - x^2 + c)$$

is holomorphic on $\mathbb{C}$ for any $c \in \mathbb{R}$.

Theorem 3.4 *Let u (resp. v) be a harmonic function on the simply connected domain*
$\Omega \subset \mathbb{C}$. Then there exists a function $f : \mathbb{C} \to \mathbb{C}$ uniquely determined up to strictly imaginary (resp. strictly real) constant such that:

(i) f is holomorphic on Ω,

(ii) for all $x + iy \in \Omega : u(x, y) = \operatorname{Re} f(x + iy)$ (resp. $v(x, y) = \operatorname{Im} f(x + iy)$).

Exercise 3.1

1. *Find all on the domain $\mathbb{C} \setminus \{0\}$ holomorphic functions $f = u + iv$, where*

$$v(x, y) = \frac{y}{x^2 + y^2}.$$

2. *Prove that a function*

$$v(x, y) = \ln(x^2 + y^2)$$

is harmonic on the domain $\mathbb{C} \setminus \{0\}$ and that there does not exist a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $f = u + iv$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

[6] It is necessary to check the conditions from Theorem 3.2.

3.3 Remark about geometric meaning of derivatives

Assume that a function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at the point $z_0 \in \mathbb{C}$ and that

$$0 \neq f'(z_0) = |f'(z_0)|\, e^{i \arg f'(z_0)}.$$

It follows from the definition of a derivative that

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)| \in \mathbb{R}^+.$$

Therefore, for every z is close to the point z_0 the number $|f(z) - f(z_0)|$ close to the number $|f'(z_0)| \cdot |z - z_0|$. In other words, for a sufficiently small $\delta > 0$ an f–image of a circle $\{z \in \mathbb{C} : \ |z - z_0| = \delta\}$ differs a little from the circle $\{w \in \mathbb{C} : \ |w - f(z_0)| = |f'(z_0)| \cdot \delta\}$.

Now, let us demonstrate a geometrical interpretation of $\arg f'(z_0)$. Let γ be any smooth arc in $\mathbb{C}$ such that $\gamma(t_0) = z_0$. Then the number $\arg \gamma'(t_0)$ gives an angle between tangent vector $\gamma'(t_0)$ and a positive part of a real axis.[7] Now, let $\Gamma(t) = f(\gamma(t))$ be a curve on a sufficiently small neighbourhood of a point t_0. Then, explore a perturbation of a tangent vector $\Gamma'(t_0)$ from a positive part of a real axis, i.e., the argument $\Gamma'(t_0)$. Since $\Gamma'(t_0) = f'\left(\gamma(t_0)\right) \gamma'(t_0) = f'(z_0)\gamma'(t_0)$, it holds

$$\arg f'(z_0) + \arg \gamma'(t_0) \in \operatorname{Arg} \Gamma'(t_0).$$

In other words, the number $\arg f'(z_0)$ gives an angle by which it is necessary to rotate the directional vector of the tangent to the smooth arc γ at the point $\gamma(t_0) = z_0$ so that we get a directional vector of the tangent to the curve $\Gamma = f \circ \gamma$ at the point $\Gamma(t_0) = f(z_0)$ while the curve γ was arbitrarily picked.

These observations give us the following definition.

Definition 3.4 *Let $f : \mathbb{C} \to \mathbb{C}$ be a function holomorphic at a point z_0 and $f'(z_0) \neq 0$. The number $|f'(z_0)|$ is called the* **extensibility coefficient of the function f at the point z_0.**[8] *The number $\arg f'(z_0)$ is called the* **rotational angle of the function f at the point z_0.**

[7] Draw a picture!

[8] Moreover, if $|f'(z_0)| < 1$, (resp. $|f'(z_0)| > 1$) then a function f is called a **contraction,** (resp. a **dilatation) of the function f at the point** z_0.

Chapter 4

Conformal function

4.1 Basic properties

Definition 4.1 *A function* $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *is called* **conformal in an open set** $G \subset \mathbb{C}_\infty$ *if*

(i) a function f *is continuous and injective in* G*,*

(ii) a derivative f' *exists at any point of the set* G *up to finitely many ones.*

Exercise 4.1 *Find out in which domains are the functions*

$$e^z,\ \ln z,\ \sin z,\ z^2,\ z^4$$

conformal.

Definition 4.2 *Open sets* $G_1, G_2 \subset \mathbb{C}_\infty$ *are called* **conformally equivalent** *if there exists a function* $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *such that*

(i) a function f *is conformal in* G_1*,*

(ii) $f(G_1) = G_2$.

Properties of conformal functions

(i) If a function f is conformal on G, then $0 \neq f'(z) \in \mathbb{C}$ for all $z \in G$ up to at most two points: the point ∞ (in case it belongs to G) and a point (in case that such a point in G exists) whose f-image is ∞.[1]

(ii) An inverse function to the conformal function is also conformal.

(iii) An image of a domain under conformal function is also a domain.

[1] Notice that conformal function preserves angles between curves pathing through z_0 ($z_0 \in G$, $z_0 \neq \infty \neq f(z_0)$) – see geometrical interpretation of $\arg f'(z_0)$ on page 28. This property of a function f is called **conformallity at the point** z_0.

(iv) All simply connected domains in $\mathbb{C}_\infty$ can be classified into four groups:
group 1. contains only empty-set,
group 2. contains only $\mathbb{C}_\infty$,
group 3. contains all domains of the form $\mathbb{C}_\infty \setminus \{z_0\}$, where $z_0 \in \mathbb{C}_\infty$,
group 4. contains all remaining simply connected domains.[2]
Then, any two simply connected domains Ω_1 and Ω_2 are conformally equivalent if and only if they belong to the same group.

4.2 Linear fractional functions

Definition 4.3 *A function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called a* **linear fractional function** *if there are numbers $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ and*

$$f(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \in \mathbb{C}, \\ \dfrac{a}{c} & \text{if } z = \infty. \end{cases}$$

Properties of linear fractional functions

(i) Linear fractional function is the only conformal function from $\mathbb{C}_\infty$ on $\mathbb{C}_\infty$.

(ii) Inverse function to the linear fractional function is a linear fractional function.

(iii) An image of a generalized circle under linear fractional function is a generalized circle. (**Generalized circle** is defined as a circle in $\mathbb{C}$ or a line including the point ∞.)

(iv) Let each of the sets $\{z_1, z_2, z_3\}$, $\{w_1, w_2, w_3\}$ contains three different numbers from $\mathbb{C}_\infty$. Then there exists exactly one linear fractional function f such that $f(z_1) = w_1$, $f(z_2) = w_2$ and $f(z_3) = w_3$.

(v) A special case of linear fractional function is a **linear function** given by $f(z) = az + b$, where $a, b \in \mathbb{C}$, $a \neq 0$. [3]

Example 4.1 *Find an image of a circle*

$$K = \{z \in \mathbb{C} : |z - 1| = 1\}$$

under the function

$$f(z) = \frac{1}{z}.$$

[2] I.e., all non-empty simply connected domains, whose complement contains at most two points.
[3] Notice that one can get any linear function as a composition of three functions:
rotation ($z \mapsto e^{i \arg a} z$), homothety ($z \mapsto |a| z$), and shift ($z \mapsto z + b$).

<u>Solution:</u> *For* $0, 2, 1+i \in K$ *it holds that* $f(0) = \infty$, $f(2) = \frac{1}{2}$, $f(1+i) = \frac{1}{2} - \frac{1}{2}i$, *hence an image of the circle* K *is a line:* [4]

$$f(K) = \left\{ z \in \mathbb{C} : \ \operatorname{Re} z = \frac{1}{2} \right\} \cup \{\infty\} .$$

[4] See property (iii) of linear fractional functions.

Chapter 5

Complex functions integral, Cauchy's integral theorem and formulas

5.1 Integral of complex functions of a complex and real variable

Theorem 5.1 (Jordan's) *Denote by γ a simple closed curve in $\mathbb{C}$. Then*

$$\mathbb{C}_\infty \setminus \langle\gamma\rangle = \Omega_1 \cup \Omega_2,$$

where Ω_1 and Ω_2 are disjoint,[1] *non-empty, simply connected domains with common boundary $\langle\gamma\rangle$.*

Definition 5.1 *Consider the situation from aforementioned theorem. That domain of Ω_1, Ω_2, which does not contain ∞, is called an* **interior of the curve** γ *and is denoted by* $\operatorname{int}\gamma$. *The domain containing ∞ is called an* **exterior of the curve** γ *and is denoted by* $\operatorname{ext}\gamma$.

Definition 5.2 *Let $f = u + iv : \mathbb{R} \to \mathbb{C}$ be a function continuous on the interval $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$.*[2] *Then we define*

$$\int_a^b f(t)\ \mathrm{d}t = \int_a^b u(t) + iv(t)\ \mathrm{d}t = \int_a^b u(t)\ \mathrm{d}t + i\int_a^b v(t)\ \mathrm{d}t.$$

Definition 5.3 *Let $\gamma : [a, b] \to \mathbb{C}$ be a piece-wise smooth curve and let*

$$f = u + iv : \mathbb{C} \to \mathbb{C}$$

[1] I.e., $\Omega_1 \cap \Omega_2 = \emptyset$.

[2] It means that functions $u(t) = \operatorname{Re} f(t),\ v(t) = \operatorname{Im} f(t) : \mathbb{R} \to \mathbb{R}$ are continuous on $[a, b]$.

be a continuous function on $\langle\gamma\rangle$*. Then* [3]

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{(\gamma)} u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\int_{(\gamma)} v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y,$$

where integrals on the right are curve integrals of the second kind[4] *(*γ *is meant to be a curve in* $\mathbb{R}^2$*).*

Theorem 5.2 *Let* $\gamma : [a,b] \to \mathbb{C}$ *be a smooth arc and let* $f : \mathbb{C} \to \mathbb{C}$ *be a continuous function on* $\langle\gamma\rangle$*. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Proof Let $f = u + iv$ and $\gamma = \gamma_1 + i\gamma_2$. Then

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_{(\gamma)} u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\int_{(\gamma)} v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y = \\
&= \int_a^b u(\gamma_1(t),\gamma_2(t))\gamma_1'(t) - v(\gamma_1(t),\gamma_2(t))\gamma_2'(t)\,\mathrm{d}t + \\
&+ i\int_a^b v(\gamma_1(t),\gamma_2(t))\gamma_1'(t) + u(\gamma_1(t),\gamma_2(t))\gamma_2'(t)\,\mathrm{d}t = \\
&= \int_a^b \big(u(\gamma_1(t),\gamma_2(t)) + iv(\gamma_1(t),\gamma_2(t))\big)\,\gamma_1'(t) + \\
&+ i\,\big(u(\gamma_1(t),\gamma_2(t)) + iv(\gamma_1(t),\gamma_2(t))\big)\,\gamma_2'(t)\,\mathrm{d}t = \\
&= \int_a^b f\,\big(\gamma_1(t) + i\gamma_2(t)\big)\,\big(\gamma_1'(t) + i\gamma_2'(t)\big)\,\mathrm{d}t \\
&= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.
\end{aligned}$$

□

Example 5.1 *Calculate*

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z$$

[3]For easier recall:

$$f(z)\,\mathrm{d}z = (u+iv)(\,\mathrm{d}x + i\;\mathrm{d}y) = u\,\mathrm{d}x - v\,\mathrm{d}y + i(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

[4]See [2].

where $\gamma(t) = 5e^{it},\ t \in [0, 2\pi]$.
Solution: Using Definition 5.3:

$$\begin{aligned}
\int_\gamma \frac{1}{z}\,\mathrm{d}z &= \int_{(\gamma)} \frac{x}{x^2+y^2}\,\mathrm{d}x + \frac{y}{x^2+y^2}\,\mathrm{d}y + i\int_{(\gamma)} \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y = \\
&= \int_0^{2\pi} \frac{-25\sin t\cos t}{25} + \frac{25\sin t\cos t}{25}\,\mathrm{d}t + i\int_0^{2\pi} \frac{25\sin^2 t}{25} + \frac{25\cos^2 t}{25}\,\mathrm{d}t = \\
&= 0 + i\int_0^{2\pi} 1\,\mathrm{d}t = 2\pi i.
\end{aligned}$$

Using Theorem 5.2:

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{5e^{it}}\,5ie^{it}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i\,.$$

5.2 Cauchy's integral theorem

Theorem 5.3 (Cauchy's integral) *Let f be a holomorphic function on simply connected domain $\Omega \subset \mathbb{C}$. Then for each closed piece-wise smooth curve γ in Ω (i.e., $\langle\gamma\rangle \subset \Omega$) it holds that*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof We denote $f = u + iv$ and define vector fields

$$\begin{aligned}
f_1(x, y) &= (u(x, y), -v(x, y))\,, \\
f_2(x, y) &= (v(x, y), u(x, y))\,.
\end{aligned}$$

Then f_1 and f_2 are of the class C^2 on simply connected domain

$$\Omega^* = \{(x, y) \in \mathbb{R}^2 \,:\; x + iy \in \Omega\}$$

(see Theorem 3.3). Futhermore, in Ω^* it holds

$$\frac{\partial u}{\partial y} = \frac{\partial(-v)}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

(see Theorem 3.2). Hence f_1 and f_2 are potentially on Ω^* (see [1]). Therefore

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{(\gamma)} f_1(x, y)\mathrm{d}s + i\int_{(\gamma)} f_2(x, y)\mathrm{d}s = 0 + i0 = 0.$$

□

Theorem 5.4 (Cauchy's generalized) *Let $\Omega = \operatorname{int}\gamma$, where γ is a simple closed piece-wise smooth curve in $\mathbb{C}$. Then for any function $f : \mathbb{C} \to \mathbb{C}$ holomorphic on Ω and continuous on $\overline{\Omega} = \Omega \cup \langle\gamma\rangle$ it holds that*[5]

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Remark 5.1 *Let*

$$\gamma,\ \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n$$

be simple closed piece-wise smooth positively oriented curves in $\mathbb{C}$ such that for any $i, j \in \{1, 2, \ldots, n\}$ it holds

$$\begin{aligned} \langle\gamma_i\rangle &\subset \operatorname{ext}\gamma_j \text{ for } i \neq j, \\ \langle\gamma_i\rangle &\subset \operatorname{int}\gamma. \end{aligned}$$

Then a set

$$\Omega = \operatorname{int}\gamma \cap \operatorname{ext}\gamma_1 \cap \operatorname{ext}\gamma_2 \cap \cdots \cap \operatorname{ext}\gamma_n$$

is an $(n+1)$-times connected domain.[6]

Theorem 5.5 (Cauchy's integral theorem for n - times connected domains) *Let Ω be an $(n+1)$-times connected domain described above and let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function in Ω and continuous on*

$$\overline{\Omega} = \Omega \cup \langle\gamma\rangle \cup \langle\gamma_1\rangle \cup \langle\gamma_2\rangle \cup \cdots \cup \langle\gamma_n\rangle.$$

Then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\,\mathrm{d}z.$$

5.3 Cauchy's integral formula

Theorem 5.6 *Let γ be a simple closed piece-wise smooth positively oriented curve in $\mathbb{C}$ and let a function $f : \mathbb{C} \to \mathbb{C}$ be holomorphic on $\Omega = \operatorname{int}\gamma$ and continuous on $\overline{\Omega} = \Omega \cup \langle\gamma\rangle$. Then for any $z_0 \in \Omega$ it holds that*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{5.1}$$

Moreover, if $n \in \mathbb{N}$, then for any $z_0 \in \Omega$ exists $f^{(n)}(z_0)$ such that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z. \tag{5.2}$$

[5] Notice the connection with Green's theorem, see [1].

[6] Draw a picture!

Proof We will prove only assertion (5.1).
Let $z_0 \in \Omega$ be an arbitrary point. For any $r > 0$ we define a curve

$$\gamma_r(t) = z_0 + re^{it},\ t \in [0, 2\pi].$$

From Theorem 5.5 it follows

$$\int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z = \lim_{r\to 0+} \int_{\gamma_r} \frac{f(z)}{z - z_0}\,\mathrm{d}z = \lim_{r\to 0+} \left[\int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z + \int_{\gamma_r} \frac{f(z_0)}{z - z_0}\,\mathrm{d}z \right].$$

Now, from the assumption

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$$

it follows that there exists $\delta > 0$ and $k > 0$ such that for all $z \in \mathbb{C}$, $0 < |z - z_0| < \delta$ it holds that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \le k.$$

Therefore, for all sufficiently small $r > 0$ we receive[7]

$$\left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z \right| \le k2\pi r,$$

and thus

$$\lim_{r\to 0+} \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z = 0.$$

Moreover,

$$\lim_{r\to 0+} \int_{\gamma_r} \frac{f(z_0)}{z - z_0}\,\mathrm{d}z = \lim_{r\to 0+} \left(f(z_0) \int_0^{2\pi} \frac{1}{re^{it}} rie^{it}\,\mathrm{d}t \right) = \lim_{r\to 0+} (f(z_0)2\pi i) = f(z_0)2\pi i,$$

therefore

$$\int_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z = f(z_0)2\pi i.$$

$\square$

[7] We apply this estimation of the curve integral: *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth arc and let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function on $\langle\gamma\rangle$. Then*

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \le \sup_{z\in\langle\gamma\rangle} |f(z)| \cdot \underbrace{\int_a^b |\gamma'(t)|\,\mathrm{d}t}_{\text{length of curve } \gamma}.$$

Remark 5.2

(i) It follows from Theorem 5.6 that a derivative of a holomorphic function is also a holomorphic function. In other words, if a function f is holomorphic on an open set Ω and $n \in \mathbb{N}$, then a function $f^{(n)}$ is holomorphic on Ω.

(ii) Let us consider Theorem 5.6. Then values of a function f on Ω are uniquely determined by values of a function f on $\langle \gamma \rangle$.

(iii) We can obtain formula (5.2) if we formally differentiate n-times both sides of formula (5.1) with respect to z_0.

Example 5.2 *Find*

$$\int_\gamma \frac{e^z}{z(1-z)^3}\,\mathrm{d}z$$

where $\gamma(t) = \frac{3}{2}e^{it},\ t \in [0, 2\pi]$.
Solution:
It follows from Theorem 5.5 that:

$$\int_\gamma \frac{e^z}{z(1-z)^3}\,\mathrm{d}z = \int_{\gamma_1} \frac{e^z}{z(1-z)^3}\,\mathrm{d}z + \int_{\gamma_2} \frac{e^z}{z(1-z)^3}\,\mathrm{d}z,$$

where

$$\begin{aligned}\gamma_1(t) &= \frac{1}{4}e^{it},\ t \in [0, 2\pi],\\ \gamma_2(t) &= 1 + \frac{1}{4}e^{it},\ t \in [0, 2\pi].\end{aligned}$$

Now, applying Theorem 5.6

$$\begin{aligned}\int_{\gamma_1} \frac{e^z}{z(1-z)^3}\,\mathrm{d}z &= \int_{\gamma_1} \frac{\frac{e^z}{(1-z)^3}}{z-0}\,\mathrm{d}z = 2\pi i \left[\frac{e^z}{(1-z)^3}\right]_{z=0} = 2\pi i,\\ \int_{\gamma_2} \frac{e^z}{z(1-z)^3}\,\mathrm{d}z &= \int_{\gamma_2} \frac{-\frac{e^z}{z}}{(z-1)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}\left[\left(-\frac{e^z}{z}\right)''\right]_{z=1} = \pi i(-e),\end{aligned}$$

hence

$$\int_\gamma \frac{e^z}{z(1-z)^3}\,\mathrm{d}z = \pi i(2-e).$$

5.4 Primitive functions and path independent integral

Definition 5.4 *A function $F : \mathbb{C} \to \mathbb{C}$ is called a* **primitive function** *to a function $f : \mathbb{C} \to \mathbb{C}$* **in a domain** *$\Omega \subset \mathbb{C}$ if for any $z \in \Omega$ it holds that*

$$F'(z) = f(z).$$

Theorem 5.7 *Let F be a primitive function to a function f in a domain Ω. Then every primitive function to f in Ω is in the form $F + k$ where $k \in \mathbb{C}$.*

Proof We have to prove that:

(i) for every $k \in \mathbb{C}$ a function $F + k$ is primitive to f in Ω,

(ii) if Φ is a primitive function to f in Ω, then there exists $k \in \mathbb{C}$ such that $\Phi = F + k$.

Ad i) $(F + k)' = F' + 0 = f$ in Ω.
Ad ii) Let us define a function $G = u + iv : \mathbb{C} \to \mathbb{C}$ by a formula

$$G(z) = \Phi(z) - F(z).$$

Then for all $z \in \Omega$ it holds that $G'(z) = 0$, therefore [8] for all $x + iy \in \Omega$ one obtains

$$0 \quad = \quad G'(x + iy) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

Therefore, for all $x + iy \in \Omega$ it holds that

$$\frac{\partial u}{\partial x}(x, y) \quad = \quad \frac{\partial v}{\partial y}(x, y) = 0 = \frac{\partial v}{\partial x}(x, y) = -\frac{\partial u}{\partial y}(x, y) = 0.$$

Hence it follows that the functions u and v are constant on the set $\{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$ and so does the function $G = u + iv = \Phi - F$ in Ω. □

Definition 5.5 *An* **integral of a function** $f : \mathbb{C} \to \mathbb{C}$ *is called* **path independent in a domain** $\Omega \subset \mathbb{C}$ *if for any two piece-wise smooth curves γ_1 and γ_2 are such that*

(i) $\langle\gamma_1\rangle \cup \langle\gamma_2\rangle \subset \Omega$,

(ii) an initial point of γ_1 equals an initial point of γ_2,

(iii) a terminal point of γ_1 equals a terminal point of γ_2

and it holds that

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$$

where z_1 (resp. z_2) denotes initial (resp. terminal) point of curves γ_1, γ_2.

Theorem 5.8 *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on the simply connected domain $\Omega \subset \mathbb{C}$. Then an integral to a function f is path independent in Ω.*

Proof The assertion follows straightforwardly from Theorem 5.3. □

[8] See Theorem 3.2.

Theorem 5.9 (Morera's) *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function on a domain $\Omega \subset \mathbb{C}$. Assume that for any simple closed piece-wise connected curve γ in Ω it holds that*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Then f is holomorphic in Ω.

Theorem 5.10 *Assume that an integral of a continuous function $f : \mathbb{C} \to \mathbb{C}$ is path independent in a domain $\Omega \subset \mathbb{C}$. Then there exists a primitive function to f in Ω. Moreover, for any point $z_0 \in \Omega$ a primitive function F to f in Ω is given by* [9]

$$F(z) = \int_{z_0}^{z} f(\xi)\,\mathrm{d}\xi.$$

Proof Let $z_0 \in \Omega$ and $F(z) = \int_{z_0}^{z} f(\xi)\,\mathrm{d}\xi$. We want to prove that for any $z \in \Omega$ it holds that

$$\lim_{h\to 0}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \quad = \quad 0.$$

Let $z \in \Omega$ denote an arbitrary point. Let $P(0)$ be such that for all $h \in P(0)$ it is

$$z + h \quad \in \quad \Omega$$

and define for all $h \in P(0)$ a curve γ given by

$$\gamma_h(t) \quad = \quad z + th,\ t \in [0,1].$$

Then for all $h \in P(0)$ it holds that:

$$\begin{aligned}
0 \quad &\le \quad \left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \frac{1}{|h|}\left|\int_z^{z+h} f(\xi)\,\mathrm{d}\xi - f(z)h\right| = \\
&= \quad \frac{1}{|h|}\left|\int_{\gamma_h} f(\xi)\,\mathrm{d}\xi - f(z)\int_{\gamma_h} 1\,\mathrm{d}\xi\right| = \frac{1}{|h|}\left|\int_{\gamma_h} f(\xi) - f(z)\,\mathrm{d}\xi\right| \le \\
&\le \quad \frac{1}{|h|}\sup_{\xi\in\langle\gamma_h\rangle}|f(\xi)-f(z)|\,|h| = \sup_{\xi\in\langle\gamma_h\rangle}|f(\xi)-f(z)| \to 0 \text{ for } h \to 0,
\end{aligned}$$

since f is continuous in z. □

Example 5.3 *A function*

$$f(z) = \frac{1}{z}$$

[9]Formula $\int_{z_0}^{z} f(\xi)\,\mathrm{d}\xi$ denotes an integral $\int_\gamma f(\xi)\,\mathrm{d}\xi$ where γ is any piece-wise smooth curve in Ω whose initial point is $\gamma = z_0$ and terminal point is $\gamma = z$.

is holomorphic on a simply connected domain $\Omega = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$, *therefore a function (we integrate along curves in* Ω*)*

$$F(z) = \int_1^z f(\xi)\ \mathrm{d}\xi = \int_1^{|z|} \frac{1}{x}\ \mathrm{d}x + \int_{|z|}^z \frac{1}{\xi}\ \mathrm{d}\xi = [\ln x]_1^{|z|} + i \int_0^{\arg z} \mathrm{d}t = \ln z$$

is a primitive function to the function f *in* Ω.[10]

Remark 5.3 *Let* F *denote a primitive function to a function* f *on simply connected domain* Ω *and let* $z_1, z_2 \in \Omega$. *Let us look at* $\int_{z_1}^{z_2} f(z)\,\mathrm{d}z$. [11]

Let $z_0 \in \Omega$ *be an arbitrary point. Then there exists a constant* $k \in \mathbb{C}$ *such that for all* $z \in \Omega$ *it holds that*

$$F(z) \quad = \quad \int_{z_0}^z f(\xi)\,\mathrm{d}\xi + k$$

(see Theorems 5.7, 5.8 and 5.10.)

$$\begin{aligned}
\int_{z_1}^{z_2} f(z)\,\mathrm{d}z &= \int_{z_1}^{z_0} f(z)\,\mathrm{d}z + \int_{z_0}^{z_2} f(z)\,\mathrm{d}z = \\
&= -\left(\int_{z_0}^{z_1} f(z)\,\mathrm{d}z + k\right) + \left(\int_{z_0}^{z_2} f(z)\,\mathrm{d}z + k\right) = \\
&= F(z_2) - F(z_1) = [F(z)]_{z_1}^{z_2}.
\end{aligned}$$

One can generalize this as follows:

Theorem 5.11 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a function such that there exists a primitive function to* f *in a domain* $\Omega \subset \mathbb{C}$. *Then an integral of the function* f *is path independent in* Ω. *Moreover, if* F *is a primitive function to the function* f *in* Ω *and if* γ *is a piece-wise connected curve in* Ω *defined on* $[i, j]$, *then*

$$\int_\gamma f(z)\,\mathrm{d}z \quad = \quad F(\gamma(j)) - F(\gamma(i)).$$

Exercise 5.1

1. *Let* $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. *Then* $\int_\gamma \frac{1}{z^2}\,\mathrm{d}z = 0$ *since* $\left(-\frac{1}{z}\right)' = \frac{1}{z^2}$ *in the domain* $\mathbb{C} \setminus \{0\}$.

2. $\int_0^{1+i} \sin z \cos z\,\mathrm{d}z = \int_0^{1+i} \frac{1}{2} \sin(2z)\,\mathrm{d}z = \frac{1}{4}\left[-\cos(2z)\right]_0^{1+i} = \frac{1}{4}\left(1 - \cos(2 + 2i)\right).$

3. $\int_0^{2\pi i} z e^z\,\mathrm{d}z = [z e^z]_0^{2\pi i} - \int_0^{2\pi i} e^z\,\mathrm{d}z = 2\pi i - [e^z]_0^{2\pi i} = 2\pi i$ *(the integration by parts was used).*

[10] Think over it carefully!

[11] Once again we integrate along piece-wise connected curves in Ω.

Chapter 6

Number series, sequences and function series

6.1 Number series

Definition 6.1 *A* **series of complex numbers** *is given by*

$$z_1 + z_2 + \cdots + z_n + \cdots = \sum_{n=1}^{\infty} z_n, \tag{6.1}$$

where $z_n \in \mathbb{C}$ *for all* $n \in \mathbb{N}$.

A number z_n *is called an* n**-th term of a series** *(6.1). A sequence* (s_n) *defined by*

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k$$

is called a **sequence of partial sum of a series** *(6.1). A* **series** *(6.1) called* **convergent** *if there exists a finite limit* $\lim s_n \in \mathbb{C}$. *Then the number*

$$s = \lim s_n$$

is called a **sum of the series** *(6.1) and* [1]

$$s = \sum_{n=1}^{\infty} z_n.$$

A series which is not convergent is called **divergent**.

Theorem 6.1 *Let* $\sum\limits_{n=1}^{\infty} z_n$ *be a series. Then*[2]

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[1]Notice that in the following the notation $\sum\limits_{n=1}^{\infty} z_n$ is used for a series and also for its sum, which is a number.

[2]See [3].

(i) *convergence neccesary condition*

$$\sum_{n=1}^{\infty} z_n \textit{ is convergent} \quad \Rightarrow \ \lim z_n = 0.$$

(ii) *relation between convergent series and convergent real and imaginary parts*

$$\sum_{n=1}^{\infty}(x_n + iy_n) \textit{ is convergent} \quad \Leftrightarrow \quad \textit{series} \ \sum_{n=1}^{\infty} x_n \textit{ and } \sum_{n=1}^{\infty} y_n \textit{ are convergent.}$$

Moreover, if a series $\sum_{n=1}^{\infty}(x_n + iy_n)$ *is convergent, then its sum is given as*

$$\sum_{n=1}^{\infty}(x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n.$$

(iii) *Bolzano–Cauchy's condition*

$$\sum_{n=1}^{\infty} z_n \textit{ is convergent} \quad \Leftrightarrow \quad \left(\forall \varepsilon \in \mathbb{R}^+\right)\left(\exists n_0 \in \mathbb{N}\right)\left(\forall n, m \in \mathbb{N};\ n, m > n_0\right) : \ |s_n - s_m| < \varepsilon \ \left(s_n = \sum_{k=1}^{n} z_k\right).$$

(iv) *absolute convergence test*

$$\sum_{n=1}^{\infty} |z_n| \textit{ is convergent} \quad \Rightarrow \ \sum_{n=1}^{\infty} z_n \textit{ is convergent .}$$

(A **series** $\sum_{n=1}^{\infty} z_n$ *is called* **absolutely convergent** *if a series* $\sum_{n=1}^{\infty} |z_n|$ *is convergent. A series which is convergent but not absolutely convergent is called a* **conditionally convergent series**.*)*

(v) *comparison test*

$$\left.\begin{array}{l} \forall n \in \mathbb{N} : \ |z_n| \le a_n \\ \sum_{n=1}^{\infty} a_n \textit{ is convergent} \end{array}\right\} \Rightarrow \ \sum_{n=1}^{\infty} z_n \textit{ is absolutely convergent.}$$

(vi) *d'Alembert's ratio test*

$$\lim\left|\frac{z_{n+1}}{z_n}\right| < 1 \quad\Rightarrow\quad \sum_{n=1}^{\infty} z_n \textit{ is absolutely convergent,}$$

$$\lim\left|\frac{z_{n+1}}{z_n}\right| > 1 \quad\Rightarrow\quad \sum_{n=1}^{\infty} z_n \textit{ is divergent.}$$

(vii) *Cauchy's criterion, also known as the n-th root test*

$$\lim \sqrt[n]{|z_n|} < 1 \quad\Rightarrow\quad \sum_{n=1}^{\infty} z_n \textit{ is absolutely convergent,}$$

$$\lim \sqrt[n]{|z_n|} > 1 \quad\Rightarrow\quad \sum_{n=1}^{\infty} z_n \textit{ is divergent.}$$

(viii) *integral test*
Let $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative, monotone, decreasing, and continuous function defined on* $[1, +\infty)$ *and let* $|z_n| = f(n)$ *for all* $n \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} |z_n| < +\infty \quad\Leftrightarrow\quad \int_1^{+\infty} f(x)\,\mathrm{d}x < +\infty\,.$$

(ix) *Leibniz criterion*

$$\left.\begin{aligned} \forall n \in \mathbb{N} :\ 0 \le z_{n+1} \le z_n \\ \lim z_n = 0 \end{aligned}\right\} \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1} z_n \textit{ is convergent.}$$

(x) *convergence of a geometric series theorem*
A series $\sum\limits_{n=1}^{\infty} q^{n-1}$, *where* $q \in \mathbb{C}$, *is convergent if and only if* $|q| < 1$. *In that case it holds that*

$$\sum_{n=1}^{\infty} q^{n-1} = \frac{1}{1-q}\,.$$

Example 6.1

1. *A series*

$$\sum_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n$$

is absolutely convergent since

$$\left|\frac{\frac{n+1}{3^{n+1}}(1+i)^{n+1}}{\frac{n}{3^n}(1+i)^n}\right| = \left|\frac{1}{3}\,\frac{n+1}{n}\,(1+i)\right| \to \frac{\sqrt{2}}{3} < 1.$$

2. *A series*

$$\sum_{n=1}^{\infty} \frac{e^{i\frac{\pi}{n}}}{\sqrt{n}}$$

is divergent because for all $n \in \mathbb{N}$ we have $\dfrac{e^{i\frac{\pi}{n}}}{\sqrt{n}} = \dfrac{1}{\sqrt{n}} \cos\dfrac{\pi}{n} + i\dfrac{1}{\sqrt{n}} \sin\dfrac{\pi}{n}$ *and at the same time a series* $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}} \cos\dfrac{\pi}{n}$ *is divergent.* [3]

6.2 Sequence of functions, pointwise and uniform convergence

Definition 6.2 *We say that a sequence of functions (f_n)* **converges pointwise on the set** $\Omega \subset \mathbb{C}_\infty$ *to the function f if for all $z \in \Omega$*

$$\lim f_n(z) = f(z).$$

In other words, if

$$(\forall z \in \Omega)\left(\forall \varepsilon \in \mathbb{R}^+\right)\left(\exists n_0 \in \mathbb{N}\right)\left(\forall n \in \mathbb{N},\ n \geq n_0\right) :\ f_n(z) \in U(f(z), \varepsilon).$$

It is denoted by $f_n \to f$ on Ω.

The natural number n_0 mentioned above depends on the choice $z \in \Omega$ and $\varepsilon \in \mathbb{R}^+$. If it is possible to choose n_0 independently on $z \in \Omega$ and functions f_n and f are finite then one observes uniform convergence on Ω.

Definition 6.3 *Assume that f_n and f are finite functions defined on a set $\Omega \subset \mathbb{C}_\infty$ for all $n \in \mathbb{N}$. We say that a sequence of functions (f_n)* **converges uniformly on the set** Ω *to the function f if*

$$\lim \left[\sup_{z\in\Omega} |f_n(z) - f(z)|\right] = 0.$$

In other words,

$$\left(\forall \varepsilon \in \mathbb{R}^+\right)\left(\exists n_0 \in \mathbb{N}\right)\left(\forall n \in \mathbb{N},\ n \geq n_0\right)(\forall z \in \Omega) :\ f_n(z) \in U(f(z), \varepsilon).$$

It is denoted by $f_n \rightrightarrows f$ on Ω.

Theorem 6.2 *Let $f_n \rightrightarrows f$ on Ω and let f_n be a continuous function on Ω for all $n \in \mathbb{N}$. Then a function f is continuous on Ω.*

[3] Think over it carefully!

Definition 6.4 *Assume that functions f_n and f are finite and defined on a set $\Omega \subset \mathbb{C}_\infty$ for all $n \in \mathbb{N}$. We say that a* **function series**

$$f_1(z) + f_2(z) + \cdots + f_n(z) + \ldots \quad = \quad \sum_{n=1}^{\infty} f_n(z) \tag{6.2}$$

converges pointwise *(resp.* **converges uniformly) on a set** Ω *to the sum f if a* **sequence** (s_n) **of partial sum of function series** *(6.2) converges*[4] *pointwise (resp. uniformly) on Ω to the function f.*

Theorem 6.3 (Weierstrass) *Assume that for all $n \in \mathbb{N}$ a function f_n is holomorphic on $\Omega \subset \mathbb{C}$ and that a function series $\sum\limits_{n=1}^{\infty} f_n(z)$ converges locally uniformly on Ω, i.e.,*

$$(\forall z \in \Omega)\,(\exists U(z) \subset \Omega) : \sum_{n=1}^{\infty} f_n(z) \text{ converges uniformly on } U(z).$$

Then a function f defined by

$$f(z) = \sum_{n=1}^{\infty} f_n(z)$$

is holomorphic in the domain Ω and for all $p \in \mathbb{N}$ and $z \in \Omega$ it holds that

$$f^{(p)}(z) = \sum_{n=1}^{\infty} f_n^{(p)}(z).$$

Moreover, if γ is a piece-wise smooth curve in Ω, then[5]

$$\int_\gamma f(z)\mathrm{d}z = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\mathrm{d}z.$$

---

[4] $s_n(z) = \sum\limits_{k=1}^{n} f_k(z)$.

[5] Symbolically written:

$$\left(\sum \cdots\right)' = \sum (\ldots)', \quad \int \left(\sum \cdots\right) = \sum \left(\int \cdots\right).$$

Chapter 7

Power and Taylor series

7.1 Power series

Definition 7.1 *A* **power series with a centre** $z_0 \in \mathbb{C}$ *is defined as a functional series*

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad = \quad \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{7.1}$$

where $a_n \in \mathbb{C}$ *for every* $n \in \mathbb{N} \cup \{0\}$.

Let us focus on a convergence of the series (7.1). In other words, let us explore for which $z \in \mathbb{C}$ that series is convergent. Obviously, series (7.1) is convergent for $z = z_0$, i.e., in its centre, and the sum is a_0 at this point. Now, assume that the series (7.1) is convergent at the point $z_1 \neq z_0$ and that $z \in \mathbb{C}$ is such a point that $|z - z_0| < |z_1 - z_0|$. Then for all $n \in \mathbb{N}$ it holds that

$$\left|a_n(z - z_0)^n\right| \quad = \quad \left|a_n(z_1 - z_0)^n\right| \left|\frac{z - z_0}{z_1 - z_0}\right|^n. \tag{7.2}$$

Now let us apply Theorem 6.1. From the assumption that the series $\sum\limits_{n=0}^{\infty} a_n(z_1 - z_0)^n$ is convergent, it follows that

$$\lim \left(a_n(z_1 - z_0)^n\right) = 0.$$

Therefore there exists $k \in \mathbb{R}^+$ such that $\left|a_n(z_1 - z_0)^n\right| \leq k$ for all $n \in \mathbb{N}$. Moreover, since $\left|(z - z_0)(z_1 - z_0)\right| < 1$ it follows that a geometric series

$$\sum_{n=0}^{\infty} k \left|\frac{z - z_0}{z_1 - z_0}\right|^n$$

is convergent, and hence from (7.2) one can see that $\sum\limits_{n=0}^{\infty} a_n(z - z_0)^n$ is absolutely convergent. This can be generalized as follows.

Theorem 7.1 (Abel's) *Let* $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ *be a series convergent at a point* $z_1 \neq z_0$. *Then this series is absolutely convergent and locally uniformly convergent in* $U\left(z_0, |z_1 - z_0|\right)$.

Corollary 1 *If a power series* $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ *is divergent at a point* $z_2 \in \mathbb{C}$, *then it is divergent at each point of the set*

$$\{z \in \mathbb{C} : \ |z - z_0| > |z_2 - z_0|\}.$$

Theorem 7.2 *For any power series (7.1) with a centre* z_0 *there exists exactly one number* $R \in [0, +\infty) \cup \{+\infty\}$, *which is called a* **radius of convergence of the power series** *(7.1) , such that*

(i) if $|z - z_0| < R$, *then series (7.1) is absolutely convergent,*

(ii) if $|z - z_0| > R$, *then series (7.1) is divergent.*

Proof Let us define

$$R = \sup\left\{ |z - z_0| \ : \ z \in \mathbb{C} \ \wedge \ \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ is convergent} \right\}.$$

Now, one can easily prove the above assertions using Theorem 7.1. □

Definition 7.2 *Assume that a radius of convergence* R *of a power series (7.1) satisfies* $0 < R < +\infty$. *A set* $U(z_0, R)$ *is called a* **disk of the convergence of the power series** *(7.1). For* $R = +\infty$ *a disk of the convergence of the power series (7.1) is a set* $U(z_0, +\infty) = \mathbb{C}$.

Remark 7.1 *Assume that a radius of convergence* R *of a power series* $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ *satisfies* $0 < R < +\infty$. *Notice that in that case one can say nothing about the convergence of the power series at the points of the circle*

$$\{z \in \mathbb{C} : \ |z - z_0| = R\}.$$

One can demonstrate this situation with the power series[1]

$$\sum_{n=1}^{\infty} z^n, \ \sum_{n=1}^{\infty} \frac{z^n}{n}, \ \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

[1]Power series of the form $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ where $z_0 = 0$ and $a_0 = 0$ are considered.

Since

$$\left|\frac{z^{n+1}}{z^n}\right| \to |z|, \quad \left|\frac{\frac{z^{n+1}}{n+1}}{\frac{z^n}{n}}\right| \to |z|, \quad \left|\frac{\frac{z^{n+1}}{(n+1)^2}}{\frac{z^n}{n^2}}\right| \to |z|,$$

then (use d'Alembert's ratio test) the radius of convergence of each of these power series equals 1. Moreover,

(i) the series $\sum_{n=1}^{\infty} z^n$ *is divergent at every point of the circle* $\{z \in \mathbb{C} : |z| = 1\}$ *since any* $z \in \mathbb{C}$, $|z| = 1$ *does not fulfill the necessary convergence condition for* $\sum_{n=1}^{\infty} z^n$, *i.e., the condition* $\lim z^n = 0$,

(ii) the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ *is uniformly convergent for* $z = -1$ *(use the Leibnitz criterion) and is divergent for* $z = 1$ *(use the integral test),*

(iii) the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ *is absolutely convergent for all* $z \in \mathbb{C}$, $|z| = 1$ *(use the integral test).*

Theorem 7.3 *Assume that*

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = L, \quad \textit{resp.} \quad \lim \sqrt[n]{|a_n|} = K.$$

Then for a radius of convergence R *of a power series* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *it holds*

$$R = \begin{cases} \frac{1}{L} \text{ if } L \in \mathbb{R}^+, \\ 0 \text{ if } L = +\infty, \\ +\infty \text{ if } L = 0, \end{cases} \quad \textit{resp.} \quad R = \begin{cases} \frac{1}{K} \text{ if } K \in \mathbb{R}^+, \\ 0 \text{ if } K = +\infty, \\ +\infty \text{ if } K = 0. \end{cases}$$

Proof Notice that for $z \neq z_0$ we have

$$\lim \left|\frac{a_{n+1}(z - z_0)^{n+1}}{a_n(z - z_0)^n}\right| = L|z - z_0|, \quad \text{resp.} \quad \lim \sqrt[n]{|a_n(z - z_0)^n|} = K|z - z_0|.$$

Then it is enough to use d'Alembert's ratio test, resp. Cauchy's criterion. □

Example 7.1 *Find a* **domain of convergence of a power series** [2]

$$\sum_{n=0}^{\infty} \frac{n}{2^n}\, z^n.$$

Solution:

$$\lim \sqrt[n]{\frac{n}{2^n}} = \lim \frac{\sqrt[n]{n}}{2} = \frac{1}{2},$$

hence $R = 2$. *The series is absolutely convergent for every* $z \in U(0,2)$ *and is divergent for every* $z \in \mathbb{C},\ |z| > 2$.

For $|z| = 2$ *it holds*

$$\lim \left|\frac{n}{2^n} z^n\right| = \lim n = \infty \neq 0,$$

therefore the series $\sum\limits_{n=0}^{\infty} \frac{n}{2^n}\, z^n$ *is divergent (the necessary convergence condition is not fulfilled).*

Example 7.2 *Find a radius of convergence of a power series*

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\, z^n.$$

Solution:

$$\frac{\dfrac{(2(n+1))!}{((n+1)!)^2}}{\dfrac{(2n)!}{(n!)^2}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \to 4,$$

hence $R = 1/4$.

Theorem 7.4 *Assume that a power series* $\sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ *has a radius of convergence* $R > 0$. *Then a function* f *defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is holomorphic in the domain $U(z_0, R)$. *Moreover, for all* $p \in \mathbb{N}$ *and* $z \in U(z_0, R)$ *the following equality holds*

$$f^{(p)}(z) = \sum_{n=p}^{\infty} n\,(n-1)\,\cdots\,(n-p+1)\,a_n\,(z-z_0)^{n-p}$$

[2] It means, find a set of all $z \in \mathbb{C}$ for which the series converges.

and the power series $\sum_{n=p}^{\infty} n\,(n-1)\,\cdots\,(n-p+1)\,a_n\,(z-z_0)^{n-p}$ *has the same radius of convergence* R.

Proof The above theorem follows immediately from Weierstrass (see Theorem 6.3) and Abel's theorems (see Theorem 7.1). □

Example 7.3 *Find a sum of a power series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{z^n}{n}$$

in a disk of convergence.
Solution: Since

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} \to 1,$$

a domain $U(0,1)$ *is a disk of convergence of the above power series. Let us define a function* f *by*

$$f(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{z^n}{n}.$$

Then for all $z \in \mathbb{C},\ |z| < 1$ *it holds that*

$$f'(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\,z^{n-1} = \sum_{n=1}^{\infty}(-z)^{n-1} = \frac{1}{1-(-z)} = \frac{1}{1+z}.$$

Therefore there exists $c \in \mathbb{C}$ *such that for any* $z \in U(0,1)$ *we have*

$$f(z) = \ln(1+z) + c.$$

Obviously,

$$0 = f(0) = \ln 1 + c = c$$

then for all $z \in U(0,1)$ *holds*

$$f(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{z^n}{n} = \ln(1+z).$$

Theorem 7.5 (Abel's) *Assume that a power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has a radius of convergence* $R \in (0,+\infty)$ *and that this power series is convergent at a point*

$$z_1 = z_0 + Re^{i\varphi},\ \text{where } \varphi \in \mathbb{R}.$$

Then a function f defined by

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

is continuous on an interval $[z_0,z_1]$, i.e., on a set

$$\left\{z_0+re^{i\varphi}\ :\ r\in[0,R]\right\}=\left\{z_0+(z_1-z_0)\,t\ :\ t\in[0,1]\right\}.$$

Specially:

$$f(z_1)=f\left(z_0+Re^{i\varphi}\right)=\lim_{r\to R-}f\left(z_0+re^{i\varphi}\right)=\lim_{t\to 1-}f\left(z_0+(z_1-z_0)\,t\right).$$

Example 7.4 *Find a sum of a series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{n}.$$

Solution: First of all, let us notice that the above series is convergent.[3] *Now, let f be a function defined by*

$$f(z)=\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{z^n}{n}.$$

From Theorem 7.5 and the above example it follows that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{1}{n}=f(1)=\lim_{z\to 1-}f(z)=\lim_{z\to 1-}(\ln(1+z))=\ln 2.$$

7.2 Taylor series

As mentioned above, a sum of a power series (in a disk of convergence) is a holomorphic function. The following theorem outlines that every holomorphic function is (at least locally) a sum of a certain power series.

Theorem 7.6 (Taylor's expansion of holomorphic function) *Let f be a holomorphic function on $U(z_0,R)$ where $z_0\in\mathbb{C}$ and $R\in(0,+\infty)\cup\{+\infty\}$. Then there exists exactly one power series $\sum\limits_{n=0}^{\infty}a_n(z-z_0)^n$ such that for all $z\in U(z_0,R)$ it holds that*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n.$$

[3] Use the Leibniz criterion.

Moreover, if ϱ denotes a real number such that $0 < \varrho < R$, *then the coefficients of the above* **Taylor series** *satisfy*

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z,$$

where

$$\gamma(t) = z_0 + \varrho\, e^{it},\ t \in [0, 2\pi].$$

Remark 7.2 *If f denotes a holomorphic function in $\mathbb{C}$, then a radius of convergence of its Taylor series (with a centre at any point $z_0 \in \mathbb{C}$) equals $+\infty$. Examples of such functions and their Taylor series with a centre at* 0 *are*

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},\quad \sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},\quad \cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}.$$

Example 7.5 *Find a Taylor series of a function f with a centre z_0 if*

a) $f(z) = \dfrac{1}{3-z},\ z_0 = 0,$

b) $f(z) = \dfrac{1}{3-z},\ z_0 = -1+3i,$

c) $f(z) = \ln z,\ z_0 = 2.$

Solution: *Ad a) Notice that a function f is holomorphic on $U(0,3)$. Using the theorem about convergence of a geometric series is a crucial tool to find its Taylor series:* [4]

$$\forall z \in U(0,3):\ f(z) = \frac{1}{3-z} = \frac{1}{3}\frac{1}{1-\frac{z}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}.$$

Ad b) Let us apply the same process as above. For all

$$z \in U\left(-1+3i, |3-(-1+3i)|\right) = U(-1+3i, 5)$$

it holds that

$$f(z) = \frac{1}{3-z} = \frac{1}{4-3i-(z-(-1+3i))} = \frac{1}{4-3i}\frac{1}{1-\frac{z+1-3i}{4-3i}} = \sum_{n=0}^{\infty}\frac{(z+1-3i)^n}{(4-3i)^{n+1}}.$$

Ad c) Obviously, the function f is holomorphic on $U(2,2)$. Then all $z \in U(2,2)$ satisfy

$$f'(z) = \frac{1}{z} = \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n.$$

[4] See Theorem 6.1.

Therefore there exists $c \in \mathbb{C}$ such that for all $z \in U(2,2)$ it holds that

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \frac{(z-2)^{n+1}}{n+1} + c.$$

Since

$$f(2) = \ln 2 = c,$$

then for all $z \in U(2,2)$ it holds that

$$f(z) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n\, n} (z-2)^n.$$

Theorem 7.7 (Liouville's) *Let f be a holomorphic and bounded (i.e., there exists $M \in \mathbb{R}^+$ such that for every $z \in \mathbb{C}$ it holds $|f(z)| \le M$) function on $\mathbb{C}$. Then f is constant on $\mathbb{C}$.*

Proof It is known from Theorem 7.6 that for all $z \in \mathbb{C}$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where for all $n \in \mathbb{N} \cup \{0\}$ and all $\varrho \in (0, +\infty)$ it holds that

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \mathrm{d}z = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\varrho\, e^{it})}{(\varrho\, e^{it})^{n+1}}\, \varrho\, i e^{it}\, \mathrm{d}t, \\ \gamma(t) &= \varrho\, e^{it},\ t \in [0, 2\pi]. \end{aligned}$$

Using above given, for all $n \in \mathbb{N} \cup \{0\}$ and all $\varrho \in (0, +\infty)$ it holds

$$|a_n| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(\varrho\, e^{it})}{(\varrho\, e^{it})^n}\, i\, \mathrm{d}t \right| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{\varrho^n}\, \mathrm{d}t = \frac{M}{\varrho^n}.$$

Now, since one could choose a constant $\varrho \in \mathbb{R}^+$ arbitrarily small, then it follows from theestimation $|a_n| \le M/\varrho^n$ that for any $n \in \mathbb{N}$ it holds $a_n = 0$. Therefore for every $z \in \mathbb{C}$ is $f(z) = a_0$ and the function f is constant. □

Theorem 7.8 (Fundamental theorem of algebra) *Every positive degree polynomial has at least one root in $\mathbb{C}$. In other words, let $f : \mathbb{C} \to \mathbb{C}$ denote a function defined by*

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where

$$n \in \mathbb{N},\ a_0, a_1, \ldots, a_n \in \mathbb{C},\ a_n \ne 0.$$

Then there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

Proof Let assume that for all $z \in \mathbb{C}$ it holds

$$f(z) \neq 0$$

and let F denote a function given by

$$F(z) = \frac{1}{f(z)}.$$

Then obviously

1. F is holomorphic on $\mathbb{C}$, i.e., for all $z \in \mathbb{C}$ it holds

$$F'(z) = -\frac{f'(z)}{f^2(z)},$$

2. F is bounded on $\mathbb{C}$, i.e.,

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{1}{z^n\,(a_n + a_{n-1}\frac{1}{z} + \cdots + a_0\frac{1}{z^n})} = \frac{1}{\infty\,(a_n)} = \frac{1}{\infty} = 0.$$

Therefore the function F on $\mathbb{C}$ is constant (see Theorem 7.7). This is a contradiction with a definition of F. □

Definition 7.3 *Let f denote a holomorphic function at a point $z_0 \in \mathbb{C}$ and let $p \in \mathbb{N}$. The point z_0 is called a p* **– times root** *(or p* **– times zero point***)* **of a function** *f if*

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(p-1)}(z_0) = 0 \neq f^{(p)}(z_0).$$

Theorem 7.9 *Let f denote a holomorphic function at a point $z_0 \in \mathbb{C}$ and let $f(z_0) = 0$. Then there exists $U(z_0)$ such that only exactly one of the following properties holds*

(i) f is a zero function on $U(z_0)$,

(ii) $f(z) \neq 0$ for all $z \in U(z_0) \setminus \{z_0\}$.

Proof Obviously, a function f equals to a sum of its Taylor series (with a centre at z_0) on some neighbourhood of z_0. In case that this series does not equal zero (i.e., f is not a zero function on any neighbourhood of z_0), then there exists $p \in \mathbb{N}$ such that z_0 is a p – times root of the function f. In other words, on some neigbourhood of the point z_0 it holds that

$$f(z) = \sum_{n=p}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n = (z-z_0)^p \sum_{n=p}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^{n-p} = (z-z_0)^p\,\varphi(z),$$

where a function
$$\varphi(z) = \sum_{n=p}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-p}$$
is holomorphic (and continuous) and non-zero at z_0.[5] Therefore, there exists $U(z_0)$ such that the function φ is non-zero in $U(z_0)$ and then for all $z \in U(z_0) \setminus \{z_0\}$ we have
$$f(z) = (z - z_0)^p \varphi(z) \neq 0.$$
□

Let us introduce an important corollary of Theorem 7.9.

Theorem 7.10 *Let f and g denote holomorphic functions in a domain $\Omega \subset \mathbb{C}$ and let γ denote a simple curve in Ω for which $f = g$ on $\langle\gamma\rangle$. Then $f = g$ on Ω.*

Exercise 7.1 *Using Theorem 7.10 prove that for all $z \in \mathbb{C}$ it holds that:*

1. $\sin^2 z + \cos^2 z = 1,$
2. $\sin(2z) = 2 \sin z \cos z,$
3. $\cos^2 z = \dfrac{1 + \cos(2z)}{2}, \ \sin^2 z = \dfrac{1 - \cos(2z)}{2},$
4. $\operatorname{Re} z > 0 \ \Rightarrow \ \ln(z^2) = 2 \ln z.$

[5] $\varphi(z_0) = \frac{f^{(p)}(z_0)}{p!} \neq 0$

Chapter 8

Laurent series and classification of singularities

8.1 Laurent series

Definition 8.1 *A* **Laurent series with a centre at a point** $z_0 \in \mathbb{C}$ *is defined by*

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \tag{8.1}$$

where $a_n \in \mathbb{C}$ *for all* $n \in \{\dots,-3,-2,-1,0,1,2,3,\dots\}$. *A power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is called a **regular part of a Laurent series** *(8.1), a function series*

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} = \sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-z_0)^n}$$

is called a **principal part of a Laurent series** *(8.1). We say that a* **Laurent series** *(8.1)* **is convergent on a set** $\Omega \subset \mathbb{C}$ *if their regular and principal part converge on* Ω. *Then a function* f *defined by* $f(z) = f_1(z) + f_2(z)$ *on* Ω, *where* f_1*(resp.* f_2*) is a sum of the regular (resp. principal) part of Laurent series (8.1), is called a* **sum of Laurent series** *(8.1).*

Now, let us focus on convergence of Laurent series (8.1). First of all, let us investigate a convergence of its principal part. If

$$\xi = \frac{1}{z-z_0}$$

then

$$\sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-z_0)^n} = \sum_{n=1}^{\infty} a_{-n}\xi^n,$$

where on the right hand side there is a power series with a centre at 0 (in ξ). Let us denote by ρ its radius of convergence. Then it holds that [1]

(i) if $|\xi| < \rho$, then the series $\sum_{n=1}^{\infty} a_{-n}\xi^n$ is absolutely convergent,

(ii) if $|\xi| > \rho$, then the series $\sum_{n=1}^{\infty} a_{-n}\xi^n$ is divergent.

If we define a number r as

$$r = \begin{cases} \frac{1}{\rho} \text{ if } 0 < \rho < +\infty, \\ 0 \text{ if } \rho = +\infty, \\ +\infty \text{ if } \rho = 0, \end{cases}$$

then from the above it follows that

(i) if $|z - z_0| > r$, then the series $\sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-z_0)^n}$ is absolutely convergent,

(ii) if $|z - z_0| < r$, then the series $\sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-z_0)^n}$ is divergent.

Now, let us denote by R a radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

i.e., the regular part of Laurent series (8.1). Then exactly one possibility may be fulfilled

$$r < R, \quad r = R, \text{ or } \quad r > R.$$

1. If $r < R$, then Laurent series (8.1) is absolutely convergent (and locally uniformly) on an **annulus**

$$P(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

and is divergent at every point of the set

$$\{z \in \mathbb{C} : |z - z_0| < r \text{ or } |z - z_0| > R\}.$$

Moreover, one can demonstrate that a sum f of Laurent series (8.1) is a holomorphic function on $P(z_0, r, R)$ and that for all $p \in \mathbb{N}$ and $z \in P(z_0, r, R)$ it holds that

$$f^{(p)}(z) = \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{d}^p\left((z - z_0)^n\right)}{\mathrm{d}z^p}.$$

[1] See Theorem 7.2.

2. In the case that $r = R$ Laurent series (8.1) is divergent in every point of the set
$$\{z \in \mathbb{C} : \ |z - z_0| \neq r = R\}.$$

3. Finally, if $r > R$, then there does not exist any $z \in \mathbb{C}$ for which Laurent series (8.1) is convergent.

Similarly as in Section 7.1 it has been shown that a sum of Laurent series (under the assumption $r < R$) is a holomorphic function on annulus $P(z_0, r, R)$. Vice versa, the following theorem states that every holomorphic function on annulus $P(z_0, r, R)$ is a sum of some Laurent series.

Theorem 8.1 (holomorphic function expansion in Laurent series) *Let f be a holomorphic function on $P(z_0, r, R)$, where $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq +\infty$. Then there exists exactly one Laurent series $\sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$ such that for every $z \in P(z_0, r, R)$ it holds that*
$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$
Moreover, if ϱ denotes some real number such that $r < \varrho < R$, then the coefficients of the above mentioned Laurent series (which is called a **Laurent expansion of a function** *f) are given by*
$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z,$$
where
$$\begin{aligned} \gamma(t) &= z_0 + \varrho\, e^{it},\ t \in [0, 2\pi] \\ n &\in \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \end{aligned}$$

Example 8.1 *Find a Laurent expansion of a function*
$$f(z) = \frac{1}{(z - 1)(z - 2)}$$
on every annulus with a centre at $z_0 = 0$ on which f is holomorphic.
Solution: Obviously, one has to find Laurent series of f on the annuli
$$P(0, 0, 1), \quad P(0, 1, 2), \quad P(0, 2, +\infty).$$
First of all, notice that for all $z \in \mathbb{C} \setminus \{1, 2\}$ it holds
$$f(z) = \frac{1}{z - 2} - \frac{1}{z - 1}.$$
Now, let us deal individually with each annulus.

1. *From the implications*

$$|z| < 2 \quad \Rightarrow \quad \frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}\, z^n,$$
$$|z| < 1 \quad \Rightarrow \quad -\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

one obtains $z \in P(0,0,1) = \{z \in \mathbb{C} : \; 0 < |z| < 1\}$

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n.$$

Notice that we have found a Taylor series.

2. *As we already know from the above part, for every* $z \in \mathbb{C}$ *such that* $1 < |z| < 2$ *it holds that*

$$\frac{1}{z-2} = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}\, z^n.$$

Additionally, it holds that

$$|z| > 1 \quad \Rightarrow \quad -\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} -\frac{1}{z^{n+1}}$$

therefore for every $z \in P(0,1,2) = \{z \in \mathbb{C} : \; 1 < |z| < 2\}$ *we get*

$$f(z) = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}}\, z^n + \sum_{n=1}^{\infty} -\frac{1}{z^n}.$$

3. *From the implication in the above part one obtains*

$$|z| > 2 \quad \Rightarrow \quad \frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}.$$

Therefore for all $z \in P(0,2,+\infty) = \{z \in \mathbb{C} : \; 2 < |z|\}$ *it holds that*

$$f(z) = \sum_{n=1}^{\infty}\left(2^{n-1} - 1\right)\frac{1}{z^n}.$$

Exercise 8.1 *Find a Laurent expansion of a function*

$$f(z) = \frac{1}{(z-1)(z-2)}$$

on each annulus with a centre at $z_0 = 1$ *on which* f *is holomorphic.*

8.2 Isolated singularities and their classification

Definition 8.2 *A point $z_0 \in \mathbb{C}$ is called an* **isolated singularity of a function** f *if*

(i) function f is not holomorphic at z_0,

(ii) there exists a ring neighbourhood $P(z_0)$ on which f is holomorphic.

If z_0 is an isolated singularity of a function f, then there exists a number $R \in \mathbb{R}^+$ such that f is holomorphic on $P(z_0, R) = P(z_0, 0, R)$ and therefore[2] *for all $z \in P(z_0, R)$ it holds that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Depending on a number of non-zero coefficients of a principal part of this Laurent series one can determine three cases:

(i) every coefficient of a principal part equals zero (i.e., $a_{-n} = 0$ for all $n \in \mathbb{N}$),

(ii) there exists at least one and at most finitely many non-zero coefficients of a principal part (i.e., there exists $n \in \mathbb{N}$ such that $a_{-n} \neq 0$ and for all $k \in \mathbb{N}$, $k > n$ is $a_{-k} = 0$),

(iii) there exist infinitely many non-zero coefficients of a principal part.

In the first case (i) we call a point z_0 a **removable singularity of a function** f*, in the second case (ii) a point z_0 is called a* **pole of order** n **of a function** f*,*[3] *and in the third case (iii) a point z_0 is called an* **essential singularity of a function** f.

Theorem 8.2 *Let $z_0 \in \mathbb{C}$ be an isolated singularity of a function f. Then*

(i) z_0 is a removable singularity of a function f if and only if

$$\lim_{z \to z_0} f(z) \in \mathbb{C},$$

(ii) z_0 is a pole of a fuction f (resp. pole of order n of a function f) if and only if

$$\lim_{z \to z_0} f(z) = \infty$$

$$\left(\textit{resp. if } \lim_{z \to z_0} \left[(z - z_0)^n f(z) \right] \in \mathbb{C} \setminus \{0\} \right),$$

(iii) z_0 is an essential singularity of a function f if and only if $\lim\limits_{z \to z_0} f(z)$ does not exist.

[2]See Theorem 8.1.

[3]Pole of order 1 is also called a **simple pole.**

Theorem 8.3 (great Picard's) *Assume that $z_0 \in \mathbb{C}$ is an essential singularity of a function f. Then f takes on any ring neighborhood of z_0 all possible complex values with at most a single exception. In other words, for all $P(z_0)$ there exists $z \in \mathbb{C}$ such that*

$$\mathbb{C} \setminus \{z\} \subset f\left(P(z_0)\right).$$

8.3 Laurent series with a centre ∞, classification of a point ∞

Definition 8.3 *A* **Laurent series with a centre** ∞ *is given by*

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}, \tag{8.2}$$

where $a_n \in \mathbb{C}$ for every $n \in \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

A power series

$$\sum_{n=1}^{\infty} a_{-n} z^n$$

is called a **principal part of Laurent series** *(8.2), a function series*

$$\sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

is called a **regular part of Laurent series** *(8.2).* [4]

Similarly as in Laurent series with a centre at $z_0 \in \mathbb{C}$ one can introduce a term of convergence of a Laurent series with a centre ∞ and its sum. One can analogously introduces an annulus of convergence, it this case of the form

$$P(\infty, r, R) = \left\{ z \in \mathbb{C} : \ \frac{1}{R} < |z| < \frac{1}{r} \right\}.$$

Again, it holds that if Laurent series (8.2) is convergent on annulus $P(\infty, r, R) \neq \emptyset$, then a sum of the series on $P(\infty, r, R)$ is a holomorphic function. It also holds as an analogy of Theorem 8.1.

Theorem 8.4 *Let f be a holomorphic function on $P(\infty, r, R) \neq \emptyset$. Then there exists exactly one Laurent series $\sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}$ such that for every $z \in P(\infty, r, R)$ it holds that*

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}.$$

[4] Notice that formally there is no difference between Laurent series with the centre 0 and Laurent series with the centre ∞. If one needs to distinguish between these two cases it is necessary to mark out the centre of the series or to establish its principal resp. regular part.

Moreover, if ϱ is some real number such that

$$\frac{1}{R} < \varrho < \frac{1}{r},$$

it holds for coefficients that the above mentioned Laurent series

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{-n+1}} \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma f(z) z^{n-1} \mathrm{d}z,$$

where

$$\begin{aligned} \gamma(t) &= \varrho e^{it},\ t \in [0, 2\pi], \\ n &\in \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \end{aligned}$$

Definition 8.4 *We call ∞ an* **isolated singularity of a function** *f if there exists $P(\infty)$ on which f is holomorphic.*

If ∞ is an isolated singularity of a function f, then it is possible to expand f on some $P(\infty)$ in a Laurent series with a centre ∞, i.e., for all $z \in P(\infty)$

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}.$$

Similarly as for finite isolated singularities, one can classify a point ∞ according to the number of non-zero coefficients of a principal part of this Laurent series. An analogy of Theorem 8.2 remains true.

Theorem 8.5 *Let ∞ be an isolated singularity of a function f. Then it holds that*

(i) ∞ is a removable singularity of a function f if and only if

$$\lim_{z\to\infty} f(z) \in \mathbb{C},$$

(ii) ∞ is a pole of a function f (resp. a pole of order n of a function f) if and only if

$$\lim_{z\to\infty} f(z) = \infty$$

$$\left(\textit{resp. if } \lim_{z\to\infty} \frac{f(z)}{z^n} \in \mathbb{C} \setminus \{0\}\right),$$

(iii) ∞ is an essential singularity of a function f if and only if $\lim\limits_{z\to\infty} f(z)$ does not exist.

Chapter 9

Residue and the residue theorem

9.1 Residue of a function and it's calculation

Definition 9.1 *Let $z_0 \in \mathbb{C}$ (resp. ∞) be an isolated singularity of a function f and let $\sum\limits_{n=-\infty}^{\infty} a_n (z - z_0)^n$ (resp. $\sum\limits_{n=-\infty}^{\infty} a_n / z^n$) be a Laurent series of the function f on some ring neighbourhood of a point z_0 (resp. ∞). A number a_{-1} (resp. $-a_1$) is called a* **residue of a function** f **at the point** z_0 *(resp. ∞), it is denoted by res $f(z_0)$ (resp. res $f(\infty)$).* [1]

One could naturally ask why we call a number a_{-1} (resp. $-a_1$) a residue of a function. To answer this question it is enough to realize that from the above definition it follows that [2]

$$\begin{aligned} \operatorname{res} f(z_0) &= \frac{1}{2\pi i}\int_\gamma f(z)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \left(\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n\right)\mathrm{d}z \\ \text{resp.}\quad \operatorname{res} f(\infty) &= -\frac{1}{2\pi i}\int_\gamma f(z)\mathrm{d}z = -\frac{1}{2\pi i}\int_\gamma \left(\sum_{n=-\infty}^{\infty} \frac{a_n}{z^n}\right)\mathrm{d}z. \end{aligned}$$

Theorem 9.1 *It holds*

(i) if a point $z_0 \in \mathbb{C}$ is a removable singularity of a function f, then[3] *res $f(z_0) = 0$,*

(ii) if f is a holomorphic function at a point $z_0 \in \mathbb{C}$ and if a function g has a simple pole at z_0, then

$$\operatorname*{res}_{z=z_0} (f(z)g(z)) = f(z_0) \operatorname*{res}_{z=z_0} g(z),$$

[1] Notation $\operatorname*{res}_{z=z_0} f(z)$ (resp. $\operatorname*{res}_{z=\infty} f(z)$) will be used sometimes.

[2] See Theorem 8.1 (resp. Theorem 8.4).

[3] Warning! Let us assume a function $f(z) = 1/z$. Then ∞ is a removable singularity of a function f but despite this it holds that res $f(\infty) = -1 \neq 0$.

(iii) if f and g are holomorphic functions at a point $z_0 \in \mathbb{C}$ and if the point z_0 is a simple root of a function g,[4] then

$$\operatorname*{res}_{z=z_0} \left(\frac{f(z)}{g(z)} \right) = \frac{f(z_0)}{g'(z_0)},$$

(iv) if $z_0 \in \mathbb{C}$ (resp. ∞) is a pole of k-order of a function f, then

$$\operatorname*{res}_{z=z_0} f(z) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left(f(z)(z-z_0)^k \right) \right),$$

resp.

$$\operatorname*{res}_{z=\infty} f(z) = \frac{(-1)^k}{(k+1)!} \lim_{z \to \infty} \left(z^{k+2} \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} f(z) \right),$$

(v) if f is a holomorphic function on $\mathbb{C} \setminus \{z_1, z_2, \dots, z_n\}$ where $z_1, z_2, \dots, z_n \in \mathbb{C}$ are different isolated singularities of a function f, then

$$\operatorname{res} f(\infty) + \sum_{i=1}^{n} \operatorname{res} f(z_i) = 0.$$

Exercise 9.1 *Prove Theorem 9.1.*

Example 9.1 *Find*

1. $\displaystyle\operatorname*{res}_{z=0} \left(z^2 \sin \frac{1}{z} \right)$,
2. $\displaystyle\operatorname*{res}_{z=\frac{\pi}{4}} \frac{z^3 \sin z}{\cos(2z)}$,
3. $\displaystyle\operatorname*{res}_{z=2\pi i} \frac{1}{(e^z - 1)^2}$.

Solution:
Ad 1. For all $z \in \mathbb{C} \setminus \{0\}$ it holds that

$$z^2 \sin \frac{1}{z} = z^2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{1}{z^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n-1}},$$

therefore

$$\operatorname*{res}_{z=0} \left(z^2 \sin \frac{1}{z} \right) = -\frac{1}{3!} = -\frac{1}{6}.$$

Ad 2. Obviously, $\pi/4$ is a simple root of a function

$$g(z) = \cos(2z),$$

[4] I.e., $g(z_0) = 0 \neq g'(z_0)$.

therefore [5]

$$\underset{z=\frac{\pi}{4}}{res}\ \frac{z^3 \sin z}{\cos(2z)} = \left[\frac{z^3 \sin z}{-2\sin(2z)}\right]_{z=\frac{\pi}{4}} = -\frac{\sqrt{2}\pi^3}{256}.$$

Ad 3. In this case, a investigated function has a pole of the second order at $2\pi i$*. Therefore* [6]

$$\underset{z=2\pi i}{res}\ \frac{1}{(e^z-1)^2} = \frac{1}{1!}\lim_{z\to 2\pi i}\left[\frac{(z-2\pi i)^2}{(e^z-1)^2}\right]' = \cdots = -1.$$

9.2 The residue theorem

Theorem 9.2 (Residue) *Assume that* $\Omega \subset \mathbb{C}$ *denotes a simply connected domain and* γ *is a simple closed piece-wise smooth positively oriented curve in* Ω*. Let* f *be a holomorphic function on* $\Omega \setminus \{z_1, z_2, \dots, z_n\}$ *where* $z_1, z_2, \dots, z_n \in \operatorname{int}\gamma$ *are different isolated singularities of a function* f*. Hence it holds that*

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{n} res\, f(z_i)\,.$$

Proof The proof is a simple corollary of the definition of residue and Theorems 5.5 and 8.1. □

Example 9.2 *Find*

$$\int_\gamma z^2 \sin\frac{1}{z+1}\,\mathrm{d}z,$$

where

$$\gamma(t) = 2e^{it},\ t \in [0, 2\pi].$$

Solution: It follows from Theorem 9.2 that

$$\int_\gamma z^2 \sin\frac{1}{z+1}\,\mathrm{d}z = 2\pi i\ \underset{z=-1}{res}\ z^2 \sin\frac{1}{z+1}.$$

Since it holds that for every $z \in \mathbb{C} \setminus \{-1\}$

$$z^2 \sin\frac{1}{z+1} = \left((z+1)^2 - 2(z+1) + 1\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\frac{1}{(z+1)^{2n+1}}$$

is [7]

$$\int_\gamma z^2 \sin\frac{1}{z+1}\,\mathrm{d}z = 2\pi i\ \underset{z=-1}{res}\ z^2 \sin\frac{1}{z+1} = 2\pi i\left(1\frac{(-1)^1}{3!} + 0 + 1\frac{(-1)^0}{1!}\right) = \frac{5}{3}\pi i.$$

[5] See Theorem 9.1 – part(iii).
[6] See Theorem 9.1 – part(iv).
[7] Think over it carefully!

9.3 Calculation integrals of real variable function with the aid of the residue theorem

a) Integrals of the form $\int_0^{2\pi} R(\sin x, \cos x)\, \mathrm{d}x$,
where $R: \mathbb{R}^2 \to \mathbb{R}$ denotes a rational function of two variables and a function $x \mapsto R(\sin x, \cos x))$ is continuous on $[0, 2\pi]$.

Let us use the substitution

$$e^{ix} = z.$$

Then one obtains

$$\sin x = \frac{z - \frac{1}{z}}{2i}, \quad \cos x = \frac{z + \frac{1}{z}}{2}, \quad \mathrm{d}z = e^{ix}\, i\, \mathrm{d}x, \text{ that is } \mathrm{d}x = \frac{1}{iz}\, \mathrm{d}z,$$

and therefore,

$$\int_0^{2\pi} R(\sin x, \cos x)\, \mathrm{d}x = \int_\gamma R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \frac{1}{iz}\, \mathrm{d}z, \tag{9.1}$$

where

$$\gamma(x) = e^{ix}, \quad x \in [0, 2\pi].$$

Then equality (9.1) we derived by formal substitution directly follows from Theorem 5.2. One can compute the integral on the right hand side of the above equality using Theorem 9.2.

Example 9.3

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}x}{\frac{5}{4} - \cos x}\, \mathrm{d}x &= \int_\gamma \frac{1}{\frac{5}{4} - \frac{z + \frac{1}{z}}{2}} \frac{1}{iz}\, \mathrm{d}z = \\
&= -\frac{2}{i} \int_\gamma \frac{\mathrm{d}z}{(z-2)(z-\frac{1}{2})} = -\frac{2}{i}\, 2\pi i \operatorname*{res}_{z=\frac{1}{2}} \left(\frac{1}{(z-2)(z-\frac{1}{2})}\right) = \\
&= -4\pi \frac{1}{\frac{1}{2} - 2} = \frac{8}{3}\pi \\
\gamma(x) &= e^{ix},\ x \in [0, 2\pi]
\end{aligned}$$

b) Integrals of the form $\int_{-\infty}^{+\infty} P(x)/Q(x)\, \mathrm{d}x$, where $P, Q: \mathbb{R} \to \mathbb{R}$ are polynomials satisfying

(i) Q does not have any real root,

(ii) the degree of the polynomial Q is at least 2 greater than the degree of the polynomial P.

From the above mentioned assumptions it coincides that

$$\lim_{k\to+\infty}\int_{\alpha_k}\frac{P(z)}{Q(z)}\,\mathrm{d}z=\lim_{k\to+\infty}\int_{-k}^{k}\frac{P(x)}{Q(x)}\,\mathrm{d}x=\int_{-\infty}^{+\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x,$$

where

$$\alpha_k(t)=t,\ t\in[-k,k]$$

and that

$$\lim_{k\to+\infty}\int_{\beta_k}\frac{P(z)}{Q(z)}\,\mathrm{d}z=0,$$

where

$$\beta_k(t)=ke^{it},\ t\in[0,\pi].$$

Therefore it holds that

$$\int_{-\infty}^{+\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x=\lim_{k\to+\infty}\int_{\gamma_k}\frac{P(z)}{Q(z)}\,\mathrm{d}z,$$

where[8]

$$\gamma_k(t)=\begin{cases}\alpha_k(t+k), & \text{je-li } t\in[-2k,0),\\[2ex] \beta_k(t), & \text{if } t\in[0,\pi].\end{cases}$$

Now, let $U(0,r)\subset\mathbb{C}$ denote a disk big enough to include all roots of the integral Q (one certainly exists!). Then it holds that for every real number $k>r$

$$\int_{\gamma_k}\frac{P(z)}{Q(z)}\,\mathrm{d}z=\int_{\gamma_r}\frac{P(z)}{Q(z)}\,\mathrm{d}z,$$

and therefore,

$$\int_{-\infty}^{+\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x=\lim_{k\to+\infty}\int_{\gamma_k}\frac{P(z)}{Q(z)}\,\mathrm{d}z=\int_{\gamma_r}\frac{P(z)}{Q(z)}\,\mathrm{d}z.$$

Now let us apply Theorem 9.2

$$\int_{-\infty}^{+\infty}\frac{P(x)}{Q(x)}\,\mathrm{d}x=\int_{\gamma_r}\frac{P(z)}{Q(z)}\,\mathrm{d}z=2\pi i\sum_{\substack{z_k\in\mathbb{C}:\\ Q(z_k)=0,\\ \operatorname{Im} z_k>0}}\operatorname*{res}_{z=z_k}\left(\frac{P(z)}{Q(z)}\right).$$

Example 9.4 *Calculate*

$$\int_{-\infty}^{+\infty}\frac{1}{(x^2+2x+2)^2}\,\mathrm{d}x.$$

[8]Draw geometrical images of curves $\alpha_k,\ \beta_k,\ \gamma_k$.

Solution:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{(x^2+2x+2)^2}\, \mathrm{d}x &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{((x-(-1+i))\,(x-(-1-i)))^2}\, \mathrm{d}x = \\
&= 2\pi i \mathop{res}_{z=-1+i} \frac{1}{((z-(-1+i))\,(z-(-1-i)))^2} = \\
&= 2\pi i \lim_{z\to -1+i} \left(\frac{1}{(z-(-1-i))^2}\right)' = \\
&= 2\pi i \left[-2\frac{1}{(z-(-1-i))^3}\right]_{z=-1+i} = \frac{\pi}{2}.
\end{aligned}$$

Chapter 10

Exercise

Exercise 10.1 *Find the real and imaginary parts of a complex number*

1. $z = (1 + i)(3 - 2i)$,
2. $z = \dfrac{2 - 3i}{3 + 4i}$,
3. $z = \dfrac{1 + i}{1 - i}$,
4. $z = 2i - \overline{\dfrac{2 - 4i}{\overline{2}}}$.

Exercise 10.2 *Find a polar form of a complex number*

1. $z = -1 + \sqrt{3}i$,
2. $z = i$,
3. $z = -8$,
4. $z = -1 - \sqrt{3}i$,
5. $z = \dfrac{2 + i}{3 - 2i}$,
6. $z = \dfrac{3 - i}{2 + i}$.

Exercise 10.3 *Prove Moivre's theorem:*

$$(\forall n \in \mathbb{N})\,(\forall \varphi \in \mathbb{R}) : \ \big(\cos\varphi + i \sin\varphi\big)^n = \cos(n\varphi) + i\sin(n\varphi)$$

using mathematical induction.

Exercise 10.4 *Let* $\varphi \in \mathbb{R}$. *Express* $\sin(4\varphi)$ *and* $\cos(4\varphi)$ *using* $\sin\varphi$ *a* $\cos\varphi$.

Exercise 10.5 *Find* $\operatorname{Re} z$ *and* $\operatorname{Im} z$ *for* $z = \left(\frac{1-i}{1+\sqrt{3}i}\right)^{24}$.

Exercise 10.6 *Find Arg z and arg z for*

1. $z = \left(\sqrt{3}+i\right)^{126}$,
2. $z = (1+i)^{137}$,
3. $z = -1-5i$.

Exercise 10.7 *Illustrate in the Gauss plane a set*

1. $\{z \in \mathbb{C}_\infty : \operatorname{Re} z \leq 1\}$,
2. $\{z \in \mathbb{C}_\infty : \operatorname{Re}(z^2) = 2\}$,
3. $\{z \in \mathbb{C}_\infty : \operatorname{Im} 1/z = 1/4\}$,
4. $\{z \in \mathbb{C}_\infty : |\operatorname{Im} z| < 1\}$,
5. $\{z \in \mathbb{C}_\infty : |z| = \operatorname{Re} z + 1\}$,
6. $\{z \in \mathbb{C}_\infty : |z-2| = |1-2\overline{z}|\}$,
7. $\{z \in \mathbb{C}_\infty : |(z-2)/(z-3)| = 1\}$,
8. $\{z \in \mathbb{C}_\infty : |1+z| < |1-z|\}$,
9. $\{z \in \mathbb{C}_\infty : |z+1| = 2|z-1|\}$,
10. $\{z \in \mathbb{C}_\infty : 2 < |z+2-3i| < 4\}$,
11. $\{z \in \mathbb{C}_\infty : \pi/4 \leq arg\,(z+2i) \leq \pi/2\}$,
12. $\{z \in \mathbb{C}_\infty : |z| + \operatorname{Re} z \leq 1 \ \wedge \ \pi/2 \leq arg\, z \leq \pi/4\}$.

Exercise 10.8 *Let* $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. *Prove the following implications:*

1. $$\left.\begin{matrix} \varphi_1 \in Arg\, z_1 \\ \varphi_2 \in Arg\, z_2 \end{matrix}\right\} \Rightarrow \varphi_1 + \varphi_2 \in Arg\,(z_1 z_2),$$

2. $$\left.\begin{matrix} \varphi_1 \in Arg\, z_1 \\ \varphi_2 \in Arg\, z_2 \end{matrix}\right\} \Rightarrow \varphi_1 - \varphi_2 \in Arg\left(\frac{z_1}{z_2}\right).$$

Exercise 10.9 *Calculate (if it exists) a limit for*

1. $\lim (3-4i)^n$,
2. $\lim \left((-1)^n + \dfrac{i}{n}\right)$,
3. $\lim \left(\dfrac{1+i}{\sqrt{2}}\right)^n$,
4. $\lim \left(\dfrac{1-\sqrt{3}i}{2}\right)^{6n}$.

Exercise 10.10 *Let (z_n) be a sequence of complex numbers. Prove the following propositions:*

1. $z_n \to 0 \;\Leftrightarrow\; \frac{1}{z_n} \to \infty$,
2. $$\left.\begin{array}{r} |z_n| \to r \in \mathbb{R} \\ arg\, z_n \to \varphi \in \mathbb{R} \end{array}\right\} \Rightarrow\; z_n \to r\big(\cos\varphi + i\sin\varphi\big),$$

and prove that in a second case an opposite implication does not hold.

Exercise 10.11 *Find all $z \in \mathbb{C}_\infty$ for which it holds that*

1. $z^3 = 1$,
2. $z^2 = i$,
3. $z^2 = 24i - 7$,
4. $\left(\dfrac{z-1}{z+1}\right)^2 = 2i$,
5. $z^4 = -1$,
6. $z^3 = i - 1$,
7. $z^5 = 1$,
8. $z^2 = -11 + 60i$,
9. $z^2 = 3 + 4i$.

Exercise 10.12 *Find a set $M = \left\{\frac{1}{z} \;:\; z \in \Omega\right\}$ if*

1. $\Omega = \{z \in \mathbb{C} \;:\; arg\, z = \alpha\},\ \alpha \in (-\pi, \pi]$,

2. $\Omega = \{z \in \mathbb{C} : \ |z-1| = 1\}$,
3. $\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z = \operatorname{Im} z\}$,
4. $\Omega = \{x + iy \in \mathbb{C} : \ x = 1\}$,
5. $\Omega = \{x + iy \in \mathbb{C} : \ y = 0\}$.

Exercise 10.13 *Find a set* $M = \{f(z) : \ z \in \Omega\}$ *if*

1. $\Omega = \left\{z \in \mathbb{C} : \ |arg\, z| \le \frac{\pi}{6}\right\}, \ f(z) = z^2$,
2. $\Omega = \left\{z \in \mathbb{C} : \ |\operatorname{Im} z| < \frac{\pi}{2}\right\}, \ f(z) = e^z$,
3. $\Omega = \{z \in \mathbb{C} : \ 0 < \operatorname{Re} z < \pi \ \wedge \ \operatorname{Im} z > 0\}, \ f(z) = e^{iz}$,
4. $\Omega = \{z \in \mathbb{C} : \ \operatorname{Im} z = 1/2\}, \ f(z) = z^2$.

Exercise 10.14 *Calculate*

1. $\sin(2-3i)$,
2. $\cos i$,
3. $\cosh i$,
4. $Ln\,(-5+3i)$ *a* $\ln(-5+3i)$,
5. $Ln\,(-4-\sqrt{3}i)$ *a* $\ln(-4-\sqrt{3}i)$,
6. $Ln\,(ie^2)$.

Exercise 10.15 *Find all* $z \in \mathbb{C}$ *for which it holds that*

1. $\sin z = 3$,
2. $\cos z = \frac{\sqrt{3}}{2}$,
3. $\sin z + \cos z = 2$,
4. $\sin z - \cos z = 3$,
5. $z^2 + 2z + 9 + 6i = 0$.

Exercise 10.16 *Calculate*

1. 2^i,
2. $(-2)^{\sqrt{2}}$,

3. $\left(\dfrac{1-i}{\sqrt{2}}\right)^{1+i}$,

4. $i^{\frac{3}{4}}$,

5. $(-1)^{\sqrt{3}}$,

6. $(-\sqrt{3}i+1)^{-3}$.

Exercise 10.17 *Find the real and imaginary parts of a function* $f : \mathbb{C} \to \mathbb{C}$ *defined by*

1. $f(z) = \sin z$,

2. $f(z) = z^2 \cos z$,

3. $f(z) = z^3 + 5z - 1$,

4. $f(z) = |z|\,\overline{z}$,

5. $f(z) = z^2\,\overline{z}$,

6. $f(z) = \dfrac{1}{z}$.

Exercise 10.18 *Find out if the function* $f(z) = z^3$ *is injective on a set* Ω *in case that*

1. $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$,

2. $\Omega = \left\{z \in \mathbb{C} : \arg z \in \langle 0, \frac{\pi}{4})\right\}$.

Exercise 10.19 *Find a limit (if it exists):*

1. $\lim\limits_{z\to 0} \dfrac{\operatorname{Re} z}{z}$,

2. $\lim\limits_{z\to 0} \dfrac{\operatorname{Im}(z^2)}{z\overline{z}}$,

3. $\lim\limits_{z\to 0} \dfrac{z \operatorname{Im} z}{|z|}$,

4. $\lim\limits_{z\to 0} \dfrac{z^2}{|z|^2}$,

5. $\lim\limits_{z\to 0} \dfrac{z^3}{|z|^2}$,

6. $\lim\limits_{z\to i} \dfrac{z^2 + z(2-i) - 2i}{z^2+1}$,

7. $\lim_{z\to 0} \frac{\operatorname{Re} z}{1+|z|}$.

Exercise 10.20 *Draw a set* $\langle\varphi\rangle = \{\varphi(t) : t \in D\varphi\}$ *if*

1. $\varphi(t) = 1 - it,\ D\varphi = [0,2]$,
2. $\varphi(t) = t - it^2,\ D\varphi = [-1,2]$,
3. $\varphi(t) = 1 + e^{-it},\ D\varphi = [0,2\pi]$,
4. $\varphi(t) = e^{2it} - 1,\ D\varphi = [0,2\pi]$,
5. $\varphi(t) = \begin{cases} e^{i\pi t},\ t \in [0,1], \\ t-2,\ t \in [1,3], \end{cases}$
6. $\varphi(t) = \begin{cases} e^{it},\ t \in [-\pi/2,\pi], \\ 3t/\pi - 4,\ t \in [\pi,2\pi]. \end{cases}$

Exercise 10.21 *Find a curve* $\varphi : \mathbb{R} \to \mathbb{C}$ *such that* $\langle\varphi\rangle = \Omega$*) if*

1. $\Omega = \{z \in \mathbb{C} : |z - 2 + 3i| = 2\}$,
2. Ω *is a line with endpoints* $a, b \in \mathbb{C}$, $a \neq b$,
3. $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z = 2 \operatorname{Im} z\}$,
4. $\Omega = \{z \in \mathbb{C} : \operatorname{Re}\left(\frac{1}{z}\right) = 2\}$.

Exercise 10.22 *Draw a set* Ω *and find out if it is a domain and an open set, where*

1. $\Omega = \{z \in \mathbb{C} : |z - i| < 1 \ \vee\ |z + i| < 1\}$,
2. $\Omega = \{z \in \mathbb{C} : |z - 1| < 1 \ \wedge\ |z - 2| < 2\}$,
3. $\Omega = \{z \in \mathbb{C} : |z - 1| < |z + 1|\}$,
4. $\Omega = \{z \in \mathbb{C} : |z + 1| > 2|z|\}$,
5. $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$,
6. $\Omega = \left\{z \in \mathbb{C} : |z| < 1 \ \wedge\ \arg z \in (-\pi,\pi] \setminus \{0\}\right\}$,
7. $\Omega = \{z \in \mathbb{C} : |2z| < |1 + z^2|\}$.

Exercise 10.23 *Find all points at which a function* f *has a derivative and is holomorphic if*

1. $f(z) = \operatorname{Re} z$,
2. $f(z) = |z^2|$,

3. $f(z) = ze^z$,
4. $f(z) = \overline{z}|z|$,
5. $f(z) = \dfrac{\operatorname{Re} z}{z}$,
6. $f(z) = z^2\overline{z}$,
7. $f(z) = z^2 + 2z - 1$.

Exercise 10.24 *Find out if a function Φ is harmonic in a domain Ω, where*

1. $\Phi(x, y) = x^2 - y^2 + 2020$, $\Omega = \mathbb{C}$,
2. $\Phi(x, y) = \frac{x}{x^2+y^2} + x^2 - y^2 + x - y$, $\Omega = \mathbb{C} \setminus \{0\}$.

Exercise 10.25 *Find (if it exists) a holomorphic function $f = u + iv$ on Ω if*

1. $u(x, y) = x^3 - 3xy^2 - 2y$, $\Omega = \mathbb{C}$,
2. $u(x, y) = \dfrac{x}{x^2 + y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$,
3. $u(x, y) = 3x^2 - y^2 + 3x + y$, $\Omega = \mathbb{C}$,
4. $u(x, y) = x^2 - y^2 + 5x + y - \dfrac{y}{x^2 + y^2}$, $\Omega = \mathbb{C} \setminus \{0\}$.

Exercise 10.26 *Let $u(x, y) = x^3 - 3xy^2 - 2y + 2$. Find (if it exists) a holomorphic function $f = u + iv$ on $\mathbb{C}$ for which it holds that*

1. $f(0) = i$,
2. $f(1) = 3 - i$.

Exercise 10.27 *Find (if it exists) a holomorphic function $f = u + iv$ on Ω if*

1. $v(x, y) = -3xy^2 + x^3 + 5$, $\Omega = \mathbb{C}$,
2. $v(x, y) = \arctan \dfrac{y}{x}$, $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Exercise 10.28 *Let $v(x, y) = 1 + \arctan \frac{y}{x}$. Find (if it exists) a holomorphic function $f = u + iv$ in a domain $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ for which it holds that*

1. $f(3) = \ln 3 + 6 + i$,
2. $f(e) = 1 - i$.

Exercise 10.29 *Prove that a function*

$$v(x, y) = \ln(x^2 + y^2)$$

is harmonic in two-times connected domain $\mathbb{C} \setminus \{0\}$ *and that there does not exist a function* $u : \mathbb{R}^2 \to \mathbb{R}$ *such that a function* $f = u + iv$ *is holomorphic on* $\mathbb{C} \setminus \{0\}$.

Exercise 10.30 *Find a rotation angle and an explosion coefficient of a function* f *at a point* z_0 *if*

1. $f(z) = e^z$, $z_0 = -1 - \frac{\pi}{2}i$,
2. $f(z) = z^3$, $z_0 = -3 + 4i$,
3. $f(z) = \frac{z+i}{z-i}$, $z_0 = 2i$.

Exercise 10.31 *Find all points of the Gauss plane in which a function* f *is a contraction*

1. $f(z) = \frac{2}{z}$,
2. $f(z) = \ln(z + 4)$.

Exercise 10.32 *Draw sets* Ω *and* $f(\Omega) = \{f(z) : z \in \Omega\}$ *if*[1]

1. $\Omega = U(1, 2)$, $f(z) = 1 - 2iz$,
2. $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$, $f(z) = (1 + i)z + 1$,
3. $\Omega = U(1, 2)$, $f(z) = \frac{1}{z}$,
4. $\Omega = U(1, 2)$, $f(z) = \frac{2iz}{z+3}$,
5. $\Omega = U(1, 2)$, $f(z) = \frac{z-1}{2z-6}$,
6. $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$, $f(z) = \frac{1}{z}$,
7. $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$, $f(z) = \frac{z}{z-1+i}$,

[1] Use (and also prove) the following statement:

$$\left.\begin{array}{c} f \text{ is conformal in a domain } \Omega \subset \mathbb{C}_\infty, \\ A,\ B \subset \Omega \end{array}\right\} \Rightarrow f(A \cap B) = f(A) \cap f(B).$$

8. $\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z < 1\}, \ f(z) = \dfrac{z}{z-2}$,

9. $\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z < 0 \ \wedge \ \operatorname{Im} z < 0\}, \ f(z) = \dfrac{1}{z}$,

10. $\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z > 0 \ \wedge \ \operatorname{Im} z > 0\}, \ f(z) = \dfrac{z-1}{z+1}$,

11. $\Omega = \{z \in \mathbb{C} : \ -1 < \operatorname{Re} z < 0 \ \wedge \ \operatorname{Im} z < 0\}, \ f(z) = \dfrac{z-i}{z+i}$,

12. $\Omega = \{z \in \mathbb{C} : \ |z| < 1 \ \wedge \ \operatorname{Re} z < 0 \ \wedge \ \operatorname{Im} z > 0\}, \ f(z) = \dfrac{z}{z-i}$.

Exercise 10.33 *Find a linear fractional function* f *such that*

1. $f(-1) = 0$, $f(i) = 2i$, $f(1+i) = 1-i$,

2. $f(i) = \infty$, $f(6) = 0$, $f(\infty) = 3$,

3. $f(0) = i$, $f(i) = 0$, $f(-1) = -i$.

Exercise 10.34 *Find a linear function mapping the square with vertices* $0, \ 1-i, \ 2, \ 1+i$ *on the square with vertices* $1+i, \ -1+i, \ -1-i, \ 1-i$.

Exercise 10.35 *Let*

$$\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z > \operatorname{Im} z\}.$$

Find a linear fractional function f *such that* $f(\Omega) = U(0,1)$.

Exercise 10.36 *Find a conformal function which maps the domain*

$$\{z \in \mathbb{C} : \ |z| < 1 \ \wedge \ \operatorname{Re} z > 0\}$$

on the domain

$$\{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}.$$

Exercise 10.37 *Let*

$$\Omega = \{z \in \mathbb{C} : \ \operatorname{Re} z > 0 \ \wedge \ \operatorname{Im} z < 0\}.$$

Find a linear fractional function f *such that*

$$f(\Omega) = \{z \in \mathbb{C} : \ |z| < 1 \ \wedge \ \operatorname{Re} z < 0\}.$$

Exercise 10.38 *Find a conformal function which maps the domain*

$$\{z \in \mathbb{C} : \ \operatorname{Re} z > \operatorname{Im} z > 0\}$$

on the domain $U(0,1)$.

Exercise 10.39 *Find images of lines parallel to a real, resp. imaginary axis under a function* $f(z) = \dfrac{1}{z}$.

Exercise 10.40 *Find images of the sets*

$$M_\alpha = \{z \in \mathbb{C} : \arg z = \alpha\}, \quad N_r = \{z \in \mathbb{C} : |z| = r\},$$

where $\alpha \in (-\pi, \pi]$, $r \in \mathbb{R}^+$ *under the function* $f(z) = \ln z$.

Exercise 10.41 *Find*

$$\int_\gamma |z| \,\mathrm{d}z$$

if

$$\gamma(t) = \begin{cases} 3e^{it}, \ t \in \left[0, \frac{\pi}{2}\right], \\ i\left(3 + \frac{\pi}{2} - t\right), \ t \in \left[\frac{\pi}{2}, \frac{\pi}{2} + 3\right], \\ t - \frac{\pi}{2} - 3, \ t \in \left[\frac{\pi}{2} + 3, \frac{\pi}{2} + 6\right]. \end{cases}$$

Exercise 10.42 *Calculate*

$$\int_\gamma z^3 \,\mathrm{d}z$$

if

$$\gamma(t) = \begin{cases} e^{it}, \ t \in \left[-\frac{\pi}{2}, \pi\right], \\ \frac{3}{\pi}t - 4, \ t \in [\pi, 2\pi], \\ -\frac{2+i}{\pi}t + 6 + 2i, \ t \in [2\pi, 3\pi]. \end{cases}$$

Exercise 10.43 *Calculate*

$$\int_\gamma |z|\overline{z} \,\mathrm{d}z,$$

where γ *denotes a simple piece-wise smooth positively oriented curve such that* $\langle\gamma\rangle$ *is a border of a set*

$$\{z \in \mathbb{C} : |z| < 2 \ \wedge \ \operatorname{Im} z > 0\}.$$

Exercise 10.44 *Using Cauchy's integral formulas calculate the following integrals* [2]

1. $$\int_k \frac{z^2 + i}{z} \,\mathrm{d}z, \quad \textit{where} \ k = \{z \in \mathbb{C} : |z - 2i| = 1\},$$

[2] Notice that by a formula $\int_k f(z)\,\mathrm{d}z$, where $k \subset \mathbb{C}$, one means $\int_\gamma f(z)\,\mathrm{d}z$, where γ is a simple closed piece-wise smooth positively oriented curve such that $\langle\gamma\rangle = k$.

2.
$$\int_k \frac{\sin z}{z+i}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z+i| = 1\},$$

3.
$$\int_k \frac{\sin z}{z^2 - 7z + 10}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 3\},$$

4.
$$\int_k \frac{\sin z}{(z-2i)^3}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 3\},$$

5.
$$\int_k \frac{\cos z}{z^2 - \pi^2}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 4\},$$

6.
$$\int_k \frac{e^{\frac{1}{z}}}{(z^2-4)^2}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z-2| = 1\},$$

7.
$$\int_\gamma \frac{e^z \cos(\pi z)}{z^2 + 2z}\,\mathrm{d}z, \quad \textit{where} \;\; \gamma(t) = \frac{3}{2}e^{it}, \; t \in [0, 2\pi],$$

8.
$$\int_\gamma \frac{\mathrm{d}z}{(z^2-1)^3}, \quad \textit{where} \;\; \gamma(t) = \frac{-2 + e^{-4\pi i t}}{2}, \; t \in [0, 4],$$

9.
$$\int_\gamma \frac{\mathrm{d}z}{(1-z)(z+2)(z-i)^2},$$

where γ denotes a simple closed piece-wise smooth positively oriented curve such that $-2 \in int\,\gamma, \; i \in int\,\gamma, \; 1 \in ext\,\gamma$.

Exercise 10.45 *Find*

1. $\int_0^{1+i} e^z \,\mathrm{d}z$,
2. $\int_0^{1+i} z^3 \,\mathrm{d}z$,
3. $\int_0^{i} z^2 \sin z \,\mathrm{d}z$,
4. $\int_0^{i} z \sin z \,\mathrm{d}z$.

Exercise 10.46 *Find out if a series is convergent:*

1. $\sum_{n=1}^{\infty} \frac{i^n}{n2^n}$,

2. $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}(1+i)^n$,

3. $\sum\limits_{n=1}^{\infty} \frac{(-i)^n}{3n-17}$.

Exercise 10.47 *Find a domain of convergence of a series* [3]

1. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z+1}{z-1}\right)^n$,

2. $\sum\limits_{n=1}^{\infty} \left(\frac{z^n}{n!}+\frac{n^2}{z^n}\right)$.

Exercise 10.48 *Find a radius of convergence of a power series*

1. $\sum\limits_{n=1}^{\infty} d\,\frac{z^n}{n^{2011}}$,

2. $\sum\limits_{n=1}^{\infty} n^n(z-1)^n$,

3. $\sum\limits_{n=1}^{\infty} \frac{3^n(z-1)^n}{\sqrt{(3n-2)2^n}}$,

4. $\sum\limits_{n=0}^{\infty} \frac{(z+1+i)^n}{3^n(n-i)}$,

5. $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!}z^n$,

6. $\sum\limits_{n=0}^{\infty} \big(\cos(in)\big)z^n$,

7. $\sum\limits_{n=0}^{\infty} (n^2-n-2)z^n$,

8. $\sum\limits_{n=0}^{\infty} \frac{z^n}{(n+8)!}$.

Exercise 10.49 *Find the sum of a power series in a disk of convergence*

1. $\sum\limits_{n=1}^{\infty} nz^n$,

2. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$,

[3] It means, find all $z \in \mathbb{C}$ for which the series is convergent.

3. $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$,

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n+1}$,

5. $\sum_{n=0}^{\infty} (n^2 - n - 2) z^n$.

Exercise 10.50 *Find the sum of a series*

1. $\sum_{n=1}^{\infty} \frac{1}{n2^n}$,

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$.

Exercise 10.51 *Find a Taylor series of a function f with a centre z_0 and find its radius of convergence if*

1. $f(z) = \frac{z+1}{z^2+4z-5}$, $z_0 = -1$,

2. $f(z) = \frac{z}{z^2+i}$, $z_0 = 0$,

3. $f(z) = \ln \frac{1+z}{1-z}$, $z_0 = 0$,

4. $f(z) = e^{3z-2}$, $z_0 = 1$,

5. $f(z) = \sin(3z^2+2)$, $z_0 = 0$,

6. $f(z) = \frac{1}{(z-1)^3}$, $z_0 = 3$,

7. $f(z) = \sin^2 z$, $z_0 = 0$.

Exercise 10.52 *Find a domain of convergence of a Laurent series*[4]

1. $\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n$,

2. $\sum_{n=-\infty}^{\infty} \frac{(z-i)^n}{n^2+1}$.

Exercise 10.53 *Find a Laurent series of a function f on an annulus*

1. $f(z) = \frac{\cos z}{z^2}$, $0 < |z| < 1$,

[4]That is, find all $z \in \mathbb{C}$ for which the series is convergent.

2. $f(z) = \dfrac{1}{z^2+1},\ |z| > 1,$

3. $f(z) = \dfrac{z^2+1}{z(z-i)},\ \frac{1}{2} < |z-i| < 1,$

4. $f(z) = \dfrac{1}{2z-5},\ |z| > \frac{5}{2},$

5. $f(z) = \dfrac{1}{z(z-2)},\ 1 < |z-2| < 2,$

6. $f(z) = \dfrac{z}{(z^2+1)^2},\ 0 < |z-i| < 2,$

7. $f(z) = \dfrac{z - \sin z}{z^4},\ 0 < |z| < \infty,$

8. $f(z) = \dfrac{z+2}{z^2-4z+3},\ 2 < |z-1| < \infty,$

9. $f(z) = \dfrac{1}{z(z-3)^2},\ 1 < |z-1| < 2.$

Exercise 10.54 *Find a Laurent series of a function f on all maximal annuli with a centre at z_0 on which f is a holomorphic fucntion if*

1. $f(z) = \dfrac{z^2-z+3}{z^3-3z+2},\ z_0 = 0,$

2. $f(z) = \dfrac{z+1}{z^2},\ z_0 = 1+i.$

Exercise 10.55 *Determine the type of each isolated singularity of a function f if*

1. $f(z) = z^5 + 4z^3 - 2 + \dfrac{2}{z} + \dfrac{3}{z^2},$

2. $f(z) = \dfrac{z^2-4}{z-2},$

3. $f(z) = \dfrac{1}{z-z^3},$

4. $f(z) = \dfrac{z^4}{z^4+1},$

5. $f(z) = \dfrac{e^z}{z^2+4},$

6. $f(z) = \dfrac{z^2+4}{e^z},$

7. $f(z) = \dfrac{1-e^z}{2+e^z},$

8. $f(z) = e^{\frac{1}{z^2}}$,
9. $f(z) = \dfrac{1}{(z-3)^2(2-\cos z)}$,
10. $f(z) = \dfrac{z}{\sin z}$,
11. $f(z) = z^2 \sin \frac{z}{z+1}$,
12. $f(z) = \dfrac{1-\cos z}{\sin^2 z}$.

Exercise 10.56 *Prove L'Hôpital's rule:*
Let f and g be holomorphic non-constant functions on some ring neighbourhood of a point $z_0 \in \mathbb{C}$ and let $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} g(z) = 0$. Then it holds that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}.$$

Exercise 10.57 *Find a residue of a function f at all its isolated singularities if*

1. $f(z) = \dfrac{1}{z+z^3}$,
2. $f(z) = \dfrac{z^2}{(1+z)^3}$,
3. $f(z) = \dfrac{1}{(z^2+1)^3}$,
4. $f(z) = \dfrac{z^3+1}{z-2}$,
5. $f(z) = \dfrac{1}{z^6(z^2+1)^2}$,
6. $f(z) = \tan z$,
7. $f(z) = \dfrac{1}{\sin z}$,
8. $f(z) = \cot^3 z$,
9. $f(z) = \sin z \sin \dfrac{1}{z}$,
10. $f(z) = \dfrac{\sin(\pi z)}{(z-1)^3}$.

Exercise 10.58 *Using the residue theorem calculate an integral*

1.
$$\int_{\gamma} \frac{\cos z}{z^3}\,\mathrm{d}z, \quad \textit{where} \;\; \gamma(t) = 3e^{it},\; t \in [0, 2\pi],$$

2.
$$\int_{\gamma} \frac{1}{z+2} \cos\frac{1}{z}\,\mathrm{d}z, \quad \textit{where} \;\; \gamma(t) = 18e^{it},\; t \in [0, 2\pi],$$

3.
$$\int_{k} \frac{z^3}{z^4 - 1}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 2\},$$

4.
$$\int_{k} \frac{z^3}{z+1} e^{\frac{1}{z}}\,\mathrm{d}z, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 2\},$$

5.
$$\int_{\gamma} z \sin\frac{z+1}{z-1}\,\mathrm{d}z, \quad \textit{where} \;\; \gamma(t) = 2e^{-it},\; t \in [0, 6\pi],$$

6.
$$\int_{\gamma} \frac{e^{\pi z}}{2z^2 - i}\,\mathrm{d}z,$$

where γ is a simple closed piece-wise smooth positively oriented curve such that
$$\textit{int}\,\gamma = \left\{ z \in \mathbb{C} : \; |z| < 1 \;\wedge\; 0 < \arg z < \frac{\pi}{2} \right\},$$

7.
$$\int_{k} \frac{\mathrm{d}z}{z^5(z^{10} - 2)}, \quad \textit{where} \;\; k = \{z \in \mathbb{C} : \; |z| = 2\}.$$

Exercise 10.59 *Using the residue theorem calculate an integral* [5]

1.
$$\int_{-\pi}^{\pi} \frac{\mathrm{d}x}{5 + 3\cos x},$$

2.
$$\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{x^4 + 6x^2 + 25},$$

3.
$$\int_{0}^{\infty} \frac{x^4 + 1}{x^6 + 1}\,\mathrm{d}x,$$

4.
$$\int_{0}^{\infty} \frac{x^2}{(x^2 + 1)^3}\,\mathrm{d}x,$$

[5] Introduced integrals are meant to be integrals of a real variable.

5.

$$\int_{-\pi}^{\pi} \frac{\cos x}{3 + 2\sin x}\, \mathrm{d}x,$$

6.

$$\int_{0}^{2\pi} \frac{\cos^2(2x)}{5 - 4\cos x}\, \mathrm{d}x,$$

7.

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1 + x^6},$$

8.

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + x + 1}.$$

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