**Physical quantities, their measurements and uncertainty**

**Brief overview**

**Physical quantity** is the property of an object that can be quantified. For instance, the physical quantities are the length of a rod or the mass of a body. **Measurement** is the act of comparing a physical quantity with its unit. **Measurement result** is the value of a physical quantity obtained by means of measurement. This value can be expressed as the product of numerical value and corresponding unit. The complete presentation of the measurement result must contain the information about the **uncertainty** (see below).

**Uncertainty** gives the range of possible values of the measurand, which covers the true value of the measurand. Thus uncertainty characterizes the spread of measurement results. The interval of possible values of measurand is commonly accompanied with the confidence level. Therefore, the uncertainty also indicates a doubt about how well the result of the measurement presents the value of the quantity being measured.

The symbol for uncertainty is \( u \) or \( U \), depending on what kind of uncertainty is regarded. The uncertainty expressed as the experimental standard deviation is called the standard uncertainty \( (u) \). The product of the standard uncertainty and the factor larger than one, known as coverage factor (symbol \( k \)), is called the expanded uncertainty \( (U) \). Consequently \( U=k\cdot u \). Commonly expanded uncertainty is equal to the half-width of an interval having a stated level of confidence. (The coverage factor depends on the probability distribution and on the confidence level.)

**Direct measurement** is a measurement in case of which the value of a physical quantity in question is obtained directly from the instrument scale. **Indirect measurement** is a measurement in case of which the value being sought for a quantity is calculated by a known relationship (formula) between that quantity and the quantities subject to a direct measurement.

**Measurement units**
By making the measurements the units are of great importance. Nowadays the most common system of units is the International System of Units (Systeme International d’Unites, abbreviation SI).

The International System of Units (SI) specifies a set of seven **base units** from which all other units of measurement are built up. Derived units are based on those seven base units.

**SI base units are:**
- for length \((l)\) metre (m),
- for mass \((m)\) kilogram (kg),
- for time \((t)\) second (s),
- for electric current \((I,i)\) ampere (A),
- for thermodynamic temperature \((T)\) kelvin (K),
- for amount of substance \((n)\) mole (mol),
- for luminous intensity \((I_v)\) candela (cd).

In the physics laboratory almost all units must be converted into the SI system.

**Treatment of the results of direct measurements and the evaluation of uncertainty**

The individual measurement result is regarded as a random variable. In order to obtain more reliable evaluation of the physical quantity being determined the arithmetic mean is used:
\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \]  

where \( n \) is the number of trials and \( x_i \) is the result of the \( i \)-th measurement.

The deviation of the \( i \)-th measurement from the arithmetic mean is called the random deviation:

\[ x_i - \bar{x} = \Delta x_i \]

**Evaluation of the uncertainty of the measurement results**

Pursuant to the International Standard and the Standard of the Republic of Estonia, the quality of any measurement result must be evaluated using the concept of uncertainty. According to this, the aim of a measurement is the reliable estimation of characteristics (basic parameters) of the probability distribution of a measurable quantity (measurand). These characteristics are usually the mathematical expectation (mean value) and the standard deviation (square root of the variance). The variance is a measure of the random variability which indicates the extent of the variation of the quantity values being studied.

The uncertainty consists of many components which are divided into two categories according to the method used to estimate their numerical values:

**Type A uncertainty** \( U_A (x) \) which is evaluated from the actual repeated measurements by the statistical methods;

**Type B uncertainty** \( U_B (x) \) which is evaluated by other means, e.g. by assigning a probability distribution to the corresponding deviations. (In this case the actual repeated measurements haven’t been done.)

In the physics laboratory the necessary information for finding such type of uncertainty is found from the instrument specification (manual) or dial, from the corresponding table on the wall or from the measurement procedure. Besides, estimation of this type of uncertainty can be based on experience or reasonable considerations.

From now on, the notion of uncertainty always signifies the expanded uncertainty.

**Type A uncertainty evaluation**

Type A uncertainty of the arithmetic mean is calculated by the formula:
\[ U_A(x) = t_{v, \beta} \sqrt{\frac{\sum (x_i - \bar{x})^2}{n(n-1)}} \] 

where \( t_{v, \beta} \) is the Student’s coefficient which values are given in the table below:

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>0,5</th>
<th>0,68</th>
<th>0,95</th>
<th>0,975</th>
<th>0,9973</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,00</td>
<td>1,8</td>
<td>12,7</td>
<td>12,7</td>
<td>235,8</td>
</tr>
<tr>
<td>2</td>
<td>0,82</td>
<td>1,3</td>
<td>4,3</td>
<td>4,3</td>
<td>19,2</td>
</tr>
<tr>
<td>3</td>
<td>0,77</td>
<td>1,2</td>
<td>3,2</td>
<td>3,2</td>
<td>9,2</td>
</tr>
<tr>
<td>4</td>
<td>0,74</td>
<td>1,1</td>
<td>2,8</td>
<td>2,8</td>
<td>6,6</td>
</tr>
<tr>
<td>5</td>
<td>0,73</td>
<td>1,1</td>
<td>2,6</td>
<td>2,6</td>
<td>5,5</td>
</tr>
<tr>
<td>6</td>
<td>0,72</td>
<td>1,1</td>
<td>2,5</td>
<td>2,4</td>
<td>4,9</td>
</tr>
<tr>
<td>7</td>
<td>0,71</td>
<td>1,1</td>
<td>2,4</td>
<td>2,4</td>
<td>4,5</td>
</tr>
<tr>
<td>8</td>
<td>0,71</td>
<td>1,1</td>
<td>2,3</td>
<td>2,3</td>
<td>4,3</td>
</tr>
<tr>
<td>9</td>
<td>0,70</td>
<td>1,1</td>
<td>2,3</td>
<td>2,3</td>
<td>4,1</td>
</tr>
<tr>
<td>10</td>
<td>0,70</td>
<td>1,1</td>
<td>2,2</td>
<td>2,2</td>
<td>4,0</td>
</tr>
<tr>
<td>20</td>
<td>0,69</td>
<td>1,0</td>
<td>2,1</td>
<td>2,1</td>
<td>3,4</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0,67</td>
<td>1,0</td>
<td>2,0</td>
<td>2,0</td>
<td>3,0</td>
</tr>
</tbody>
</table>

Index \( \nu = n - 1 \) is the number of independent deviations (degrees of freedom) and index \( \beta \) is the confidence level – the probability that the true value of the measured quantity is situated in a predetermined range which half-width is given by formula (3).

**Type B uncertainty evaluation**

- uncertainty caused by the maximum permissible deviation or bias \( (e_p) \) of measuring instrument is found by the formula:

\[ U_B(x)_m = t_{v, \beta} \frac{e_p}{3}, \]  

where \( \infty \) is infinity;

- uncertainty caused by the rounding of the instrumental reading to the smallest full subdivision on the scale is found by the formula:

\[ U_B(x)_l = \beta \cdot L, \]  

where \( \beta \) is the confidence level and \( L \) is half of the value of the scale smallest subdivision being determined by the measurement.
COMBINED UNCERTAINTY

In order to find the overall uncertainty of a measurement result one must evaluate combined uncertainty \( U_c(\bar{x}) \) by using the simplified approximate formula that incorporates type A and type B uncertainties:

\[
U_c(\bar{x}) = \sqrt{U_A^2(\bar{x}) + U_B^2(\bar{x})},
\]

(6)

- In case of the repeated direct measurements the combined uncertainty appears as follows:

\[
U_c(\bar{x}) = \sqrt{U_A^2(\bar{x}) + U_B^2(\bar{x})}_m,
\]

(7)

- In case of the single direct measurement the combined uncertainty appears as follows:

\[
U_c(x) = \sqrt{U_A^2(x) + U_B^2(x)},
\]

(8)

One must remember, that by applying approximate formulas (6), (7) and (8) individual uncertainties \( U_A \) and \( U_B \) must be calculated at the same confidence level. Then \( U_c \) is considered having just the same confidence level.

- The indirect measurement.

If a physical quantity \( y \) (output) being sought is a function of several independent variables \( y = f(x_1, x_2, ..., x_k) \), where \( x_1, x_2, ..., x_k \) are directly measured quantities (inputs) with their respective combined uncertainties \( U_c(x_1), U_c(x_2), ..., U_c(x_k) \), then the overall combined uncertainty for \( y \) is calculated by the following approximate formula:

\[
U_c(y) = \sqrt{\left(\frac{\partial y}{\partial x_1} U_c(x_1)\right)^2 + \left(\frac{\partial y}{\partial x_2} U_c(x_2)\right)^2 + \ldots + \left(\frac{\partial y}{\partial x_k} U_c(x_k)\right)^2}
\]

(9)

where \( \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, ..., \frac{\partial y}{\partial x_k} \) are partial derivatives or sensitivity coefficients. In order to apply approximate formula (9) all individual combined uncertainties \( U_c(x_1), U_c(x_2), ..., U_c(x_k) \) must be found at the same confidence level. Then the confidence level of \( U_c(y) \) is considered equal to the confidence level of all individual combined uncertainties.
UNCERTAINTY OF PHYSICAL QUANTITIES GIVEN AT WORKPLACE

When a certain value of a physical quantity given at the workplace or in the table on the wall does not have uncertainty specified, then the uncertainty is by default considered to be the half of its last decimal place.

Example. The diameter of a capillary tube given at the workplace is \( d = 0.078 \text{ mm} \). Here the last decimal place is a thousandth of millimetre. Therefore the uncertainty of diameter will be the half of a thousandth of millimetre, i.e. \( U(d) = 0.0005 \text{ mm} \).

PRESENTATION OF FINAL RESULT

**Significant figures**
The significant figures are all numbers 1, 2, 3, ..., 9, and 0 if it lies between the numbers 1 … 9 or at the end of an integer or decimal. The zeros situated at the beginning of decimal are not considered as significant figures.
Example. In the number 0.06503 there are four significant figures (the first significant digit is 6 and the last one is 3), but the number 20.620 has five significant figures (all digits are significant).
The integer 5200 has four significant figures.

**Significant figures in calculations**

1. When performing multiplication, division, exponentiation or root extraction, leave in the result one significant figure more than there are in the quantity that has the smallest number of significant figures.
   Example. 
   \[
   \rho = \frac{m}{V} = \frac{0,62 \text{ g}}{1,42 \text{ cm}^3} = 0,437 \text{ g} \cdot \text{cm}^3
   \]

2. When adding or subtracting quantities, leave in the result one decimal place more than there are in the quantity with largest last decimal place.
   Example. 
   \[
   \sum_{i=1}^{3} x_i = 12,1 + 4,34 - 0,402 = 16,04
   \]

As a rule the uncertainty of final measurement result must have two significant figures.
The measurement result is rounded off to the last decimal place of its uncertainty. The confidence level should be added to the final result.

ASSIGNING UNCERTAINTY TO SINGLE READING FROM POINTER MEASUREMENT INSTRUMENT

Depending on their instrumental bias, measuring instruments can be classified as belonging to particular accuracy class. Commonly the accuracy class of a pointer measurement instrument indicates the maximum permissible deviation (bias) percentage from the upper limit of measuring range.
Let’s have a DC ammeter (DC = direct current) which upper limit of measuring range is 3 A and the number of subdivisions on its scale is 100. The pointer is at the subdivision 62.
In our case, the accuracy class of DC ammeter is 1.5 and it can be found at the bottom right corner of the instrument dial. The current intensity corresponding to the given pointer position is \( \frac{3.00 \text{ A}}{100} = 1.86 \text{ A} \)

To find uncertainty we must calculate 1.5% from 3.00 A. The result is 0.045 A (confidence level of 99.7%, the Gaussian distribution of reading deviations is assumed).

The uncertainty caused by rounding to the scale smallest subdivision can be calculated as follows:

\[
U_b(I) = \pm \frac{3.00 \text{ A}}{100} \cdot \frac{1}{2} = \pm 0.015 \text{ A} , \text{ confidence level 100%}
\]

(The maximum rounding deviation makes half of the smallest subdivision. Rounding deviations are considered to have rectangular distribution.)

The combined uncertainty can be calculated as follows:

\[
U_c(I) = \pm \sqrt{U^2_B(I) + U^2_b(I)} = \pm \sqrt{0.045^2 + 0.015^2} = 0.0474 \text{ A}
\]

(with the approximate confidence level of 100%).

Result: \( I = 1.860 \pm 0.047 \text{ A} \)

To change the confidence level, formulas (4) and (5) must be used.

ASSIGNING UNCERTAINTY TO SINGLE READING FROM DIGITAL INSTRUMENT

Let’s have an AC voltmeter (AC = alternating current) which upper limit of measuring range is 200 V. The reading from the voltmeter is 20.00 V. The known frequency of the voltage oscillation is 50 Hz. The resolution of voltmeter is 0.01 V. This means that the least significant digit on the voltmeter display is a hundredth.

The formula to calculate the maximum permissible deviation (bias) of the reading as given in the instrument specification (manual) is:

\[
\text{accuracy} = \pm (0.5\% \text{ RDG} + 20 \text{ DGT})
\]  

(10)

at frequencies up to 100 Hz.

(Here the accuracy signifies the maximum permissible deviation and it is considered as maximum uncertainty.)

In the formula (10) RDG stands for reading and DGT stands for the least significant digit.

The uncertainty of the given reading 20.00 V according to formula (10) can be calculated as follows:

\[
U_B(U) = \pm (0.5\% \cdot 20.00 \text{ V} + 20.00 \cdot 0.01 \text{ V}) = \pm (0.10 + 0.20) \text{ V} = \pm 0.30 \text{ V}
\]

(with confidence level of 100%, rectangular distribution of the indication deviations is assumed).

The abovementioned reading (20.00 V) can be thus presented as follows:

\[
U = (20.00 \pm 0.30) \text{ V} , \text{ confidence level 100%}.
\]
If it is necessary to change the confidence level, one must apply formula (5), \( l \) considered as accuracy.
When reading off the indication from the display of a digital instrument, there is no uncertainty.

**ASSIGNING UNCERTAINTY TO SINGLE READING FROM DECADE BOX**

In the physics laboratory there are two categories of decade boxes – of Russian origin and of Western origin. Russian decade boxes are provided with the accuracy class and with the corresponding formula to calculate the maximum permissible deviation (bias). Western decade boxes are only provided with the formula to calculate the maximum permissible deviation.
The uncertainty with high confidence level can be calculated using the maximum permissible deviation which is regarded as expanded uncertainty with the confidence level of 99.7% (= 100%). (The Gaussian distribution of indication deviations is assumed.)

When reading off the indication from the decade box, there is no uncertainty.

For example, if the accuracy class 0.2 is written on the resistance decade box of Russian origin and nothing more is known, then the maximum permissible deviation is 0.2% from the indication. As agreed the uncertainty with the confidence level of 99.7% is considered equal to the maximum permissible deviation.
If it is necessary to change the confidence level, one must apply formula (4).