Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/"(weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Lecture overview

Chapter Graph colorings and graph drawing

- motivation
- graph coloring
- drawing graphs in the plane
- recognizing planar graphs
- map coloring and planar graphs coloring

Graph colorings

We mention two problems that can be solved naturally using graph colorings.

Storing goods

There are many different food products in a storehouse. By regulations several of them certain have to be stored separately. E.g. fruit salads cannot be in the same department as raw eggs or salami cannot share department with raw meat.
What is the smallest number of departments necessary?

Set up a graph whose vertices represent stored goods and an edges joins two vertices whenever the corresponding two commodities have to be stored separately. Compartments are distinguished by colors.

Question

What is the least number of different colors necessary to color the vertices of the graph so that any two adjacent vertices have the different colors?

Optimization of traffic lights

A crossing has several corridors for both cars and pedestrians. Corridors (even of different directions) may not interfere and the traffic can flow simultaneously. On the other hand corridors that do interfere with each other have to have green within non-overlapping time slots. Time slots are distinguished by colors.
What is the least number of time intervals necessary in one "cycle" of the traffic lights?

In the graph model the vertices will represent corridors and edges will join vertices that correspond to corridors that do interfere.

Question

What is the least number of colors necessary to color the vertices of the graph so that any two adjacent vertices have the different colors?

We show some special cases and prove a couple of simpler theorems. First we introduce several definitions.

Definition

Graph coloring of G by k colors is such a mapping

$$c : V(G) \to \{1, 2, \ldots, k\},$$

that any two adjacent vertices have different colors, i.e. $c(u) \neq c(v)$ for every edge $uv \in E(G)$.

Note

Graph coloring is called also a proper vertex coloring of a graph.

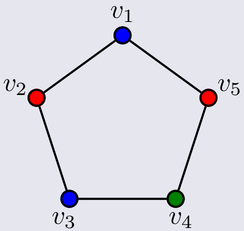
There exists a proper coloring of every graph by $|V(G)|$ colors. We are interested in the lowest possible number of colors, for which a graph coloring of G exists.

Definition

The chromatic number $\chi(G)$ of G is the least k, such that there exists a proper coloring of G by $\chi(G)$ colors.

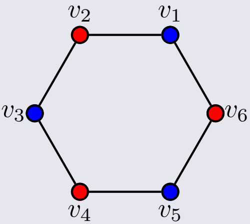
Example

What is the chromatic number of C_5?



Example

What is the chromatic number of C_6?



Upper bounds on the number of colors

Lemma

Let G by a simple graph on n vertices. Then $\chi(G) \leq n$. Equality holds if and only if G is a complete graph.

Proof In any graph G with n vertices it is enough to color every vertex by a different color and we get a proper vertex coloring of G by n different colors. Thus, $\chi(G) \leq n$.

If $G \simeq K_n$, then no two adjacent vertices can have the same color Thus, $\chi(K_n) = n$.

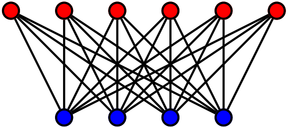
If some edge uv is missing in G, we can color $c(u) = c(v) = 1$ and color the remaining vertices by colors $2, 3, \ldots, n-1$. We obtain a proper vertex coloring by less than n colors, thus $\chi(G) < n$. □

Brook's Theorem

For every graph G with n vertices different from K_n and different from odd cycles C_n is $\chi(G) \leq \Delta(G)$.

Proof is beyond the level of this course, you can find it in the textbook „Teorie grafů" (in Czech) or on-line.

Notice, not in every graph G as many as $\Delta(G)$ color have to be used. For example to color the vertices of a complete bipartite graph only two colors are necessary.



Algorithms for finding a proper vertex coloring by the least number of colors are complex and are not included in this text. For general graphs there are algorithms with complexity $O(n2^n)$, where n is the number of vertices.

Lower bounds on the number of colors

Brooks Theorem says *at most* how many colors are necessary to color the graph properly. Now we show a couple of simple bounds on how many colors are necessary *at least* for a proper edge coloring.

Theorem

Graph G has chromatic number 1 if and only if it has no edge.

Proof If a graph has no edge, we color all vertices by color 1. If all vertices have the same color, no edge can be in the graph. □

The next theorem we mention without proving it.

Theorem

If there is a complete subgraph on k vertices in a given graph G, then any proper vertex coloring of G requires at least k colors.

We prove one particular case of the theorem.

Theorem

Graph G has chromatic number 2 if and only if it contains no cycle of odd length (as a subgraph).

Proof (idea of "$\Leftarrow$") An odd cycle cannot be properly colored by two colors. We choose any vertex v in G and color it by color 1. Vertices in odd distance from v we color by color 2. Vertices in even distance from v we color by color 1.

If any two vertices x, y in even distance from v are joined be edge xy, then $v, \dots x, y, \dots v$ is a walk of odd length. From the walk we obtain an odd cycle be deleting repeated parts which contradicts the premise. For vertices in odd distance from v we reason similarly. Thus, in this coloring no adjacent vertices have the same color and we have a proper coloring by two colors. □

Graphs without cycles of odd lengths are bipartite. The vertices of each such graph can be partitioned into two independent (partite) sets.

In each partite set it is enough to use only one color for all the vertices in the set.

How to determine chromatic number

To determine the chromatic number of a graph means to find the smallest number of colors required for a proper vertex coloring.

There is no theorem that would yield the chromatic number “easily”.

The chromatic number can be found by algorithms with complexity $O(n2^n)$, where n is the number of vertices of the given graph.

Here we have shown

- upper and lower bound of the chromatic number,
- applications (warehouse problem, scheduling).

Drawing graphs in the plane

In some cases it is important how the drawing looks like. Printed circuit boards can be represented as graphs and when designing the board crossings have to be avoided.

Question: „Is it possible to draw a given graph without crossing edges?“

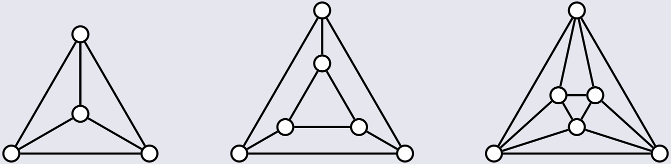
Definition

Planar drawing of a graph G is such a drawing in the plane, in which vertices are different points and edges are lines connecting the points of their end-vertices and no two edges intersect save their end-points.
We say a graph is planar if there exists its planar drawing.

Not every graph has a planar drawing!

Examples

Examples of planar graphs are graphs of polyhedrons (tetrahedron, cube, octahedron, dodecahedron, prisms, ...)

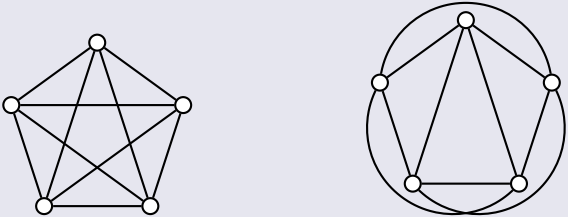


All graphs of polyhedrons are planar and (at least) 3-connected.

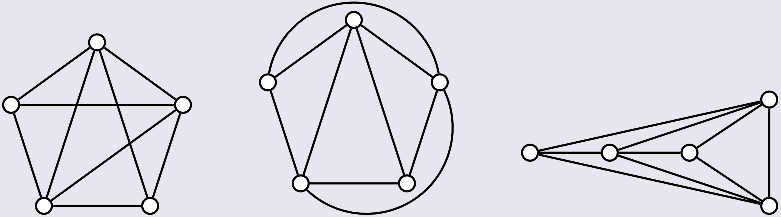
On contrary every planar 3-connected simple graph is a graph of some polyhedron.

Example

Are the graphs a) K_5, b) $K_5 - e$ planar (drawn without crossing edges)?



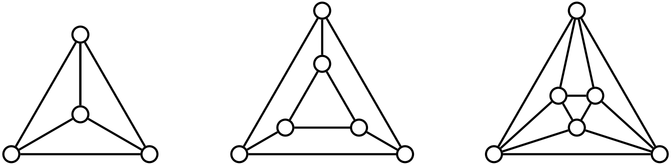
Graph K_5 and its drawing with a single crossing of edges.



Graph K_5 with an edge removed and two its planar drawings.

Definition

Faces in a planar drawing of a graph are connected regions in a plane bounded by edges and points of the drawing.



Faces in a planar drawing.

We show an important formula that counts graph elements of a planar graph: Euler's formula.

Theorem Euler's formula

A planar drawing of a *nonempty* connected graph G has f faces. The the following holds

$$v + f - e = 2.$$

Proof Proof goes by induction on the number of edges e.

Basis step: If G is a tree, it contains no cycle and the planar drawing has only one face. By a theorem a tree has $e = v - 1$ edges and be evaluate that $v + f - e = v + 1 - (v - 1) = 2$.

Inductive step: Suppose the claim holds for all graphs with $e - 1$ edges. If G contains a cycle C, then by omitting one edge uv of cycle C the number of edges decreases by 1. At the same time the number of faces decreases by 1, since the edge uv separated two faces (neighboring to uv) and by deleting uv these faces merge. The number of vertices remains the same.

By the induction hypothesis is $v + (f - 1) - (e - 1) = 2$, thus also $v + f - e = 2$. □

Note

Euler's formula is independent of a particular drawing, only on the graph structure.

Though it is a simple formula it has many applications and corollaries.

Corollary

A simple planar graph on $v \geq 3$ vertices has at most $3v - 6$ edges.

Proof Suppose we have a connected graph G, otherwise we can add more edges. By v we denote the number of vertices in G, by f the number of faces and by e the number of edges.
Since there are no loops or multiple edges, each face of G (in any planar drawing) is bounded by at least three edges. Each edge is counted at most twice (for both neighboring faces). Thus $2e \geq 3f$, from which follows $\frac{2}{3}e \geq f$. Substituting into the Euler's formula we get

$$2 = v + f - e \leq v + \frac{2}{3}e - e = v - \frac{1}{3}e$$
$$e \leq 3(v - 2) = 3v - 6.$$

□

If there are no faces with only three edges in G (a triangle-free graph) the number of edges is even smaller.

Corollary

A simple triangle-free planar graph on $v \geq 3$ vertices has at most $2v - 4$ edges.

Proof The proof is similar. By v we denote the number of vertices in G, by f the number of faces and by e the number of edges. Now we know there are no triangles in G, thus each face is bounded by at least four edges. Thus $2e \geq 4f$, which implies $\frac{2}{4}e \geq f$. Substituting into the Euler's formula we get

$$2 = v + f - e \leq v + \frac{2}{4}e - e = v - \frac{1}{2}e$$

$$e \leq 2(v - 2) = 2v - 4.$$

□

We can also bound the smallest degree of a planar graph!

Corollary

Every planar graph has a vertex of degree at most 5.
Every triangle-free planar graph has a vertex of degree at most 3.

Proof By contradiction. If all vertices would be of degree at least 6, there would be at least $\frac{1}{2} \cdot 6v = 3v$ edges in a planar graph, which contradicts previous Corollary. Thus there has to be a vertex of degree smaller than 6.

Similarly, if in a triangle-free graph all vertices would be of degree at least 4, there would be at least $\frac{1}{2} \cdot 4v = 2v$ edges in the graph, which contradicts previous Corollary. Thus there has to be a vertex of degree smaller than 4. □

Notice, that a planar graph can have vertices of high degree, but not all of them. There has to be some vertex (or vertices) of small degree as well.

Recognizing planar graphs

To "be planar", or "non-planar" is an important property of a graph with many applications. Among the most important are

- printed circuit boards of single layer (the circuits form a graph, need solder wires?)
- well drawn graphs (no unnecessary crossings)

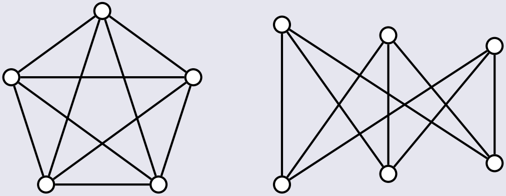
We show that Euler's formula and its corollaries can help when determining whether a graph is or is not planer.

In comparison to Hamiltonian cycles or graph colorings there are relatively fast algorithms.
We focus only on small graphs, the algorithm mentioned above go beyond the scope of this course.

We show two important graphs, that are not planar.

Example

Graphs K_5 and $K_{3,3}$ are non planar (are non-planar).



Graphs K_5 and $K_{3,3}$.

Proof We use the Corollary on the number of edges.
Notice that K_5 has 5 vertices and 10 edges. But by the Corollary a planar graph on five vertices has at most $e \leq 3 \cdot 5 - 6 = 9$ edges, hence K_5 is non-planar.

Similarly $K_{3,3}$ has 6 vertices and 9 edges. Moreover, it is triangle-free. But by the Corollary a triangle-free planar graph on six vertices has at most $2 \cdot 6 - 4 = 8$ edges, thus $K_{3,3}$ is non-planar. □

Corollary

Graphs K_5 and $K_{3,3}$ are not planar.

It can be shown that both K_5 and $K_{3,3}$ are special among all non-planar graphs. Their structure does not allow their planar drawing.

Moreover, no other such structure exists.

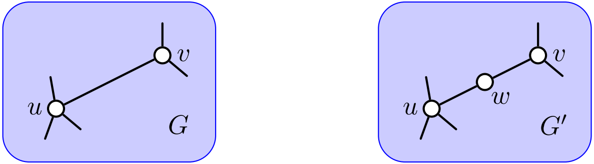
We introduce the notion of subdivision of a graph, that is a graph with similar structure, with some vertices of degree 2 added.

Definition

A subdivision of a graph G is a graphs that is obtained by replacing some edges by internally-disjoint paths.

We replace the edge uv of a graph G by a pair of edges uw and wv. We obtain a new graph G', which is a *subdivision* of the original graph G.

$$G' = (V(G) \cup \{w\}, (E(G) \setminus \{uv\}) \cup \{uw, wv\})$$

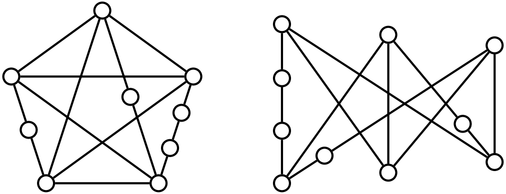


Graph G with a selected edge uv and a subdivision G' of graph G.

In 1930 K. Kuratowski proved the following simple theorem.

Theorem

Graph G is planar if and only if it does not contain a subgraph isomorphic to a subdivion of K_5 or $K_{3,3}$.



A subdivision of graphs K_5 and $K_{3,3}$.

It can be shown, that there exists a "nice" drawing of every planar graph:

Theorem

Every simple planar graph can be drawn in a plane without crossing edges so that all edges are straight lines.

Graph colorings and graph drawing

One of the best known problems in graph theory is the Four color theorem. Though the formulation is easy, correct solution took more than 100 years.

Four color problem

Given any political map, the regions may be colored using no more than four colors in such a way that no two adjacent (sharing a borderline) regions receive the same color.

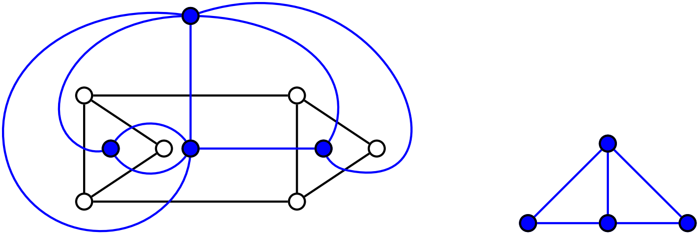
The solution required besides substantial theoretical work also a large scale computer search.

Example

A coloring of a political map can be translated into a proper vertex coloring of a graph.
Each region becomes a vertex (the capital).
Two vertices are joined by an edge if the corresponding states are neighboring.

Definition

Dual graph of a planar graph G we obtain by replacing every region by a vertex. Two vertices of the new graph are connected by an edge if and only if the corresponding regions share an edge.



Graph G with blue dual multigraph and a redrawn dual graph.

It can be shown, that the dual graph to a planar graph is again planar.

The process of transforming a political map into a graph is similar to constructing a dual graph.

In 1976 Appel and Haken, and later in 1993 again Robertson, Seymour, Sanders, and Thomas proved the theorem, which solved the four color problem. It is one of the most famous results in discrete mathematics.

Theorem Four Color Theorem

Every planar graph without loops has a proper coloring by at most 4 colors.

Proof . . . definitely exceed the scope of this course :-) □

But easily we can show a weaker result for 6 colors.

Theorem

Every planar graph can be properly colored by at most 6 color.
Every triangle-free planar graph can be properly colored by at most 4 colors.

Proof We show the second part, the first part is shown in the textbook.

We proceed by induction on the number of vertices of G.

Basis step: The trivial graph with one vertex is surely planar and can be colored by one color.

Inductive step: We have a planar graph with at least two vertices. Suppose all smaller planer graphs we can color by at most four colors. By a previous corollary we find in G a vertex v of degree at most 3. The graph $G - v$ is again planar and triangle-free. By the induction hypothesis we can color the graph $G - v$ by at most four colors. At most three of them will be used to color the neighbors of v, thus *always* there is a fourth color available to color v. □

Notice, the proof is constructive – we can find such coloring.

Next chapter

Chapter Flow in a network

- motivation
- definition of a network
- maximal flow algorithm
- network generalization
- further applications