Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/" (weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Lecture overview

Chapter Eulerian and hamiltonian graphs

- motivation
- eulerian graphs traversable in “one trail”
- hamiltonian graphs traversable in “one path”

Eulerian graphs

Historically first problem solved by graph theory approach in 1736:
Seven bridges of Königsberg – search for a trail uv, such that it contains all edges of a given graph G.

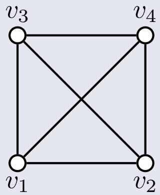
Example

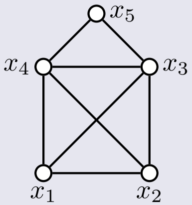
A postman has to deliver mail along each street in his district. Suppose he can traverse each street only once – he travels the shortest distance and delivers mail sooner.

Take a graph representing the district in which streets correspond to edges and junctions to vertices. On optimal solution to the postman problem corresponds to finding a trail that traverses each edge precisely once.

Similarly one can suggest an optimal route for snowplows, garbage cars, etc.

Example





Is it possible to draw the edges of G in one stroke?

Definition

A trail in a graph G which originates and stops in the same vertex is called a closed trail. Moreover, if it contains all edges of a connected graph G, it is a closed eulerian trail. A graph that allows a closed eulerian is called an eulerian graph.

A trail in a connected graph G which originates in one stops in another vertex and contains all edges of G is called an open eulerian trail.

We say that each such graph can be drawn in a single stroke.

Theorem

Graph G can be traversed by a single closed trail, if and only if G is *connected* and all vertices of G are of *even degree*.

Using an elegant argument one can show easily:

Corollary

Graph G can be traversed by a single open trail, if and only if G is *connected* and *precisely two* vertices of G are of *odd degree*.

Eulers' Theorem

Graph G can be traversed by a single closed trail, if and only if G is *connected* and all vertices of G are of *even degree*.

Proof By induction on the number of edges (just a sketch of "$\Leftarrow$").

Basis step:

We can start with the trivial graph. For non-trivial connected graph G is every vertex of degree at least 2. The smallest such graph is $G \simeq C_n$. Graph G contains a closed trail traversing all edges (why?).

Inductive step:

Suppose, that every connected graph with less than $|E|$ edges and with all vertices of even degree contains a closed trail traversing all edges. In G we take any cycle C (each vertex is of degree at least 2). In $G - C$ are vertices of even degree (or isolated vertices). If $G - C$ is not connected, each component contains by induction hypothesis a closed trail traversing all edges of the component.

Now we insert into the closed trail C a closed "sub-trail" at vertex v_i, one in each component. We obtain a closed trail traversing all edges of G.

The claim of "$\Leftarrow$" follows by (strong) mathematical induction. $\square$

Corollary

The edges of a graph G can be drawn in a single (open) stroke if and only if G is *connected* and *precisely two* of its vertices are of *odd degree*.

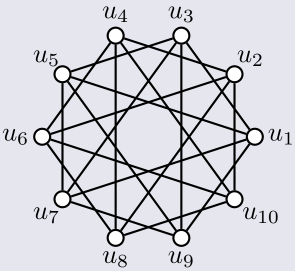
Proof

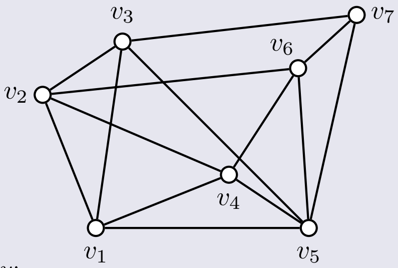
"$\Rightarrow$" Suppose the edges of a graph can be drawn in a single stroke (via an open eulerian trail). Then G is connected and all its vertices are of even degree with the exception of the first and the last vertex of the open eulerian trail.

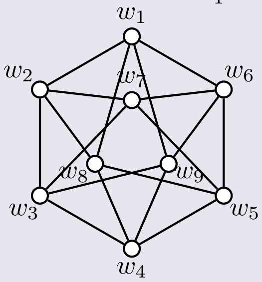
"$\Leftarrow$" Suppose G is connected wit precisely two vertices u and v of odd degree. We can add a new vertex x to G and join it with pair of edges to vertices u and v. We obtain a connected graph G' in which each vertex (u, v and x included) is of even degree.

By Eulers' Theorem there exists a *closed* eulerian trail T' in G'. After removing vertex x and both edges incident to x we obtain an open eulerian trail T from vertex u to v in G. □

Examples







Which of these graphs are eulerian?

Eulerian trail can be used to solve other problem besides traveling.
One nice application of eulerian trails:

Example

The vertices of the state graph (corresponding to some system) represent states which may occur. We join two states by an edge if one state can lead to the other – e.g. Finite Automaton.

When designing a test of the system, we would like to check all states and all possible transitions. An optimal test may run along an eulerian trail.

Hamiltonian graphs

Hamiltonian cycle

(or hamiltonian circuit) in a given graph is a cycle that contains all vertices of G.
A graph for which a hamiltonian cycle exists is a hamiltonian graph.

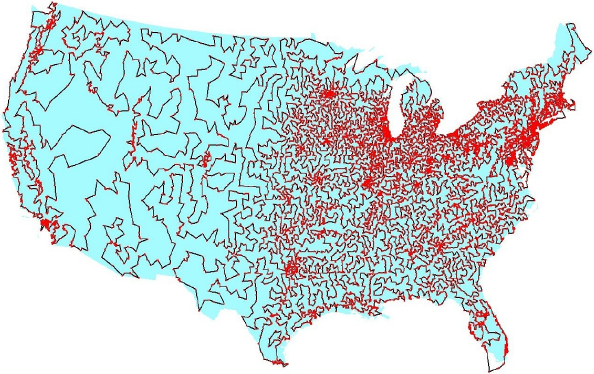
(Hamiltonian cycle visits every vertex of the graph.)

It may seem that constructing hamiltonian cycles is related to constructing eulerian trails. This is not the case!
While there is an easy necessary and sufficient condition of even degrees for the existence of eulerian trails in connected graphs, for the existence of hamiltonian cycles no such easy condition is known (some think it may not exists).

Corollary: it is not easy to decide whether a graph is hamiltonian or not.

Example

The traveling salesman problem is a well known motivation. The salesman wants to visit each city in his region, return to the starting city and travel the shortest possible distance during his travel.
Simplified version: can he visit every city wit at least 500 citizens precisely once and return back?



Optimal solution to the travelling salesman problem for 13 509 cities in the USA

Example

A postman in a village delivers mail each day only to some of the houses. Rather than traverse each street, he has to visit each address (house) to which a letter has to be delivered.

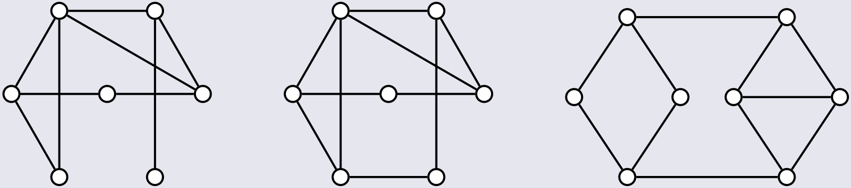
Example

In a warehouse when goods are to be deposited or picked up, the forklift (or pallet) has to visit each of the spot where a certain good is stored in pallets. The forklift has to visit all selected locations and travel the shortest distance possible.

All the problem mentioned above can be formulated as finding a hamiltonian cycle in a graph, or a shortest hamiltonian path, respectively.

Examples

Which of the following graph are/are not hamiltonian?



Hamiltonian an non-hamiltonian graphs.

Examples

More problem leading to hamiltonian cycles

- family travel plan for visiting several places of interest
- theater or circus tour through the country
- Hamilton game

Theorem (Dirac)

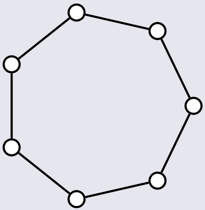
Let G be a graph on n vertices, where $n \geq 3$.
If the smallest degree is at least $n/2$, then G is hamiltonian.

Proof In another course “Teorie Grafů I”.

Notice, the statement has the form of an implication, not an equivalence. Thus, each graph in which the smallest degree high enough is hamiltonian, but not each hamiltonian graph has to have a large small degree.

Example

A hamiltonian graph does not have to have “many” edges.



Cycle C_7.

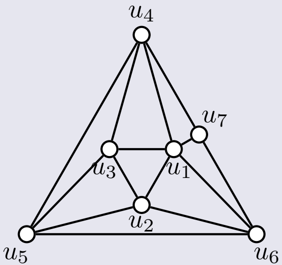
Theorem (Ore)

Let G be a graph on n vertices, where $n \geq 3$.
If for each independent (nonadjacent) vertices u and v in a graph G is $\deg(u) + \deg(v) \geq n$, then G is hamiltonian.

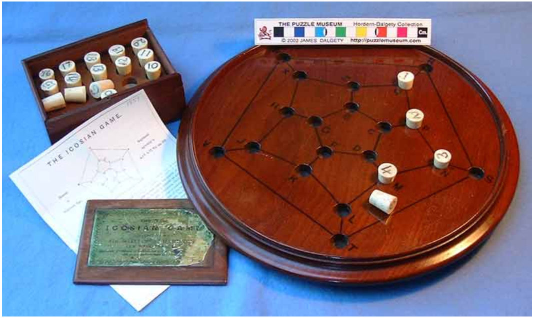
Diracs' Theorem is a special case of Ores' Theorem.

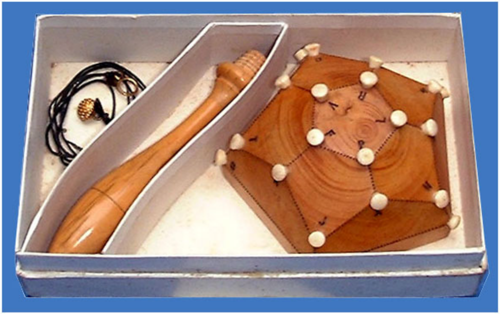
Example

Is this graph hamiltonian?



Why are the graphs called "hamiltonian"?





Next lecture

Chapter Distance and measuring in graphs

- motivation
- distance in graphs
- measuring in graphs
- weighted distance
- shortest path algorithm