Discrete mathematics

Petr Kovář & Tereza Kovářová
petr.kovar@vsb.cz

VŠB – Technical University of Ostrava

Winter Term 2022/2023
DiM 470-2301/02, 470-2301/04, 470-2301/06

EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research,
Development and Education

MINISTRY OF EDUCATION,
YOUTH AND SPORTS

The translation was co-financed by the European Union and the Ministry of Education, Youth and Sports from the Operational Programme Research, Development and Education, project "Technology for the Future 2.0", reg. no. CZ.02.2.69/0.0/0.0/18_058/0010212.

About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/" (weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Lecture overview

Kapitola 5. Recurrence relations

- motivation
- sequences given by recurrences
- main problem
- methods of solving
- examples

5. Recurrence relations

Last chapter already mentioned that not all selections and arrangements can be expressed in simple “closed” formulas mentioned in Section 2.
Today we mention several typical problems that we encounter when using recursive algorithms.
We say, how the complexity of certain such algorithms can be expressed.

Typical examples of recursive algorithms or recursive approaches

- merge sort
- dynamic programming
- using n pairs of parentheses on $n+1$ terms
- number of “ordered rooted trees” in chapter UTG 4

5.1. Motivation examples

The classic Fibonacci sequence is notoriously known.

Fibonacci sequence

A young pair of rabbits has been released on an island. The rabbits are mature at the age of two months, after that they raise another pair of rabbits each month. What is the number f_n of pairs of rabbits after n months?

Clearly $f_1 = f_2 = 1$.
For $n \geq 3$ is the number of pairs given by

- the number of pairs in the previous months,
- the number of pairs of two months age f_{n-2}, that became mature and can breed.

Altogether we have $f_n = f_{n-1} + f_{n-2}$ pairs, if dying of age is neglected.

The solution, i.e. the formula for f_n we derive at the end of the lecture.

Towers of Hanoi

Example

We have three pegs and a set of discs of different sizes. All discs are on one peg arranged according their size. The task is to move all discs to another peg while

- always one disc is moved,
- never a larger disc can be on top of a smaller one.

What is the smallest number of moves H_n to move the entire tower of n discs?

Towers of Hanoi

To move the largest disc, $n-1$ smaller discs have to be moved to another peg using H_{n-1} moves.

We divide the total number of moves H_n into three parts.

- First using H_{n-1} moves transfer $n-1$ smaller discs on the third peg,
- then using a single move transfer the largest disc to the desired peg,
- finally using H_{n-1} moves transfer $n-1$ smaller discs on top of the largest disc.

The total number of moves is given by the recurrence relation

$$H_n = 2H_{n-1} + 1,$$

while clearly $H_1 = 1$.

The solution, i.e. the formula for H_n we derive at the end of the lecture.

Bit strings without adjacent zeroes

Example

How many bit strings of length n are there, that have no two adjacent zeroes? (important in bar codes)

Denote the number of required bit strings with n bits by a_n.
We distinguish, if a string of n bits end with a 0 or a 1 (assume $n \geq 3$).

- if the last bit is 1, then there are precisely a_{n-1} such strings with an additional bit 1,
- if the last bit is 0, then the next-to-the-last bit has to be 1 and there are a_{n-2} such strings with additional bits 10 at the end.

These are all the options, therefore the total number of strings with n bits, where no two zeroes are adjacent, is

$$a_n = a_{n-1} + a_{n-2}.$$

It remains to figure out that $a_1 = 2$, $a_2 = 4 - 1 = 3$.

At the end of the lecture we derive the formula for a_n.
Notice: a_n is similar to, yet different from the Fibonacci sequence.

Code words with an even number of zeroes

Example

A computer system works with keywords made from digits 0, 1, ..., 9. A valid code word has an even number of zeroes. How many such code words of length n exist?

Let x_n denote the number of such code words with n digits.
We distinguish if the n-th digit of a code word is 0 or no (suppose $n \geq 2$).

- code words with last digit not 0 are precisely $9x_{n-1}$, where the last digit 1,2,..., 9 was added to some of x_{n-1} code words of length $n-1$,
- code words with last digit 0, are precisely those that are not code words x_{n-1}.

These are all possibilities, therefore the total of code words of length n with an even number of zeroes is

$$x_n = 9x_{n-1} + (10^{n-1} - x_{n-1}) = 8x_{n-1} + 10^{n-1}.$$

It remains to notice that $x_1 = 9$. We search for the formula expressing x_n.

Number of ways to parenthesize $n+1$ terms with n parentheses

Example

We have an expression with $n+1$ terms, priority of operation $\oplus$ is given by n pairs of parentheses. In how many different ways can C_n be parethesized?

$C_0 = 1$, since x_1 is unique.
$C_1 = 1$, since $(x_1 \oplus x_2)$ is unique.
$C_2 = 2$, since $((x_1 \oplus x_2) \oplus x_3)$, $(x_1 \oplus (x_2 \oplus x_3))$ are two possibilities.
$C_3 = 5$, there are 5 ways $(((x_1 \oplus x_2) \oplus x_3) \oplus x_4)$, $((x_1 \oplus (x_2 \oplus x_3)) \oplus x_4)$, $((x_1 \oplus x_2) \oplus (x_3 \oplus x_4))$, $(x_1 \oplus ((x_2 \oplus x_3) \oplus x_4))$, $(x_1 \oplus (x_2 \oplus (x_3 \oplus x_4)))$.

In general the most our parenthesis have only one operator "$\oplus$". Notice the operation is between two smaller terms, there are n different numbers of terms to the left of the operator in the outer parenthesis.

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

Recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ appears in a number of different real life problems, so called Catalan numbers.

5.2. Sequences given by recurrence relations

Recall:

Sequences are given by

- a list of first elements: $1, 3, 7, 15, 31, \ldots$
- a recurrence relation: $a_n = 2a_{n-1} + 1$, $a_0 = 1$
- a formula for n-th term: $a_n = 2^n - 1$

Now we deal with recurrence relations, every subsequent term can be evaluated based on previous terms.

Main problem

Find the formula for the n-th term.

- if it exists,
- if it is possible,
- and if we can do so.

Linear homogeneous recurrence relations of order k with constant coefficients

Linear homogeneous recurrence relations of order k with constant coefficients is a sequence given by a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$.

Let us explore the definition

- it is linear, because it is a linear combination of the previous terms,
- it is homogeneous, because there is **no** term without a_i,
- it is of order k, because a_n is given by at most k previous terms,
- it has constant coefficients, because each coefficient at a_i is a constant independent on n.

For a unique description of the sequence given by a recurrence relation of order k we have to *provide k first terms*.

Fibonacci sequence

Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of second order with constant coefficients.
First two terms are $f_1 = 1$, $f_2 = 1$.

Bit strings without adjacent zeroes

Sequence of the number of bit strings of length n, which have no adjacent zeroes $a_n = a_{n-1} + a_{n-2}$, is a linear homogeneous recurrence relation of order 2 with constant koeficients.
First two terms are $a_1 = 2$, $a_2 = 3$.

Hanoi tower

Sequence of the number of steps H_n *necessary to move the entire tower of n discs* $H_n = 2H_{n-1} + 1$ is a linear recurrence relation of the first order with constant coefficients, which is not homogeneous.
First term is $H_1 = 1$.

Code words with an even number of zeroes

The number of codewords made from digits 0, 1, . . . , 9, where each codeword has an even number of zeroes is a linear recurrence relation of the first order with constant coefficients $x_n = 8x_{n-1} + 10^{n-1}$. First term is $x_1 = 9$.
This relation is not homogeneous, since 10^{n-1} *is not* a coefficient at a_i.

Catalan numbers

The sequence of Catalan numbers $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ is given by a homogeneous recurrence relation. First terms are $C_1 = 1$, $C_2 = 2$.
This recurrence relation is not linear, because we multiply terms C_k, C_{n-k} and has no fixed order, because the number of terms grows with n.

Example

Recurrence relation $a_n = a_{n-1} \cdot a_{n-2}$ is a homogeneous recurrence relation of second order with constant coefficients. First two terms are $a_1 = 1$, $a_2 = 2$.
This recurrence relation is not linear, since we multiply terms a_{n-1}, a_{n-2}.

5.3. Methods for solving recurrence relations

Main problem of solving recurrence relations

If a linear recurrence relation of (small) order k with constant coefficients is given by a recurrence relation and sufficient first terms, we can "solve" this recurrence relation. This means, we find a formula for the n-th term, which evaluates a_n *without the knowledge of previous terms*.

We provide a general framework:

- first we set up a so called characteristic equation,
- we find the roots of the characteristic equation,
- based on the roots we set up a general solution,
- based on the value of the given first terms of the sequence we evaluate coefficients of the general solution.

We start with simple examples.

Characteristic equation and its roots

One can show (e.g. using so called generating functions), that the solution of the linear homogeneous recurrence relations with constant coefficients will have the form $a_n = r^n$, where r is a constant.
Substituting into the recurrence relation we obtain

$$\begin{aligned} a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \\ r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\ r^k &= c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k r^{k-k} \\ 0 &= r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k. \end{aligned}$$

The last equation is the characteristic equation of the recurrence relation. Clearly, the solution of this equation in variable r are the roots r_i. We call them characteristic roots.

Generalization of the solution

We split the solution of linear homogeneous recurrence relation with constant coefficients into several steps.

The next step includes solutions to a larger family of recurrence relations:

- first we show how a general form of the solution of a linear homogeneous recurrence relation of order 2 with constant coefficients looks like,
  - if there are two distinct real characteristic roots,
  - if there are two identical real characteristic roots.
- Next we show a general form of the solution of a linear homogeneous recurrence relation of order k with constant coefficients, with different roots.
- Then we provide a general form of the solution of a linear homogeneous recurrence relation of order k with constant coefficients.
- and finally we provide a general form of the solution of a non-homogeneous linear recurrence relation of order k with constant coefficients.
- We only mention further generalizations.

Form of the solution

There exists a general solution in a specific form.

Theorem

Let c_1, c_2 be two real numbers. If the characteristic equation $r^2 - c_1 r - c_2 = 0$ has two distinct (real) roots r_1, r_2, then the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for $n = 0, 1, 2, \ldots$.

We omit the proof.

There is a stronger claim, holds even if the roots are complex numbers.

Example

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, where $a_0 = 2$, $a_1 = 7$.

We follow the steps suggested earlier:
We expect the solution of the form $a_n = r^n$. Substituting to the recurrence relation we get the characteristic equation

$$\begin{aligned} r^2 - r - 2 &= 0 \\ (r+1)(r-2) &= 0. \end{aligned}$$

Characteristic roots are $r_1 = 2$, $r_2 = -1$. The general solution has the form

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

Substituting a_0, a_1 we get two equations in two variables α_1, α_2.

$$\begin{aligned} a_0 = 2 &= \alpha_1 \cdot 1 + \alpha_2 \cdot 1 \\ a_1 = 7 &= \alpha_1 \cdot 2 + \alpha_2 \cdot (-1) \end{aligned}$$

Solving the equation yields $\alpha_1 = 3$, $\alpha_2 = -1$, thus the general solution is

$$a_n = 3 \cdot 2^n - 1 \cdot (-1)^n.$$

Indeed,

- the formula $a_n = 3 \cdot_1 2^n - 1(-1)^n$ for $n = 0, 1, 2, \ldots$
- the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, where $a_0 = 2$, $a_1 = 7$

describe the same sequence:

$$2, 7, 11, 25, 47, 97, 191, 385, 767, 1\ 537, 3\ 071, \ldots$$

Now we examine the case with two identical characteristic roots.

Theorem

Let c_1, c_2 be two real numbers, where $c_2 \neq 0$. If the characteristic equation $r^2 - c_1 r - c_2 = 0$ has a double (real) root r_0, then the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is of the form $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$.

Notice, the second term of the general solution is a multiple of n.

Example

Solve the recurrence relation $a_n = 10a_{n-1} - 25a_{n-2}$, where $a_0 = 3$, $a_1 = 5$.

We follow the same steps:
We expect the solution of the form $a_n = r^n$. Substituting to the recurrence relation we get the characteristic equation

$$\begin{aligned} r^2 - 10r + 25 &= 0 \\ (r-5)(r-5) &= 0. \end{aligned}$$

Characteristic roots are $r_1 = r_2 = 5$, we denote $r_0 = 5$. The general solution has the form

$$a_n = \alpha_1 5^n + \alpha_2 n 5^n.$$

Substituting a_0, a_1 we get two equations in two variables α_1, α_2.

$$\begin{aligned} a_0 = 3 &= \alpha_1 \cdot 1 + 0 \\ a_1 = 5 &= \alpha_1 \cdot 5 + \alpha_2 \cdot 5 \end{aligned}$$

Solving the equation yields $\alpha_1 = 3$, $\alpha_2 = -2$, thus the general solution is

$$a_n = 3 \cdot 5^n - 2n5^n.$$

We found the formula for the n-th term

- $a_n = 3 \cdot 5^n - 2n5^n$ describes the same sequence as
- the recurrence relation $a_n = 10a_{n-1} - 25a_{n-2}$, where $a_0 = 3$, $a_1 = 5$.

Sequence is

$$3, 5, -25, -375, -3\,125, -21\,875, -140\,625, \ldots$$

Solution of linear recurrence relations can be generalized to higher orders.

Theorem

Let $c_1, c_2, \ldots, c_k$ be k real numbers. If the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has k distinct (real) roots $r_1, r_2, \ldots, r_k$, then the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$, for $n = 0, 1, 2, \ldots$.

Example

Solve the recurrence relation $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$, where $a_0 = 6$ $a_1 = 5$, $a_2 = 13$.

We get the characteristic equation

$$r^3 - 4r^2 + r + 6 = 0.$$

Characteristic roots are $r_1 = -1$, $r_2 = 2$, $r_3 = 3$. The general solution has the form

$$a_n = \alpha_1(-1)^n + \alpha_2 2^n + \alpha_3 3^n.$$

Substituting a_0, a_1, a_2 we get three equations in three variables α_1, α_2, α_3.

$$\begin{aligned} a_0 = 6 &= \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 = 5 &= -\alpha_1 + 2\alpha_2 + 3\alpha_3 \\ a_2 = 13 &= \alpha_1 + 4\alpha_2 + 9\alpha_3 \end{aligned}$$

Solving the equation yields $\alpha_1 = 2$, $\alpha_2 = 5$, $\alpha_3 = -1$, thus the general solution is

$$a_n = 2 \cdot (-1)^n + 5 \cdot 2^n - 3^n.$$

Solving general linear homogeneous recurrence relations with constant coefficients

Theorem

Let $c_1, c_2, \ldots, c_k$ be k real numbers. If the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, then the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ for $n = 0, 1, 2, \ldots$ has the form

$$\begin{aligned} a_n &= (\alpha_{1,1} + \alpha_{1,2} n + \cdots + \alpha_{1,m_1} n^{m_1 - 1}) r_1^n + \\ &\quad + (\alpha_{2,1} + \alpha_{2,2} n + \cdots + \alpha_{2,m_2} n^{m_2 - 1}) r_2^n + \\ &\quad + \cdots + (\alpha_{t,1} + \alpha_{t,2} n + \cdots + \alpha_{t,m_t} n^{m_t - 1}) r_t^n \end{aligned}$$

To find the solution, we

1. get the characteristic equation,
2. find characteristic roots (if possible),
3. set up the general form of the solution with coefficients $\alpha_{i,j}$,
4. substitute k first (known) terms,
5. solve the system of equations with k variables,
6. set up the general solution.

Solving general linear non-homogeneous recurrence relations with constant coefficients

So far only homogeneous recurrence relations ...

Solving non-homogeneous recurrence relations in two steps:

- general solution of the associated homogeneous recurrence relation,
- one particular solution of the linear non-homogeneous recurrence.

Theorem

Let $c_1, c_2, \ldots, c_k$ be k real numbers, let $F(n)$ be a function not identically zero.
If $a_n^{(p)}$ is a particular solution to the linear non-homogeneous recurrence relations with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $a_n^{(p)} + a_n^{(h)}$, where $a_n^{(h)}$ is the general solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Example

Show that $a_n^{(p)} = -n - 2$ is a (particular) solution of the recurrence relation $a_n = 2a_{n-1} + n$.

To verify a solution is easy: substitute and compare:

$$\begin{aligned} a_n &= 2a_{n-1} + n \\ -n-2 &= 2\left(-(n-1)-2\right)+n \\ -n-2 &= -n-2. \end{aligned}$$

Notice: the particular solution has the form $a_n^{(p)} = cn + d$.

Example

Show that $a_n^{(p)} = c \cdot 7^n$ is the form of a (particular) solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Again substitute and compare:

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} + 7^n \\ c \cdot 7^n &= 5c \cdot 7^{n-1} - 6c \cdot 7^{n-2} + 7^n \\ c &= \frac{49}{20}. \end{aligned}$$

Theorem

Let $c_1, c_2, \ldots, c_k$ be k real numbers, let $F(n)$ be a function not identically zero.
Suppose $a_n^{(p)}$ is a solution of the linear non-homogeneous recurrence relations with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$.

1. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relations, then $a_n^{(p)}$ has the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

2. When s is a root with multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relations, then $a_n^{(p)}$ has the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Example

Solve the recurrence relation $a_n = 2a_{n-1} + n2^n$.

First we find the solution of the associated linear homogeneous recurrence relation

$$a_n = 2a_{n-1}.$$

The characteristic equation $r^n = 2r^{n-1}$ has a nonzero root $r = 2$.
Therefore the general solution has the form $a_n^{(h)} = \alpha 2^n$.

Next we find a particular solution of the original linear non-homogeneous recurrence relation. By the previous theorem is

$$a_n^{(p)} = n(cn + d)2^n,$$

since base 2 is the root of the characteristic equation.

To find the constants we substitute the particular solution $a_n^{(p)} = n(cn + d)2^n$ into the recurrence relation $a_n = 2a_{n-1} + n2^n$.

Example continued

We get

$$\begin{aligned} n(cn+d)2^n &= 2\cdot(n-1)(c(n-1)+d)2^{n-1} + n2^n \\ (cn^2+dn)2^n &= 2\cdot(c(n-1)^2 + d(n-1))2^{n-1} + n2^n \\ (cn^2+dn)2^n &= (cn^2 - 2cn + c + dn - d + n)2^n \\ dn &= (-2c+d+1)n + (c-d). \end{aligned}$$

Comparing coefficients of the polynomials at n^1 and n^0 we get a system of linear equations

$$\begin{aligned} n^1: \qquad d &= -2c+d+1 \\ n^0: \qquad 0 &= c-d. \end{aligned}$$

The solution is $c = \frac{1}{2}$, $d = \frac{1}{2}$ and thus the particular solutions is $a_n^{(p)} = n(\frac{1}{2}n + \frac{1}{2})2^n = (n^2+n)2^{n-1}$.

The solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha\cdot 2^n + (n^2+n)2^{n-1} = (n^2+n+2\alpha)2^{n-1}.$$

Value of α depends on the initial term a_1.

5.4. Solving the motivation examples from the first section

Fibonacci sequence

Solve the recurrence relation $f_n = f_{n-1} + f_{n-2}$, where $f_0 = 0$, $f_1 = 1$.

We obtain the characteristic equation $r^2 - r - 1 = 0$.
Characteristic roots are $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$. The general solution has the form

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Substituting $f_0 = 0$, $f_1 = 1$ we get two equations in two variables α_1, α_2.

$$\begin{aligned} 0 &= \alpha_1 \cdot 1 + \alpha_2 \cdot 1 \\ 1 &= \alpha_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right) \end{aligned}$$

Solving the system yields $\alpha_1 = \frac{\sqrt{5}}{5}$, $\alpha_2 = -\frac{\sqrt{5}}{5}$, thus, the general solution is

$$f_n = \frac{\sqrt{5}}{5} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Towers of Hanoi

Solve the recurrence relation $H_n = 2H_{n-1} + 1$, where $H_1 = 1$.

It is a linear non-homogeneous recurrence relation.
On the other hand it is a first order recurrence, we can obtain the solution differently.
Notice

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-2} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

The solution of the linear non-homogeneous recurrence relation of the Towers of Hanoi is

$$H_n = 2^n - 1.$$

Bit strings with no adjacent zeroes

Solve the recurrence relation $a_n = a_{n-1} + a_{n-2}$, where $a_1 = 2$, $a_2 = 3$.

The characteristic equation is $r^2 - r - 1 = 0$.
Characteristic roots are $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$. The general solution has the form

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Substituting $a_1 = 2$, $a_2 = 3$ we get two equations in two variables α_1, α_2.

$$\begin{aligned} 2 &= \alpha_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right) \\ 3 &= \alpha_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 + \alpha_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2 \end{aligned}$$

Solving the system yields $\alpha_1 = \frac{5+\sqrt{5}}{10}$, $\alpha_2 = \frac{5-\sqrt{5}}{10}$, the general solution is

$$a_n = \frac{5+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Code words with an even number of zeroes

Solve the recurrence relation $x_n = 8x_{n-1} + 10^{n-1}$, where $x_1 = 9$.

This is a linear non-homogeneous recurrence relation with constant coefficients.

First we find the general solution of the associated homogeneous recurrence relation

$$x_n = 8x_{n-1}.$$

Its characteristic equation $r^n = 8r^{n-1}$ has a non-zero root $r = 8$.
Therefore the general form of the solution is $x_n^{(h)} = \alpha 8^n$.
Constant α can be evaluated only after we get the particular solution.

Next we find a particular solution of the linear non-homogeneous recurrence relation. By the theorem we expect a solution of the form

$$x_n^{(p)} = c \cdot 10^n,$$

since base 10 is not the root of the characteristic equation.

To evaluate c we substitute the particular solution $x_n^{(p)} = c \cdot 10^n$ to the recurrence relation $x_n = 8x_{n-1} + 10^{n-1}$.

We get

$$\begin{aligned} c10^n &= 8 \cdot c10^{n-1} + 10^{n-1} \\ 10c &= 8c + 1 \\ 2c &= 1 \\ c &= \frac{1}{2}. \end{aligned}$$

Now the general solution of the recurrence relation $x_n = 8x_{n-1} + 10^{n-1}$ is

$$x_n = x_n^{(h)} + x_n^{(p)} = \alpha \cdot 8^n + \frac{1}{2} \cdot 10^n.$$

To evaluate α we substitute $x_1 = 9$ and $n = 1$ to the general solution $x_n = \alpha \cdot 8^n + \frac{1}{2} \cdot 10^n$. We get

$$\begin{aligned} 9 &= \alpha \cdot 8 + \frac{1}{2} \cdot 10 \\ 9 - 5 &= 8\alpha \\ \alpha &= \frac{1}{2}. \end{aligned}$$

The solution of the recurrence relation including the initial term is

$$a_n = \frac{1}{2} \cdot 8^n + \frac{1}{2} \cdot 10^n.$$

Merge sort

Merge sort is a well known algorithm for sorting a sequence of n numbers. It is a recursive algorithm.
Knowing the algorithm, it is easy to see, that the number of comparisons (and operations) M_n to sort a sequence of n terms can be bounded by a recurrence relation $M_n = 2M_{\lceil n/2 \rceil} + n$, where $M_1 = 1$.

This linear non-homogeneous recurrence relation we cannot solve now, it has not a constant order.

HOwever it can be shown, that the solution describing the number of steps of the Merge sort algorithm, is a function of complexity

$$M_n = O(n \log n).$$

Tower of Hanoi – another solution

Solve the recurrence relation $H_n = 2H_{n-1} + 1$, where $H_1 = 1$.

It is a linear non-homogeneous recurrence relation. First we find the general solution of the associated homogeneous relation $H_n = 2H_{n-1}$. The characteristic equations $r = 2$ has a single root, therefore the general solution has form $H_n = \alpha \cdot 2^n$. The value of α can be determined only later.

Since the fuction $F(n) = 1$, the particular solution $H_n^{(p)}$ has the form $c \cdot 1$. To evaluate c, the particular solution is substituted to the recurrence relation.

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ c &= 2c + 1 \\ -1 &= c \end{aligned}$$

The particular solution is $H_n^{(p)} = -1$ and the general solution of the non-homogeneous equation has the form $H_n = \alpha \cdot 2^n - 1$.

Now we can evaluate α by substituting the initial value $H_1 = 1$. We get $1 = \alpha \cdot 2^1 - 1$, thus $\alpha = 1$.
The solution of the linear non-homogeneous equation giving the number of moves to solve the Towers of Hanoi is $H_n = 2^n - 1$.

Next lecture

Chapter 6. Congruences and modular arithmetics

- motivation
- division and divisibility
- linear congruences in one variable
- methods of solving
- examples and applications