Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an `http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/`"(weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory `http://diestel-graph-theory.com/` (chapters 1-6), Springer, 2010.

See also `http://homel.vsb.cz/~kov16/predmety_dm.php`

Chapter 4. More counting techiques

- inclusion/exclusion principle
- combinatorial identities
- binomial theorem
- pigeon-hole principle

Recapitulation

We introduced terms and symbols

- permutation $P(n)$
- k-combination with or without repetition, $C^*(n,k)$ or $C(n,k)$
- k-permutation with or without repetition, $P^*(n,k)$ or $P(n,k)$

We derived formulas for the number of various selections and arrangements.

But not all selections or arrangements can be counted as simple selections/arrangements. For example

- size of a union of sets
- number of bijections without fixed points
- number of decompositions of an n-element set to k disjoint subsets
- number of decompositions of n to k summands, while order of summands is irrelevant

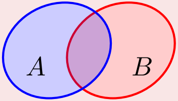
4.1. Inclusion exclusion principle

For small n we use it often intuitively:

Theorem

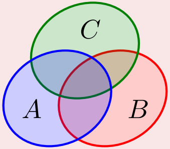
The number of elements in a union of two sets is:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



The number of elements in a union of three sets is:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$



General form of the inclusion exclusion principle

The number of elements in a union of n sets is:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\emptyset}} (-1)^{|J|-1} \cdot \left|\bigcap_{i\in J} A_i\right|.$$

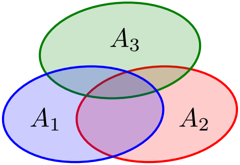
To count the cardinality of a union, we

- sum the cardinalities of all sets,
- subtract the cardinalities of intersections of all pairs of sets,
- add the cardinalities of intersections of all triples of sets,
- subtract the cardinalities of intersections of all quadruples of sets,
- ...

Cardinality of union of three sets

For example for $n = 3$ we get

$$\begin{aligned}\left|\bigcup_{i=1}^{3} A_i\right| &= \sum_{\substack{J\subseteq\{1,2,3\}\\ J\neq\emptyset}} (-1)^{|J|-1}\cdot\left|\bigcap_{i\in J} A_i\right| = \\ &= |A_1| + |A_2| + |A_3| - \\ &\quad - |A_1\cap A_2| - |A_1\cap A_3| - |A_2\cap A_3| + \\ &\quad + |A_1\cap A_2\cap A_3|.\end{aligned}$$

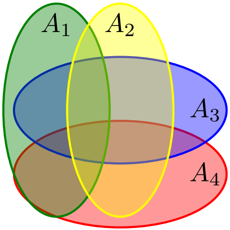


Cardinality of union of four sets

For example for $n = 4$ we get

$$\left|\bigcup_{i=1}^{4} A_i\right| = \sum_{\substack{J\subseteq\{1,2,3,4\}\\ J\neq\emptyset}} (-1)^{|J|-1}\cdot\left|\bigcap_{i\in J} A_i\right| =$$

$$\begin{aligned}
= \quad & |A_1| + |A_2| + |A_3| + |A_4| - \\
- \quad & |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_4| - |A_3 \cap A_4| + \\
+ \quad & |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - \\
- \quad & |A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}$$



Special case of inclusion exclusion principle

A simpler form (with fewer summands), if the intersections of i sets have always the same cardinality:

$$\left|\bigcup_{j=1}^{n} A_j\right| = \sum_{i=1}^{n} (-1)^{i-1} \cdot \binom{n}{i} \cdot \left|\bigcap_{j=1}^{i} A_j\right|.$$

To count the cardinality of a union, we

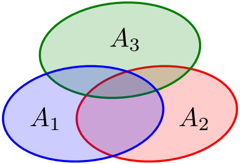
- take the number of one-element sets $\times$ size of A_1,
- subract number of two-element sets $\times$ size of pair-set intersections,
- add number of three-element sets $\times$ size of tripple-set intersections,
- subract number of four-element sets $\times$ size of quadruple-set intersections,
- ...

Cardinality of the union of three sets if each set and each intersection have the same cardinality

For $n = 3$ we have

$$\left|\bigcup_{i=1}^{3} A_i\right| = \sum_{k=1}^{3}(-1)^{k-1}\cdot\binom{3}{k}\cdot\left|\bigcap_{j=1}^{k} A_j\right| =$$

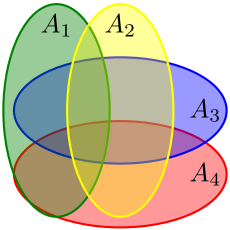
$$= \binom{3}{1}\cdot|A_1| - \binom{3}{2}\cdot|A_1\cap A_2| + \binom{3}{3}\cdot|A_1\cap A_2\cap A_3|.$$



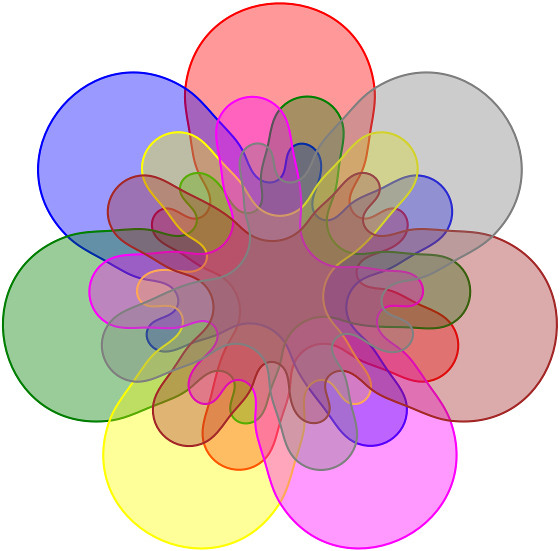
Cardinality of the union of four sets if each set and each intersection have the same cardinality

For $n = 4$ we have

$$
\begin{aligned}
\left|\bigcup_{i=1}^{4} A_i\right| &= \sum_{k=1}^{n} (-1)^{k-1} \cdot \binom{n}{k} \cdot \left|\bigcap_{j=1}^{k} A_j\right| = \\
&= \binom{4}{1} \cdot |A_1| - \binom{4}{2} \cdot |A_1 \cap A_2| + \\
&+ \binom{4}{3} \cdot |A_1 \cap A_2 \cap A_3| - \binom{4}{4} |A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}
$$



Venn diagram for seven sets – Adelaide



Example

There are 25 students in a class. 17 study English and 10 German. 4 study English and German, 4 English and French, 2 German and French and one all three languages. How many students study only French?

We denote the sets by E, G a F. We know

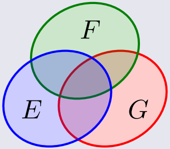
$$|E| = 17,\ |G| = 10,\ |E \cap G| = |E \cap F| = 4,\ |G \cap F| = 2,\ |E \cap G \cap F| = 1$$

From the equation

$$|E \cup G \cup F| = |E| + |G| + |F| - |E \cap G| - |G \cap F| - |E \cap F| + |E \cap G \cap F|$$

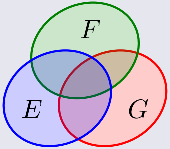
it follows

$$|F| = |E \cup G \cup F| - |E| - |G| + |E \cap G| + |G \cap F| + |E \cap F| - |E \cap G \cap F|$$
$$|F| = 25 - 17 - 10 + 4 + 4 + 2 - 1 = 7.$$



Example (continued)

But some of these 7 students study also other languages!



Just French

$$x = |F| - |E \cap F| - |G \cap F| + |E \cap G \cap F|$$
$$x = 7 - 4 - 2 + 1 = 2 \text{ students.}$$

2 students study just French.

For binomial coefficients we can derive many interesting formulas. There is an entire part of Discrete mathematics dealing with them.

Lemma

For all $n \geq 0$ the following holds

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Statement, proof of which is just a substitution and one or two simple steps we consider as obvious and their proof we do not write down.

Compare the number of corresponding subsets.

More combinatorial identities

Lemma

For all $n \geq k \geq 0$ the following holds

$$\binom{n}{k} = \binom{n}{n-k}.$$

If the proof requires some elaborate step, "trick", or genuine derivation, it is customary to give some explanation.

Compare the number of corresponding subsets.

Lemma

For all $n \geq k \geq 0$ the following holds

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof (direct by substitution and derivations)

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k! \cdot (n-k)!} + \frac{n!}{(k+1)! \cdot (n-k-1)!} =$$

$$= \frac{n! \cdot (k+1) + n! \cdot (n-k)}{(k+1)! \cdot (n-k)!} = \frac{n! \cdot (n+1)}{(k+1)! \cdot (n-k)!} =$$

$$= \frac{(n+1)!}{(k+1)! \cdot ((n+1)-(k+1))!} = \binom{n+1}{k+1}.$$

□

Combinatorial proof is explanatory:
Compare the number of $(k+1)$-element subsets of some $(n+1)$-element set. □

Notion of the binomial coefficient

These formulas are an *alternative definition of binomial coefficients*.

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad \binom{n}{k} = \binom{n}{n-k} \qquad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

The value of the binomial coefficient is uniquely determined by these equalities

- no need for factorials,
- each value can be (recursively) evaluated.

Pascal's triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

. . . .

Pascal's triangle

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \quad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4} = 1$$

$$\binom{5}{0} = 1 \quad \binom{5}{1} \qquad \binom{5}{2} \qquad \binom{5}{3} \qquad \binom{5}{4} \qquad \binom{5}{5} = 1$$

. . . .

All border elements are 1.

Pascal's triangle

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

. . . .

All border elements are 1.

All inner elements equal the sum of two elements immediately above.

Further equalities

"Hockey-Stick" equality

For all positive integers r, n, where $n \geq r$, the following holds $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.

A nice proof using Pascal's triangle.

Another equality

For all positive integers n the following holds $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$.

A nice combinatorial proof.

A more general statement is

Vandermonde's equality

For all positive integers r, m, n, where $r \leq \min\{m, n\}$, the following holds $\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}$.

A nice combinatorial proof.

Binomial Theorem

Binomial Theorem

For all $n > 0$ the following holds

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

Proof The proof can run by induction, but there is a nice argument. Multiplying through we use the rule "multiply each element with each other". Thus in $\underbrace{(1+x)(1+x)\dots(1+x)}_{n}$ each product x^k appears as many times as there are k-element selections from n parentheses. There are $\binom{n}{k}$ such different k-element subsets. $\square$

Combinatorial identities derived from Binomial Theorem

Binomial Theorem

For all $n > 0$ the following holds

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

From the Binomial theorem follows for $n \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n\,.$$

Says: the number of all subsets on an n-element set is 2^n.

From the Binomial theorem follows for $n > 0$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots - (-1)^n\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0\,.$$

The number of odd-sized subsets on an n-element set is the same as he number of even-sized subsets.

Proofs "by counting"

Sometimes we have to show that there exists an element with a certain property, but we cannot find/construct one. Such proofs are called non-constructive.

Instead to "construct" a solution, we show by "counting" there has to be at least one.

The pigeon-hole principle (Dirichlet's principle)

When distributing $\ell + 1$ (or more) objects into ℓ boxes, there has to be a box with at least two objects.

Proofs by counting

The existence of a possibility will follow from the fact that there are too few cases in which the possibility does not occur.

Example

We see three cars entering a tunnel, but only two cars leaving the tunnel. This means there is one car left in the tunnel (though we do not see it).

Example

8 friends went on a 9 day vacation. Each day some triple of them went for a trip. Show, that at least one pair of friends didn't go together on a trip.

Proof Checking of all possibilities would take long...

The proof by counting is easy: In one triple there are 3 pairs, thus after 9 days there was *at most* $9 \cdot 3$ *pairs* on trips. But $9 \cdot 3 = 27 < \binom{8}{2} = 28$, thus at least one pair is missing.

Question

Are there two people on Earth with the same number of hair?

Example

In a drawer there are 30 pairs of black socks, 10 pairs of brown socks, and 3 pairs of white socks. How many socks we have to take (without light or looking) to guarantee, that we have at least one pair of the same color?

“Boxes” in the Pigeon-hole principle are the three colors. While taking four socks (not distinguishing the right or left sock), at least two of them have to be of the same color.

Question

We have four natural numbers. Show, that among them there are two numbers difference of which is divisible by 3.

Question

We have 3 natural numbers. Show, that among them there are two numbers difference of which is divisible by some prime.

Handshaking problem

There are n people in the room, some of them shook hands. Show that there are always at least two people who performed the same number of handshakes.

Example

We have five natural numbers. Show that there are always two among them, such that their sum is divisible by 9.

Proof (incorrect!) We have a total of 9 different classes modulo 9. Among five numbers we obtain 10 different sums. Surely, there has to be at least one sum in each class, in some class there will be at least two sums. Thus the pair which is in class "0", has its sum divisible by 9. $\square$

Question

Why is the proof not correct?

Hint: try to verify the argument for the following set of five numbers: $\{0, 2, 4, 6, 8\}$.

Generalized Pigeon-hole principle

It is not difficult to seem that a more general statement holds.

Generalized Pigeon-hole principle

When distributing $k\ell + 1$ (or more) objects into ℓ boxes, there has to be a box with at least $k + 1$ objects.

Example

In a box there is a sufficient supply of marbles of four colors. What is the least number of marbles to draw to be sure to have ate least 7 marbles of the same color.

Boxes represent the colors, $\ell = 4$.
Marbles are the objects, $k = 6$.
If we draw at least $k\ell + 1 = 6 \cdot 4 + 1 = 25$ marbles, then at least 7 have the same color.

Further counting problems

Not everything can be counted using simple selections and inclusion/exclusion principle.

- Distributing k distinct objects to n identical boxes so that no box remains empty (Stirling numbers of the first kind),
- distributing k distinct objects to n distinct boxes so that no box remains empty (Stirling numbers of the second kind),
- distributing k identical objects to n identical boxes so that no box remains empty,
- distributing k identical objects to n distinct boxes so that no box remains empty.

For the problems above no closed formula exists. We can

- evaluate values by recursion,
- use sums with varying parameters,
- if you need them, you look them up.

These counting techniques are beyond this basic course.

Next lecture

Chapter 5. Recurrence relations

- motivation
- sequences given by recurrence relations
- methods of solving recurrence relations