Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/" (weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Chapter 3. Discrete probability

- describing chance: event, sample space
- independent events
- expected values
- random selections and arrangements

Discrete probability

One of the oldest motivation for counting probabilities is gambling. Questions

- "What are the chances of rolling dice with a given outcome?"
- "What are the chances of receiving a given hand of cards?"

led to formalizing the terms of chance and probability.

We will deal only with *discrete* cases, i.e. situations, that can be described with one of *finitely many* possibilities.

Overview:

- motivation problems
- sample space   (intuition often fails)
- independent events
- expected values
- random selections and arrangements

Motivation problems

Flipping a coin   2 possible outcomes: head / tail (1 / 0)
We expect that both sides are equally likely to be obtained by flipping (we say "with probability $\frac{1}{2}$")

Rolling dice   6 possible outcomes: 1, 2, 3, 4, 5, and 6
each number of points occurs with the same frequency ("probability") $\frac{1}{6}$

Shuffling a deck of cards   we expect, that the shuffling is fair, no shuffle is more likely to occur
there are $32! \doteq 2.6 \cdot 10^{35}$ possible outcomes

Powerball Winning Numbers (Tah sportky)   drawing balls (6 out of 49)
$\binom{49}{6} = 13\ 983\ 816$ possible outcomes   (Powerball: 5/55 and 1/42)

Our "expectancy of probability" is based on the expectancy of fairness. Often different outcomes are considered to be equivalent in the terms of frequency of occurrences of a random experiment.

Finite sample space

How to describe "chance" by a mathematical model?
We expect (subjectively) whether an event will occur. We compare with the previous relative frequency of that event, a number between 0 and 1.

$$\text{probability} = \frac{\text{frequence of an event}}{\text{number of experiments}}$$

Definition

Finite sample space is a pair (Ω, P), where Ω is a finite set of elements and P is probability function, which assigns to every subset of Ω a real value (probability) from $\langle 0, 1 \rangle$, s.t. the following hold

- $P(\emptyset) = 0$, $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B)$ for disjoint $A, B \subseteq \Omega$

An Event is any subset $A \subseteq \Omega$ and its probability is $P(A)$.

Note: it is enough to assign probabilities to the one element sets, called elementary events, or atomic events, or points of Ω. Then, for $A = \{a_1, \ldots, a_k\} \subseteq \Omega$, by definition $P(A) = P(\{a_1\}) + \cdots + P(\{a_k\})$.

Notice

- One element subsets are called elementary events.
  However, the elements are *not* events nor elementary events.

Example

elementary event: rolling a dice you get 1

event: rolling a dice you get an even number

- *disjoint events* cannot occur at the same time: $A \cap B = \emptyset$

Example

Rolling two dice we have events

- A: you get at least one 6
- B: you get the sum 7

are not disjoint events $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$, but

- C: you get at least one 6
- D: you get the sum 3

are disjoint events $P(C \cap D) = 0$, C and D cannot occur together.

Motivation problems based on the definition

Flipping a coin   2 possible outcomes: head / tail (1 / 0)
$\Omega = \{0, 1\}$ a $P(\{0\}) = P(\{1\}) = \frac{1}{2}$.

Rolling dice $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $P(\{1\}) = \ldots = P(\{6\}) = \frac{1}{6}$
Event "rolling an even number" is given by the subset $\{2, 4, 6\}$.

Shuffling a deck of cards Ω contains all 32! permutations of 32 cards, each permutation has the same probability $\frac{1}{32!}$.
Event "Full House" is formed by the subset of permutations with a triple of cards with the same value and a pair with another value on top.

Powerball Winning Numbers (Tah sportky)   drawing balls (6 out of 49)
Ω contains all 6-combinations of 49 numbers, each with the probability $1/\binom{49}{6}$.

Notice: in all examples the elementary events have the same probability.

Definition

If the probability function $P : \mathcal{P}(A) \to \langle 0, 1 \rangle$ is given by

$$P(A) = |A| \,/\, |\Omega|$$

for all $A \subseteq \Omega$, then P is called uniform probability and such sample space Ω is uniform.

- all elementary events have the same probability
- the probability of the event A = relative size of A with respect to Ω

One can specify the probability of elementary events only:
For each elementary event $\{e\} \subset \Omega$ let

$$P(\{e\}) = \frac{1}{|\Omega|}.$$

Example

Random experiment: sum of points while rolling two dice.

The set of all possible sums on two dice $\Omega = \{2, 3, \ldots, 12\}$.
Probabilities of the elementary events differ! There is only one way how to obtain the sum $2 = 1 + 1$, while the sum 7 can be obtained in six ways.

There are $6 \cdot 6 = 36$ possible outcomes, we mentioned that the sum 7 can be obtained in 6 ways, the sum 6 in five ways, etc. We get the following probabilities

$$P(2) = P(12) = \tfrac{1}{36},$$
$$P(3) = P(11) = \tfrac{2}{36} = \tfrac{1}{18},$$
$$P(4) = P(10) = \tfrac{3}{36} = \tfrac{1}{12},$$
$$P(5) = P(9) = \tfrac{4}{36} = \tfrac{1}{9},$$
$$P(6) = P(8) = \tfrac{5}{36},$$
$$P(7) = \tfrac{6}{36} = \tfrac{1}{6}.$$

Example

Random experiment: sum of points while rolling two dice.
(a model with a uniform sample space).
The sample space is $\Omega' = [1,6]^2$ (Cartesian power of the set $\{1,2,3,4,5,6\}$).
The probability of every elementary event is $P(A) = \frac{1}{36}$.

Events for each sum are subset in Ω'

$$
\begin{aligned}
S_1 &= \emptyset,\\
S_2 &= \{(1,1)\},\\
S_3 &= \{(1,2),(2,1)\},\\
&\vdots\\
S_7 &= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\},\\
S_8 &= \{(2,6),(3,5),(4,4),(5,3),(6,2)\},\\
&\vdots\\
S_{12} &= \{(6,6)\}.
\end{aligned}
$$

Probabilities of the events $S_1, \ldots, S_{12}$ are same as in the previous model.

We prefer sample spaces with uniform probability.

Definition

Complementary event to the even A is denoted by $\overline{A}$ and the following holds $\overline{A} = \Omega \setminus A$.

Theorem

The probability of the complementary event $\overline{A}$ to the event A is $P(\overline{A}) = 1 - P(A)$.

Example

We roll an ordinary dice.
Let A be the event "rolling a dice we got 1 or 2",
the complementary event $\overline{A}$ is the event "we rolled neither 1 nor 2", which essentially means "we rolled 3, 4, 5, or 6."

Handy for evaluating probabilities in uniform as well as not uniform samplespaces.
Usually, we count selections or arrangements.

Conditional probability of event A given B occurs

If event B has nonzero probability, then the conditional probability of event A given B occurs we denote by $P(A|B)$ and it is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The conditional probability of event A given B occurs is the probability of the event A, provided the event B occurred.

Example

What is the probability that rolling two dice we get the sum 7, provided that on some dice we got 5.
It is easy to evaluate $P(A \cap B) = \frac{2}{36}$ $(5 + 2$ or $2 + 5)$.
When evaluating $P(B)$ don't count $5 + 5$ twice, thus $P(B) = \frac{11}{36}$.
We get $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{2}{36}}{\frac{11}{36}} = \dfrac{2}{11}$.

Beware! “Intuition” is often misleading

In the previous example we showed, that when rolling two dice the conditional probability of A “the sum is 7” provided the event B “there is a 5 on at least one dice”, je $P(A|B) = \frac{2}{11}$.

But this is *not* the correct solution in the following problem:

Example

A croupier rolls two dice, checks the outcames and announces “There is at least one 5. What is the probability of the sum being 7?”

Probability $\frac{2}{11}$ is the solution to a similar but differently worded problem:

Example

A croupier rolls two dice, he will check the outcome for 5. If no 5 is obtained, he rolls again, until a 5 appears. Then he announces “There is at least one 5. What is the probability of the sum being 7?”

Reason:

It is not a conditional probability $P(A|B)$ if the event B may/did not occur and probability $P(B)$ is zero.

Independent events

It is intuitively clear what independence of events is. Informally:
The probability of one event is not influenced by the result of another event.
(similar to *independent selections/arrangements* from Chapter 2.)

Independent events:

- two consecutive rolls of a dice,
- one roll with two or more dice,
- rolling a dice and shuffling cards,
- drawing from a urn and tossing the ball back
  (two draws in Powerball)

Dependent events:

- top and bottom face numbers on a dice,
- first and second card in a deck,
- consecutive drawings from an urn while keeping drawn balls outside.

Definition

Independent events A, B are such two events, that

$$P(A \cap B) = P(A) \cdot P(B).$$

Alternative definition:

Event A is independent of event B, if the probability of occurrence of event A while event B occurs is the same as the probability of event A

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(\Omega)} \qquad \text{or} \qquad P(A|B) = P(A).$$

Both definitions are equivalent.

Questions

From the "Sportka" urn (we draw one ball and then another ball)

- what is the probability that the first ball is 1?
- what is the probability that the second ball is 2?
- what is the probability that second ball is 2, provided first ball was 1?
- how do the probabilities change if we return the first ball before drawing the second ball?

Expected value

Let us explore random experiments whose outcome is a (natural) number. Let us concentrate on problems with finitely many possible outcomes.

Definition of a random variable X

The outcome of a random experiment, which gives a number as a result will we call a random variable X.

Definition of expected value

Let X be a random variable that can have k possible outcomes from the set $\{h_1, h_2, \ldots, h_k\}$, where h_i occurs with the probability p_i, a $p_1 + p_2 + \cdots + p_k = 1$. *The expected value of* X is the number

$$E(X) = EX = \sum_{i=1}^{k} p_i h_i = p_1 \cdot h_1 + p_2 \cdot h_2 + \cdots + p_k \cdot h_k.$$

Thus, it represents the average amount one "expects" as the outcome of the random trial when identical experiments are repeated many times.

Example

What is the expected value when rolling a dice?

$$E(K) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{21}{6} = 3.5.$$

Example

What is the expected value when rolling a dice on which 6 has the probability of occurrence twice as high as any other number?

$$E(K) = \frac{1}{7} \cdot 1 + \frac{1}{7} \cdot 2 + \frac{1}{7} \cdot 3 + \frac{1}{7} \cdot 4 + \frac{1}{7} \cdot 5 + \frac{2}{7} \cdot 6 = \frac{27}{7} \doteq 3.8571.$$

Example

There is a 20\$ a 5\$ and a 1\$ banknote. We pick a banknote by random. Every bigger value has a two times bigger probability to be chosen than the smaller. What is the expected value of the banknote M?

$$p_1 = \frac{1}{7},\ p_5 = \frac{2}{7},\ p_{20} = \frac{4}{7}, \quad E(M) = \frac{1}{7} \cdot 1 + \frac{2}{7} \cdot 5 + \frac{4}{7} \cdot 20 = \frac{91}{7} = 13.$$

Sum of expected values

For any two random variables X, Y the following holds

$$E(X + Y) = E(X) + E(Y).$$

Product of expected values

For any two independent random variables X, Y the following holds

$$E(X \cdot Y) = E(X) \cdot E(Y).$$

Example

What is the expected value of the sum while rolling two dice?

Using the results from the previous example and by Sum of expected values theorem

$$E(K_1 + K_2) = E(K_1) + E(K_2) = 3.5 + 3.5 = 7.$$

Examples

What is the expected value of the product of numbers of points while rolling two dice?

Using the first example and by Product of expected values theorem

$$E(K_1 \cdot K_2) = E(K_1) \cdot E(K_2) = 3.5 \cdot 3.5 = 12.25.$$

Similarly $E(K_1 \cdot K_2) = \sum_{i=1}^{6} \sum_{j=1}^{6} \frac{1}{36} \cdot i \cdot j = \frac{441}{36} = \frac{49}{4} = 12.25$

What is the expected value of the product of the numbers of points on the top and bottom face while rolling one dice?

The expected value on the top face is 3.5 and on the bottom face also 3.5. But the expected value of their product is not $3.5 \cdot 3.5 = 12.25$, since the two variables are not independent.

We can evaluate the expected value by definition

$$\frac{1}{6}(1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1) = \frac{56}{6} \doteq 9.3333.$$

Carefully about the independence of events while multiplying expected values!

Random selections and arrangements

Frequently used *finite uniform random selections* in discrete mathematics:

Random subset Given an n-element set we pick any of the 2^n subsets, each with the probability 2^{-n}.

Random permutation Among all $n!$ permutations of a given n-element set we pick one with the probability $1/n!$.

Random combination Among all $\binom{n}{k}$ k-combinations of a given n-element set we pick one with the probability $1/\binom{n}{k}$.

Random bit sequence We obtain an arbitrary long sequence of 0 and 1 so that any subsequent bit is chosen with probability $\frac{1}{2}$ (does not depend on any previous bit, similarly to flipping a coin). Each subsequence of this random sequence is equally likely to occur.

Next lecture

Chapter 4. Counting and combinatorial identities

- inclusion exclusion principle
- double counting
- combinatorial identities
- proofs “by counting”