Discrete mathematics

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About this file

This file is meant to be a guideline for the lecturer. Many important pieces of information are not in this file, they are to be delivered in the lecture: said, shown or drawn on board. The file is made available with the hope students will easier catch up with lectures they missed.

For study the following resources are better suitable:

- Meyer: Lecture notes and readings for an http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2005/readings/"(weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory http://diestel-graph-theory.com/ (chapters 1-6), Springer, 2010.

See also http://homel.vsb.cz/~kov16/predmety_dm.php

Course number: 470-2301/02, 470-2301/04*, 470-2301/06
Credits: 6 credits (2/2/2), *5 credits (2/2/1)
Warrant: Petr Kovář
Lecturer: Petr Kovář/Tereza Kovářová

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Classification

Tests

- every week (starting with the third week)
- 2–10 minutes
- evaluation 0/1/2 (no/almost correct/completely correct)
- every other week one additional teoretical question
- we take 4 best 2-point scores and 4 best 3-point scores among 10
- total up to 20 points
- if a student skips a test: 0 points

Typical assignments available at `http://am.vsb.cz/kovar` (in Czech).

Classification (cont.)

Project

- assigned in the second half of the term
- **project**: two or four problems (discrete math & graph theory)
- Bonus **Projects for all who want to learn something**
  contains two problems (1 discrete mathematics & 1 graph theory)
- total of 10 points
- to receive credit ("zápočet") the project has to be accepted (minimum standards, see web)
- keep the deadline!
- work alone!

Credit ("Zápočet") = at least 10 points and an accepted project

Classification (cont.)

Exam

- examining dates given at the end of the term
- total of 70 points
- sample exam on the web (http://am.vsb.cz/kovar)
- you can use one page A4 with handwritten notes
  definitions, theorems a formulas, but no examples

Literature

In Czech:

- (partially M. Kubesa. Základy diskrétní matematiky, textbook on-line).
- P. Kovář: Algoritmizace diskrétních struktur on-line.
- P. Kovář. Úvod do teorie grafů, textbook on-line.
- P. Kovář: Cvičení z diskrétní matematiky, exercises on-line.
- solved examples as “pencasts” available on-line.

In English:

- Meyer: Lecture notes and readings for an open course (weeks 1-5, 8-10, 12-13), MIT, 2005.
- Diestel: Graph theory on-line preview (chapetrs 1-6), Springer, 2010.

You are free to use any major textbook, but beware: details can differ! At the exam things will be required as in the lecture.

Office hours

We 9:30–10:30 (?) EA536.

see web: http://am.vsb.cz/kovar

Sample problems

Some problem, we will learn how to solve:

- handshaking problem...
- list all possible tickets in powerball ...
- nine friends exchanging three presents each...
- three lairs and three wells...
- seven bridges of Königsberg...
- missing digits in social security number ("rodné číslo")...
- correcting UPC bar codes...
- Monty Hall...

Additional interesting problems and exercises:
http://am.vsb.cz/kovar.

Chapter 0. Review

- number sets
- set and set operations
- relations
- proof techniques
- mathematical induction
- ceiling and floor functions

Numbers and interval of integers

Natural numbers and integers

Natural numbers are denoted by $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$
notice! zero is not among them
Natural numbers with zero included denoted by $\mathbb{N}_0 = \{0, 1, 2, 3, 4, 5, \dots\}$
Integers are denoted by $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$

Intervals of integers between a and b

is the set $\{a, a+1, \dots, b-1, b\}$
we denote it by: $[a, b] = \{a, a+1, \dots, b-1, b\}$

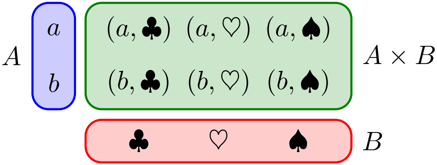
Compare to the notation used for an interval of real numbers (a, b).

Examples

$[3, 7] = \{3, 4, 5, 6, 7\}$ $\quad [-2, -2] = \{-2\}$
$[5, 0] = \emptyset$ $\quad$ (the empty set)

Cartesian product and Cartesian power

Cartesian product of two sets $A \times B = \{(a, b) : a \in A,\ b \in B\}$
is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ in this order.
$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i,\ i = 1, 2, \ldots, n\}$
For $A_1 = A_2 = \ldots = A_n$ we get the Cartesian power A^n.
We define $A^0 = \{\emptyset\}$, $A^1 = A$.



Cartesian product of sets $A \times B = \{a, b\} \times \{\clubsuit, \heartsuit, \spadesuit\}$.

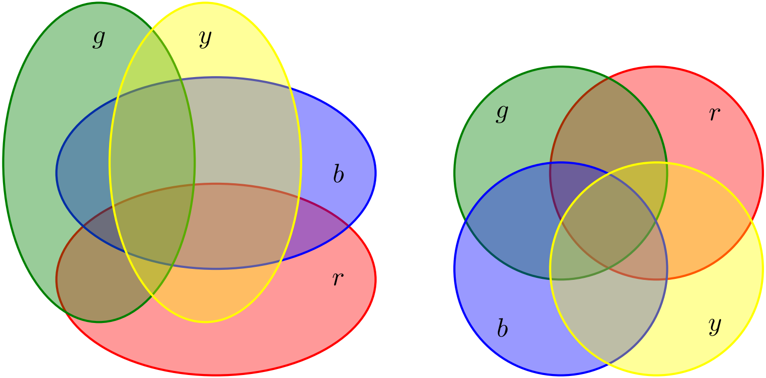
Power set of A

is the set of all subsets of A

$$2^A = \{X : X \subseteq A\}.$$

A family of sets over A

or a family of subsets of A is some $\mathcal{T} \subseteq 2^A$.
We prefer the term "family of sets" to "set of sets".



All subsets of the set of colors $C = \{r, g, b, y\}$.

Generalized unions and intersections

Generalized union $\bigcup_{i=1}^{n} X_i$ and intersection $\bigcap_{i=1}^{n} X_i$ of sets.

Given an index set J, we can write $\bigcup_{j \in J} X_j$ and $\bigcap_{j \in J} X_j$.

Examples

$$A_i = \{1, 2, \ldots, i\}, \quad \text{for each } i \in \mathbb{N}$$

$$\bigcup_{i=1}^{5} A_i = \{1, 2, 3, 4, 5\}, \quad \bigcap_{i=1}^{5} A_i = \{1\}, \quad \bigcap_{i=1}^{\infty} A_i = \{1\}$$

Questions

What is $\bigcap_{j \in J} A_j$ for $J = \{2, 5\}$?

What is $\bigcup_{j \in J} A_j$ for $J = \mathbb{N}$?

Definition

(Homogenous) binary relation R on the set A is a subset of the Cartesian product $A \times A = A^2$, i.e.

$$R \subseteq A^2.$$

Definition

(Homogenous) n-ary relation S on the set A is a subset of the Cartesian power $A \times A \times \cdots \times A = A^n$, i.e.

$$S \subseteq A^n.$$

Example

- Relation between students, with the same grade in DiM.
- Relation between pairs of students, who has a higher score.
- Relation between documents with similar terms (plagiarism)...

Binary relation is a special case of an n-ary relation. (unary, ternary, ...). (Homogenous) relations on a given set are special case of (heterogenous) relation between sets. In greater detail in another course.

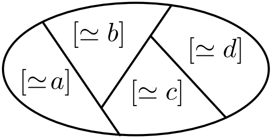
Equivalence relation

Definition

Equivalence on the set A is a *reflexive, symmetric, and transitive* binary relation on the set A. We denote it by $\simeq$.

Definition

Let $\simeq$ be an equivalence relation on the set A. An equivalence class of x (denoted by $[\simeq x]$) is the subset of A defined by $[\simeq x] = \{z \in A : z \simeq x\}$.



Equivalence relation expresses "having the same property".

Examples

- congruence relation $\equiv$ (same remainder after division by n)
- relation among students "having the same grade in DIM"
- relation "synonyms in a language" is (often) an equivalence

Partial ordering

Ordering and equivalence are among the most common relations.

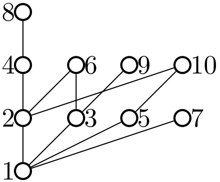
Definition

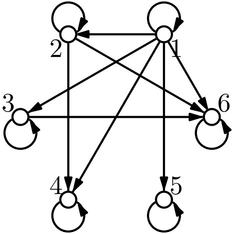
Partial ordering $\preceq$ on the set A is *reflexive, antisymmetric, and transitive* binary relation on the set A. The set with the relation is called a poset.

The word *partial* emphasizes the fact, that the relation does not have to be *linear* relation on A, i.e. not every pair of elements has necessarily to be related. Neither xRy nor yRx.

Partial orderings can be illustrated by a Hasse diagram

- if $x \preceq y$, then the element y will be drawn higher than x,
- elements x and y will be connected by a line if $x \preceq y$. We omit all lines that follow from transitivity.





0.4.2. Concept of a mathematical proof

Theorem (claims) in mathematics are usually of the form of a conditional statement: $P \Rightarrow C$

Precisely formulated premise (or hypothesis) P, under which the conclusion (consequence) C holds.

Detailed description how to obtain the conclusion from the premises is called a proof.

Mathematical proof

of some statement C is a finite sequence of steps including:

- axioms – or postulates that are considered true (the set of postulates differs for various disciplines*),
- hypothesis P is an assumption on which we work,
- statement derived from previous by some correct rule (depends on logic used).

The last step is a conditional statement with *conclusion* C.

* Discrete mathematics relies on Peano axioms, geometry is build upon five Euklid's postulates, ...

Peano axioms

Stating the axioms

- There exists a natural number (usually denoted 0), which is not a successor of any number.
- For every natural number n there exists its successor $S(n)$.
- Different natural numbers have different successors.
- If for a property X the following hold
  - number 0 has property X and
  - if from n having the property X follows that property X has also its successor $S(n)$,

  then property X have all natural numbers including 0.

What could I need a proof for?

"What is the use of a newborn?"

- correctly understand the limitations of various method
- arguments for/against a presented solution
- comparison of quality of different solutions
- 100% validity of an algorithm may be required
  (autopilot, intensive care unit)

Mathematical induction

Mathematical induction is a common proof technique used to prove propositional functions with a natural parameter n, denoted by $P(n)$.

Mathematical induction

Let $P(n)$ be a propositional function with an integer parameter n.
Suppose:

- *Basis step:*
  The proposition $P(n_0)$ is true, where $n_0 = 0$ or 1, or some integer.
- *Inductive step:*
  Assume the Inductive hypothesis: $P(n)$ holds for some n.
  Show, that for all $n > n_0$ if $P(n)$ holds, then also $P(n+1)$ holds.

Then $P(n)$ is true for all integers $n \geq n_0$.

Mathematical induction can be used also to prove validity of algorithms.

A few examples follow...

Wait a minute!

But. . .

- we verify the Basis step,
- we verify the Inductive step (using the Inductive hypothesis),

. . . how come this implies the validity for infinity many values!?!

Example

How high can you climb a ladder?

Suppose we can

- mount the first step,
- standing on rung n climb the rung $n + 1$.

. . . thus, we can reach any rung of the ladder!

Theorem

The sum of the first n even natural numbers is $n(n+1)$.

$2+4+6=12=3\cdot 4$
$2+4+6+8+10+12+14+16+18+20=110=10\cdot 11$

Proof by mathematical induction based on n:
We prove $\forall n \in \mathbb{N}$ the following holds $\sum_{i=1}^{n} 2i = n(n+1)$.

- *Basis step:* For $n=1$ claim P(1) gives "$2=1\cdot 2$".
- *Inductive step:* Does $P(n)$ imply $P(n+1)$?

  I.e. does $\sum_{i=1}^{n} 2i = n(n+1)$, imply $\sum_{i=1}^{n+1} 2i = (n+1)(n+2)$?

  We state *Inductive hypothesis* P(n):
  Suppose $\exists n \in \mathbb{N} : \sum_{i=1}^{n} 2i = n(n+1)$.
  Now
  $\sum_{i=1}^{n+1} 2i = \sum_{i=1}^{n} 2i + 2(n+1) \overset{IH}{=} n(n+1) + 2(n+1) = (n+1)(n+2)$.
  We have shown the correctness of the formula for the sum of the first $n+1$ evens using the formula for the sum of the first n evens.

By mathematical induction the claim holds $\forall n \in \mathbb{N}$. $\square$

Strong mathematical induction compared to mathematical induction

Mathematical induction

Let $P(n)$ be a propositional function with an integer parameter n.
Suppose:

- *Basis step:*
  The proposition $P(n_0)$ is true, where $n_0 = 0$ or 1, or some integer n_0.
- *Inductive step:*
  Assume the Inductive hypothesis: $P(n)$ holds for some n.
  Show, that for all $n > n_0$ if $P(n)$ holds, then also $P(n+1)$ holds.

Then $P(n)$ is true for all integers $n \geq n_0$.

Strong mathematical induction

- *Basis step:* The proposition $P(n_0)$ is true.
- *Inductive step:*
  Inductive hypothesis: Assume $P(k)$ holds for all $n_0 \leq k < n$.
  Show, that also $P(n)$ is true.

Then $P(n)$ is true for all integers $n \geq n_0$.

Example

There are always $pr - 1$ breaks necessary to split a chocolate bar of $p \times r$ squares.

By *strong* induction on $n = pr$:

- *Basis step:*
  For $n_0 = 1$ we have a bar with only one square, there are no breaks necessary $(pr - 1 = 0)$.
- *Inductive step:*
  Suppose now the claim holds for *any* chocolate bars with less than n squares. Take any bar with n squares. We break this bar into two parts of s or t squares, respectively, where $1 \leq s, t < n$ and $s + t = n$. By Inductive hypothesis we can break each part by $s - 1$ or $t - 1$ breaks, respectively. There is a total of
  $(s - 1) + (t - 1) + 1 = s + t - 1 = n - 1$ breaks necessary.

The proof is complete by strong induction for all positive p, r. $\square$

Integer part of a real number

$\lfloor x \rfloor$ floor function for a real number x
$\lceil x \rceil$ ceiling function for a real number x

Example

$\lfloor 3.14 \rfloor = 3 \qquad \lfloor -3.14 \rfloor = -4$
$\lfloor x \rfloor = \lceil x \rceil \quad \Rightarrow \quad x \in \mathbb{Z}$

Question

Gives the expression $\lceil \log n \rceil$ the number of digits of n (in decimal system)?

If not, can you find a "correct" formula?

Question

$\left\lceil \frac{n}{n+1} \right\rceil =?$, where $n \in \mathbb{N}$ (what if $n \in \mathbb{N}_0$)

Chapter 1. Sequences

- sequences
- sums and products
- arithmetic progression
- geometric progression

Sequence

is an ordered list of objects, called elements.
We denote it by $(a_i)_{i=1}^{n} = (a_1, a_2, \ldots, a_n)$.

- in real analysis defined as mappings $p : \mathbb{N} \to \mathbb{R}$
- we distinguish first, second, third, ... element in the sequence.
- indices are natural numbers, usually starting at 1
- elements in a sequence can repeat (in contrary to sets)
- sequences can be finite $(a_1, a_2, \ldots, a_n)$ and infinite $(a_1, a_2, \ldots)$, the sequence can even be empty (we focus mainly on finite sequences)

Examples

(x, v, z, v, y)
$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29)$
$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots)$
$(1, -1, 1, -1, 1, -1, 1, -1, \ldots)$

A sequence is given by: listing the elements, recurrence relations or a formula for the n-th element.

Sums

Sum of a sequence is denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$\sum_{i \in J} a_i = a_{i_1} + a_{i_2} + \cdots + a_{i_n}, \text{ where } J = \{i_1, i_2, \ldots, i_n\}.$$

Question

$$\sum_{i \in \{1,3,5,7\}} i^2 = ?$$

Example

$$\sum_{i=1}^{n} i = ?$$

Product

Product of elements in a sequence is denoted by

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \cdots \cdot a_{n-1} \cdot a_n$$

$$\prod_{i \in J} a_i = a_{i_1} \cdot a_{i_2} \cdot \ \cdots \ \cdot a_{i_n}, \text{ where } J = \{i_1, i_2, \ldots, i_n\}$$

Examples

$$\sum_{i=2}^{5} \ln(i) = \ln\left(\prod_{i=2}^{5} i\right) = \ln(2 \cdot 3 \cdot 4 \cdot 5) = \ln 120$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i \cdot j) = \sum_{i=1}^{n}\left(i \cdot \sum_{j=1}^{n} j\right) = \left(\sum_{i=1}^{n} i\right) \cdot \left(\sum_{j=1}^{n} j\right) = \left(\frac{1}{2}n(n+1)\right)^2$$

empty sum $\displaystyle\sum_{i=3}^{2} i = 0$ empty product $\displaystyle\prod_{i=3}^{2} i = 1$

Examples

$$\sum_{i=1}^{n}(i+j) = \sum_{i=1}^{n} i + \sum_{i=1}^{n} j = \frac{n}{2}(n+1) + nj$$

$$J = \{2, 8, 12, 21\}, \qquad \sum_{j \in J} j = 2 + 8 + 12 + 21 = 43$$

Questions

$$\sum_{i=1}^{5} \ln(i) = ? \qquad \sum_{i=1}^{100} i = ?$$

$$\prod_{i=1}^{6} i = ? \qquad \prod_{i=1}^{n} i = ?$$

$$\prod_{i=1}^{n}(n-i) = ? \qquad \sum_{i=1}^{n}(n+1-i) = ?$$

Question

Can you find a sequence $(a_i)_{i=1}^n$, such that $\sum_{i=1}^n a_i < \sum_{i=1}^n (-a_i)$?

Question

Can you find a sequence $(a_i)_{i=1}^n$, such that $\sum_{i=1}^n a_i > 0$ and $\prod_{i=1}^n a_i < 0$?

Question

Does there exist a sequence of *positive* numbers $(a_i)_{i=1}^n$, such that $\sum_{i=1}^n a_i > \prod_{i=1}^n a_i$?

Arithmetic progression

Certain sequences for special progressions and we know several their properties.

Arithmetic progression

The sequence (a_i) is an arithmetic progression if its terms are

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ldots$$

Real numbers a, d are the first term and the difference of the progression, respectively.

Notice that the sequence (a_i) is an arithmetic progression, if there exists a real number d, such that for all $i > 1$ is $a_i - a_{i-1} = d$.

Every subsequent term arises by adding (the same!) difference d to the previous term.

Finite arithmetic progressions are also considered.
We have n terms

$$a, a+d, a+2d, \ldots, a+(n-1)d.$$

Examples

$-2, 3, 8, 13, 18, \ldots$ first term -2, difference 5

$-3, 2, 7, 12, 17, \ldots$ first term -3, difference 5

$20, 9, -2, -13, -24, \ldots$ first term 20, difference -11

$\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots$ first term $\sqrt{2}$, difference 0

Examples

Find the n-th term of the progressions a_n from previous example

$-2, 3, 8, 13, 18, \ldots$ $a_n = -2 + (n-1)5$

$-3, 2, 7, 12, 17, \ldots$ $a_n = -3 + (n-1)5$

$20, 9, -2, -13, -24, \ldots$ $a_n = 20 - (n-1)11$

$\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots$ $a_n = \sqrt{2}$

Example

Which sequence is given by the n-th term $a_n = -8 + 5n$?

Second progression $-3, 2, 7, 12, 17, \ldots$

Summing n terms of an arithmetic progression

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

In this case

$$a + (a+d) + \cdots + a + (n-1)d = \sum_{i=1}^{n} (a_1 + (i-1)d)$$

holds

$$\sum_{i=1}^{n} (a_1 + (i-1)d) = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n-1)d) = na_1 + \frac{n(n-1)d}{2}.$$

Sum of certain consecutive n terms of an arithmetic progression

$$\sum_{i=k}^{k+n-1} a_i = \frac{n}{2}(a_k + a_{k+n-1}) = \frac{n}{2}(2a_k + (n-1)d) = na_k + \frac{n(n-1)d}{2}.$$

Notes

The sum of an infinite arithmetic progression generally does not exist.

Sequence of partial sums

- diverges to $+\infty$ for $d > 0$,
- diverges to $-\infty$ for $d < 0$,
- for $d = 0$ diverges to $+\infty$ or to $-\infty$ or converges based on a_1.

Arithmetic progression with first term a and difference d can be given by a recurrence relation

$$a_n = a_{n-1} + d, \quad a_1 = a.$$

Example savings

Example

Uncle Scrooge has 4 514 cents in his safe. Every week he adds 24 cents to the safe. What is the formula for a_n?

$$4\ 514, 4\ 538, 4\ 562, 4\ 586, \cdots = 4\ 514 + 24(n-1) = 4\ 490 + 24n.$$

Example

Uncle Scrooge has 4 514 cents in his safe. The pocket money of each of his three nephews is 1 cent, but every week he increases the pocket money by one cent.
a) Evaluate the total pocket money in the n-th week.
b) Evaluate the number of cents in the safe in the n-th week.

a) pocket money $k = 3 + 3(n-1) = 3n$
b) in safe $s = 4\ 514 - 3n$

Geometric progression

Geometric progression

The sequence (a_i) is a geometric progression if its terms are

$$a, \quad a \cdot q, \quad a \cdot q^2, \quad a \cdot q^3, \ldots$$

Real numbers a, q are the first term and the common ration of the progression, respectively.

Notice that the sequence (a_i) is a geometric progression if there exists a real number q, such that for all $i > 1$ is $\frac{a_i}{a_{i-1}} = q$.

Every subsequent term arises by multiplying the previous term by (the same!) common ratio q.

Finite geometric progressions are also considered. We have n terms

$$a, \quad a \cdot q, \quad a \cdot q^2, \ldots, \quad a \cdot q^{n-1}.$$

Question

Can a progression be both geometric and arithmetic at the same time?
If yes, can you find different solutions? Infinitely many?

Examples

$2, 10, 50, 250, 1250, \ldots$ first term 2, common ratio 5

$9, 6, 4, \frac{8}{3}, \frac{16}{9}, \ldots$ first term 9, common ratio $\frac{2}{3}$

$4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$ first term 4, common ratio $-\frac{1}{2}$

$\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots$ first term $\sqrt{2}$, common ratio 1

Examples

Find the n-th term of the progressions a_n from previous example

$2, 10, 50, 250, 1250, \ldots$ $a_n = 2 \cdot 5^{n-1}$

$9, 6, 4, \frac{8}{3}, \frac{16}{9}, \ldots$ $a_n = 9 \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{27}{2} \cdot \left(\frac{2}{3}\right)^n$

$4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots$ $a_n = 4 \cdot \left(-\frac{1}{2}\right)^{n-1} = -8 \cdot \left(-\frac{1}{2}\right)^n$

$\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots$ $a_n = \sqrt{2}$

Example

Which sequence is given by the n-th term $a_n = \left(\frac{1}{2}\right)^n$? $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Sum of n terms of a geometric progression

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

In our case

$$a + (a \cdot q) + \cdots + a \cdot q^{n-1} = \sum_{i=1}^{n} (a_1 \cdot q^{i-1})$$

for $q \neq 1$ holds

$$\sum_{i=1}^{n} (a_1 \cdot q^{i-1}) = a_1 \frac{q^n - 1}{q - 1}.$$

For $q = 1$ is the progression both arithmetic and geometric; we use a different formula.

Question

How does the sum of first n terms of a geometric progression with common ratio 1 look like?

Notes

The sum of an infinite geometric progression

- generally does not exist for $|q| \geq 1$,
- for $q = 1$ the sequence is constant; the sum depends on a_1,
- for $q = -1$ the sequence oscillates, there is no sum
- for $|q| < 1$ the sum is finite $\frac{a_1}{1-q}$

Sequence of partial sums of an infinite geometric progression

- diverges for $q \geq 1$,
- oscillates (and does not converge for $q \leq -10$,
- converges to $\frac{a_1}{1-q}$ for $|q| < 1$.

A geometric progression with first term a and common ratio q can be described recursively

$$a_n = a_{n-1} \cdot q, \quad a_1 = a.$$

Example savings

Example

Uncle Scrooge has 4 514 cents in a bank. Every year he get an interest of 2 percent (no rounding). What is the formula for the amount a_n (after n years)?

$$4\ 514, 4\ 604.3, 4\ 696.4, 4\ 790.3, 4\ 886.1, 4\ 983.8, \cdots = 4\ 514 \cdot 1.02^{n-1}.$$

Example

Uncle Scrooge has 4 514 cents in his safe. The pocket money of each of his three nephews is 1 cent, but every week he doubles the pocket money of each.

a) Evaluate the total pocket money in the n-th week.

b) Evaluate the number of cents in the safe in the n-th week.

a) pocket money $k = 3 \cdot 2^{n-1} = \frac{3}{2} \cdot 2^n$

b) in safe $s = 4\ 514 - 3 \cdot 2^{n-1}$

Example

We tilt the pendulum to 5 cm height. Due friction each sway of the pendulum looses one fifth of it energy. Describe the sequence of heights to which the pendulum rises after each sway.

$$5 \text{ cm}, \quad 4 \text{ cm}, \quad \frac{16}{5} \text{ cm}, \quad \frac{64}{25} \text{ cm}, \quad \frac{256}{125} \text{ cm}, \ldots, \quad 5 \cdot \left(\frac{4}{5}\right)^{n-1} \text{ cm}$$

first term 5 cm,
common ratio $\frac{4}{5}$.

Question

After how many tilts will the pendulum stop?

Next lecture

Arrangements and selections

- multiplication principle (of independent selections)
- method of double counting